SLOWER-THAN-LIGHT SPIN-$\frac{1}{2}$ PARTICLES ENDOWED WITH NEGATIVE MASS SQUARED

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Abstract

Extending in a straightforward way the standard Dirac theory, we study a quantum mechanical wave-equation describing free spinning particles—which we propose to call Pseudotachyons (PT’s)—which behave like tachyons in the momentum space ($p^2 = -m^2$), but like subluminal particles ($v < c$) in the ordinary space. This is allowed since, as it happens in every quantum theory for spin-$\frac{1}{2}$ particles, the momentum operator, $-i\nabla$, (that is conserved) and the velocity operator $\alpha$ (that is not) are independent operators, which refer to independent quantities: $\hat{p} \neq m\hat{v}$. As a consequence, at variance with ordinary Dirac particles, for PT’s the average velocity $\overline{v} \equiv \langle \psi^\dagger \alpha \psi \rangle / \langle \psi^\dagger \psi \rangle$ is not equal to the classical velocity $v_{c1} = p/\varepsilon$, but actually to the velocity “dual” of $v_{c1}$: $\varepsilon p/p^2$. Being reciprocal of $|v_{c1}|$, the speed of PT’s is therefore smaller than the speed of light. Since a lot of experimental data seems to involve a negative mass squared for neutrinos, we suggest that these particles might be PT’s, travelling, because of their very small mass, at subluminal speeds very close to $c$. The present theory is shown to be separately invariant under the $C, P, T$ transformations; the covariance under Lorentz transformations is also proved. Furthermore, we derive the kinematical constraints linking 4-impulse, 4-velocity and 4-polarization of free PT’s.

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1 Dirac-like equation for spin-$\frac{1}{2}$ particles endowed with negative mass squared

One of the simplest Dirac-like lagrangians, at the first order in the scalar bilinear $\overline{\psi}\psi$, hermitian and relativistically invariant, which is able to describe (as it will be shown) spin-$\frac{1}{2}$ free particles

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endowed with a negative mass squared, is the following:

\[ \mathcal{L} = i \overline{\psi} \gamma^5 \gamma^\mu \partial_\mu \psi - m \overline{\psi} \psi \]  

(1)

[as usual, we shall hereafter assume: \( \hbar = c = 1; \overline{\psi} \equiv \psi \dagger \gamma^0; \ \gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \), with \( (\gamma^5) \dagger = \gamma^5 \)], whilst the ordinary Dirac lagrangian (describing the so-called bradyons endowed with positive 4-impulse squared) writes

\[ \mathcal{L} = i \overline{\psi} \gamma^\mu \partial_\mu \psi - m \overline{\psi} \psi . \]  

(2)

For reasons which will be clarified in the fourth section henceforth we shall call Pseudotachyon\s (PT’s) the spinning particles with negative 4-impulse described by eq. (1). This lagrangian, firstly introduced by Tanaka\cite{1}, has been re-proposed in order to describe “tachyonic neutrinos” about ten years ago by Chodos \textit{et al.}\cite{2}. In this paper we study, within a new formal approach, the plane wave solutions of the PT Dirac equation entailed by this lagrangian. In so doing we shall deduce out peculiar physical results. Actually, at variance with the previous literature on spinning quantum particles endowed with \( p^2 < 0 \),\cite{3-7} we shall find that \textit{such particles are slower-than-light particles}. In the last section we shall prospect the possible observation of PT’s as ordinary neutrinos.

Before proceeding further let us advise that, for the present work, \textit{no extension of special relativity is demanded} and that \textit{only standard Lorentz transformations with subluminal boosts of the reference frames are possible}. In such a way timelike 4-impulses transform into timelike 4-impulses and spacelike 4-impulses into spacelike 4-impulses, without any change of the sign of the mass squared \( p^2 \). As a consequence, we have \textit{two distinct theories}, one generated by lagrangian (1), and one by lagrangian (2), referring to two really different types of particles. These theories entail \textit{in every reference frame} opposed sign for \( p^2 \), even if, as it will be shown in the last section, both types of particles appear to travel at subluminal speeds. Among the most noticeable differences there is, e.g., \textit{the non-existence, for PT’s, of the center-of-mass frame}, where \( p = 0 \), because we always have \( |p| \geq m \) in every frame. In fact, only through a “superluminal” Lorentz transformation with a boost larger than \( c \), the particle momentum could vanish with respect to a given reference frame. Nevertheless, we shall see below that \textit{there exists the “quiet” frame} in which, not the momentum, but the velocity vanishes.

Taking the variation with respect to the “lagrangian coordinate” \( \overline{\psi} \) we obtain the \textit{PT Dirac-like equation}:

\[ (i \gamma^5 \gamma^\mu \partial_\mu - m) \psi = 0 , \]  

(3)

in which the kinetic term differs from the one of the standard Dirac equation \( (i \gamma^\mu \partial_\mu - m) \psi = 0 \). Notice the analytic continuity of equation (3) with the Dirac equation: the “right limit” of the PT equation for \( v \to c^+ \) coincides with the “left limit” for \( v \to c^- \) of the bradyonic equation, because for \( m \to 0 \) the two equations entail the same solutions (see also the next section).

Usually\cite{3-7} the constitutive relation of tachyonic kinematics, \( p^2 = -m^2 \), is found by making recourse to non-hermitian terms in the lagrangian which involve an \textit{imaginary mass}. On the contrary, by means of lagrangian (1), this constraint is obtained simply as a direct consequence
of the non-commutativity of the hermitian matrix $\gamma^5$ with the matrices $\gamma^\mu$. In fact, multiplying eq. (3) left for $(i\gamma^5\gamma^\mu \partial_\mu + m)$,
\[
(i\gamma^5\gamma^\mu \partial_\mu + m) (i\gamma^5\gamma^\mu \partial_\mu - m) \psi = 0 ,
\]
for the anticommutativity of $\gamma^\mu$ with $\gamma^5$, which implies $(\gamma^5\gamma^\mu)^2 = -I$, we get:
\[
(\Box - m^2) \psi = 0 ,
\]
namely, the so-called Klein–Gordon equation for tachyons.$^{[8,9]}$

For the “hermitian adjoint” wave-function $\overline{\psi}$ the following equation holds (where $i\overline{\partial}$ indicates the transposed 4-impulse operator which is “left-acting”)
\[
\overline{\psi} (i\gamma^5\gamma^\mu \overline{\partial}_\mu + m) = 0 .
\]
Expliciting in (3) the product $\gamma^5\gamma^\mu \partial_\mu$, and left multiplying by $\gamma^0\gamma^5$, namely
\[
(i\partial_t - i\alpha \cdot \nabla) \psi = m\gamma^0\gamma^5 \psi
\]
(where, as usual, $\alpha \equiv \gamma^0\gamma^5$), we obtain the hamiltonian operator for free PT’s:
\[
\hat{H} = -i\alpha \cdot \nabla + m\alpha^5 .
\]
In the last expression we have renamed $\alpha^5$ the $\gamma^0\gamma^5$ matrix, with $(\alpha^5)^2 = -1$. Obviously, applying to $\psi$ the hamiltonian squared $\hat{H}^2$ we obtain:
\[
\hat{H}^2 \psi = -\partial^2_t \psi = (-\nabla^2 - m^2) \psi ,
\]
that is, the above Klein–Gordon equation for tachyons.

Like in the bradyonic theory, also for PT’s the spin component along the momentum direction, i.e. the helicity, conserves. In fact, as is easily proved, the PT hamiltonian (8) commutes with the spin×momentum operator: $[\hat{H}, \frac{1}{2}\vec{\Sigma} \cdot \vec{p}] \equiv [\hat{H}, \frac{1}{2}\alpha \gamma^5 \cdot \vec{p}] = 0$.

2 Plane waves

Let us write the general plane wave in the typical form met in the standard spinning particles theory:
\[
\psi = N_p \, w_{\pm p} e^{\mp ipx} ,
\]
where $N_p$ is a suitable normalization factor; quantity $w_{\pm p}$ denotes, as usual, the four-component “Dirac bispinor", that is a $p$-function; $px \equiv \varepsilon t - \vec{p} \cdot \vec{x}$; the superior sign refers to the “positive-energy states” for which $p^0 = +\varepsilon$ (with $\varepsilon = \sqrt{p^2 + m^2}$ for bradyons and $\sqrt{p^2 - m^2}$ for PT’s) and the momentum is $\vec{p}$, whilst the inferior sign refers to the “negative energy states” for which $p^0 = -\varepsilon$ and the momentum is $-\vec{p}$. As is known, the two superior components and the two
inferior ones in the amplitudes $w_{\pm p}$ are linked through relations whose particular form depends on the chosen representation for the matrices $\gamma^\mu$. We shall work in the so-called standard representation; for the expressions of the solutions of the PT Dirac equation in spinorial or Weyl’s representation see the Appendix. In standard representation the matrices $\gamma^\mu$, $\gamma^5$ are the following:

$$
\gamma^0 \equiv \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \gamma \equiv \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \gamma^5 \equiv \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix},
$$

(11)

where $\mathbb{I}$ is the identity $2 \times 2$ matrix, $\mathbb{O}$ is the null $2 \times 2$ matrix, the components of $\sigma \equiv (\sigma_1; \sigma_2; \sigma_3)$ are the usual Pauli matrices. In this representation the wave amplitude can be written as follows:

$$
w \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix},
$$

(12)

where $\chi$ and $\varphi$ are (two-component) Pauli spinors.

Let us now choose the normalization factor in eq. (10) as follows:

$$
\psi = \frac{1}{\sqrt{2kV}} w_{\pm p} e^{\mp ipx},
$$

(13)

where $k \equiv |\mathbf{p}|$. In the fourth section we shall see that, for PT’s, the factor $\frac{1}{\sqrt{2kV}}$ is a Lorentz invariant quantity, so that $w$ really transforms as a Dirac bispinor. For such a plane wave, the Klein–Gordon equation (5) leads to the relation which defines the tachyonic kinematics

$$
\varepsilon^2 = p^2 - m^2;
$$

(14)

while the PT Dirac equation (3) leads to the following matrix equations for the bispinorial amplitudes (with $u_p \equiv w_p$; $v_p \equiv w_{-p}$; $\hat{p} \equiv p_\mu \gamma^\mu$):

$$
(\hat{p} - m\gamma^5) u_p = 0,
$$

(15)

$$
(\hat{p} + m\gamma^5) v_p = 0.
$$

(16)

In standard representation, from eq. (15) we obtain

$$
\begin{cases}
\varepsilon \varphi - (\mathbf{p} \cdot \mathbf{\sigma}) \chi = m \chi \\
-\varepsilon \chi + (\mathbf{p} \cdot \mathbf{\sigma}) \varphi = m \varphi,
\end{cases}
$$

(17)

and from eq. (16):

$$
\begin{cases}
-\varepsilon \varphi + (\mathbf{p} \cdot \mathbf{\sigma}) \chi = m \chi \\
\varepsilon \chi - (\mathbf{p} \cdot \mathbf{\sigma}) \varphi = m \varphi.
\end{cases}
$$

(18)
From system (17) we deduce explicitly the positive-energy amplitudes and from system (18) the negative-energy ones:

\[ u_p = \begin{pmatrix} \varphi \\ \frac{p \sigma \epsilon - m}{\epsilon} \varphi \end{pmatrix}, \quad v_p = \begin{pmatrix} \frac{p \sigma \epsilon - m}{\epsilon} \chi \\ \chi \end{pmatrix}. \]  \tag{19}

We have expressed \( u_p \) as a function of \( \varphi \) rather than of \( \chi \), and \( v_p \) as a function of \( \chi \) rather than of \( \varphi \), for a mere formal analogy with the bradyonic solutions (see below). This choice is really convenient only for the Dirac amplitudes because in the center-of-mass frame (or, equivalently, in the non-relativistic approximation, \( p \to 0 \)) \( \chi \to 0 \) in \( u_p \) and \( \varphi \to 0 \) in \( v_p \) [cf. eqs. (30) with \( k = 0 \)].

Once arbitrarily chosen the \( \varphi \) (\( \chi \)) spinor, the other one, \( \chi \) (\( \varphi \)), comes out to be univocally determined. The two degrees of freedom inherent to the choice of the spinor are to be correlated to the spin orientation, and then to the two available helicity states. Thus, for every momentum \( p \) we shall have two distinct helicity eigenstates \( \psi_{p\lambda} \). Henceforth \( \lambda \) is defined as twice the helicity, \( \lambda \equiv \frac{2 p \cdot s}{|p|} \), but, for convenience, also \( \lambda \) itself will be called helicity. Let us impose that \( u_{p\lambda} \) and \( v_{p\lambda} \) be eigenstates of the operator \( \hat{\lambda} \equiv \frac{\vec{\Sigma} \cdot \vec{p}}{|p|} \), where the Dirac 4×4 spin operator \( \vec{\Sigma} \equiv \alpha \gamma^5 \) writes:

\[ \vec{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}. \]  \tag{20}

Let us also require the normalization “to one particle in the 3-volume \( V \)” (see the last section), \( \psi^\dagger \psi = 1/V \), which implies, as it is soon verified, the normalization

\[ u_{p\lambda}^\dagger u_{p\lambda} = v_{p\lambda}^\dagger v_{p\lambda} = 2k. \]  \tag{21}

From eqs. (19), exploiting also the equality \( \epsilon = \sqrt{(k + m\lambda)(k - m\lambda)} \), we have for the helicity eigenfunctions:

\[ u_{p\lambda} = \sqrt{k + m\lambda} \begin{pmatrix} \theta_{\lambda} \\ \frac{\lambda \epsilon}{k + m\lambda} \theta_{\lambda} \end{pmatrix}, \quad v_{p\lambda} = \sqrt{k - m\lambda} \begin{pmatrix} -\lambda \epsilon/k - m\lambda \theta_{-\lambda} \\ \theta_{-\lambda} \end{pmatrix}. \]  \tag{22}

In these expressions the spinors \( \theta_{\lambda}, \theta_{-\lambda} \) are 2-component column spinors, eigenfunctions of \( \hat{\lambda} \),

\[ \hat{\lambda} \theta_{\lambda} = \lambda \theta_{\lambda}, \quad \hat{\lambda} \theta_{-\lambda} = -\lambda \theta_{-\lambda}, \]  \tag{23}

and normalized to unity: \( \theta_{\lambda}^\dagger \theta_{\lambda} = \delta_{\lambda\lambda} \). Following the current notation, the amplitude \( v_{p\lambda} \) relates to a state endowed with helicity \( -\lambda \). This choice turns out to be convenient for the so-called “reinterpretation” of the negative-energy states,\(^{[10]}\) after which we shall really observe a 4-impulse \( +p \) and a helicity \( +\lambda \).
Let us now write down, for a direct comparison, the plane wave solutions of the standard bradyonic Dirac equation \( (i\gamma^\mu \partial_\mu - m) \psi = 0 \). We choose the usual normalization factor \( \frac{1}{\sqrt{2\varepsilon V}} \) which, for bradyons, constitutes a Lorentz scalar:

\[
\psi = \frac{1}{\sqrt{2\varepsilon V}} w_{\pm \varepsilon} e^{\pm i p x}.
\]  

(24)

For such a plane wave the Dirac equation leads to the following matrix equations for the bispino-rial amplitudes:

\[
(p - m) u_p = 0, \\
(p + m) v_p = 0.
\]  

(25)  

(26)

In standard representation, from the former of the above equations we get

\[
\begin{align*}
\varepsilon \varphi - (p \cdot \sigma) \chi &= m \varphi \\
\varepsilon \chi - (p \cdot \sigma) \varphi &= -m \chi,
\end{align*}
\]  

(27)

and from the latter one:

\[
\begin{align*}
-\varepsilon \varphi + (p \cdot \sigma) \chi &= m \varphi \\
-\varepsilon \chi + (p \cdot \sigma) \varphi &= -m \chi.
\end{align*}
\]  

(28)

From system (27) we deduce explicitly the positive-energy amplitudes and from (28) the negative-energy ones:

\[
\begin{align*}
u_p &= \left( \begin{array}{c}
\varphi \\
\frac{p \sigma}{\varepsilon + m} \varphi
\end{array} \right), \\
v_p &= \left( \begin{array}{c}
\frac{p \sigma}{\varepsilon + m} \chi \\
\chi
\end{array} \right).
\end{align*}
\]  

(29)

For the helicity eigenfunctions, as before, we require the normalization \( \psi^\dagger \psi = 1/V \), which now implies the usual constraint for the amplitudes \( u_{p \lambda}^\dagger u_{p \lambda} = v_{p \lambda}^\dagger v_{p \lambda} = 2\varepsilon \). Consequently, we have [now \( k = \sqrt{(\varepsilon + m)(\varepsilon - m)} \)]:

\[
u_{p \lambda} = \sqrt{\varepsilon + m} \left( \begin{array}{c}
\theta_\lambda \\
\frac{k}{\varepsilon + m} \theta_\lambda
\end{array} \right), \\
v_{p \lambda} = \sqrt{\varepsilon + m} \left( \begin{array}{c}
-\frac{k}{\varepsilon + m} \theta_{-\lambda} \\
\theta_{-\lambda}
\end{array} \right).
\]  

(30)

Let us come back to the PT wave-mechanics and to the solutions of PT Dirac equation.

The limiting-case of massless \( (p^2 = 0) \) particles —often called luxons\(^{11}\)— is immediately obtained, from solutions (22), by assuming \( m = 0; \ k = \varepsilon \). In this way we obtain the two “chiral” amplitudes \( (\gamma^5\)-eigenstates) \( w_R, w_L \):

\[
u_{p, +1} = v_{p, -1} \equiv w_R = \sqrt{k} \left( \begin{array}{c}
\theta_1 \\
\theta_1
\end{array} \right),
\]  

(31)
u_{p,-1} = -v_{p,+1} \equiv w_L = \sqrt{k} \begin{pmatrix} \theta_{-1} \\ -\theta_{-1} \end{pmatrix}.

(32)

As they must, such solutions are identical to the ones foreseen by the standard Dirac equation (and describing spinning massless particles, endowed with four distinct states if they are charged “Dirac particles”, and with two distinct states if they are uncharged “Majorana particles”).

It is interesting, at this point, to consider the so-called *trascendent tachyons*\(^{[12]}\), which, by definition, are particles endowed with nonzero mass, but zero total energy:

\[ \varepsilon = 0, \quad k = m. \]  

(33)

The trascendent *classical* tachyons travel at infinite speed \( V \equiv k/\varepsilon \)\(^#1\), representing therefore the “dual” case (for the definition of “duality” see the fourth section) of the non-relativistic case, \( v = c/V \rightarrow 0, \ p = 0, \ \varepsilon = m \). For this reason it is not surprising that, in the same way as it happens for bradyons in the center-of-mass frame, for trascendent \( PT's \) the *equations for spinors* \( \varphi \) and \( \chi \) *decouple* as well\(^#2\). As a matter of fact, making \( \varepsilon = 0 \) in system (27) we have for the bispinorial components of \( u_p \):

\[
\begin{align*}
(p \cdot \sigma + m) \chi &= 0 \\
(p \cdot \sigma - m) \varphi &= 0.
\end{align*}
\]

(34)

(here we do not consider \( v_p \), because the distinction between \( u_p \) and \( v_p \) is merely formal and concerns exclusively the sign of \( p \) and not the one of \( \varepsilon \), which is zero). For the trascendent polarized states (\( \lambda \)-eigenstates), since \( k = m \) and then \( (p \cdot \sigma) u_{\lambda} = m\lambda u_{\lambda} \), we shall have

\[
\begin{align*}
(\lambda + 1) \chi &= 0 \\
(\lambda - 1) \varphi &= 0;
\end{align*}
\]

(35)

and then:

\[
\begin{align*}
u_{\lambda=+1} &= \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \sqrt{2k} \begin{pmatrix} \theta_{+1} \\ 0 \end{pmatrix}, \\
u_{\lambda=-1} &= \begin{pmatrix} 0 \\ \chi \end{pmatrix} = \sqrt{2k} \begin{pmatrix} 0 \\ \theta_{-1} \end{pmatrix}.
\end{align*}
\]

\[
\]  

\#1 We shall show in the fourth section that the trascendent *quantum* tachyons, i.e. the trascendent \( PT's \), behave quite differently.

\#2 Of course, these two cases do differ in the phase wave: \( e^{-imt} \) for the non-relativistic case, and \( e^{ip \cdot x} \) for trascendent \( PT's \).
We are now going to study the behaviour of our theory under the main discrete symmetry operations, space-inversion $P$, charge-conjugation $C$, time-inversion $T$, 4-inversion $I$, and under the Lorentz homogeneous proper group $L$. We shall see that the PT theory is symmetric under all the above transformations; the operators related to the transformations $T$, $I$ and $L$ are the same for both bradyons and PT’s, whilst the operators related to $P$ and $C$ result multiplied by $\gamma^5$ when passing from the bradyonic to the PT sector. The procedure we shall follow to determine the symmetry operators is the one usually employed in the standard Dirac theoretical framework.

**PARITY**

Starting from the PT Dirac equation (3)

$$i\gamma^0 \partial_t - i \nabla \cdot \gamma - m\gamma^5 ) \psi(t; x) = 0,$$

we write the equation for the space-inverted wave-function:

$$(i\gamma^0 \partial_t + i \nabla \cdot \gamma - m\gamma^5 ) U_P \psi(t; -x) = 0,$$

quantity $U_P$ being a suitable unitary matrix operator. If we want $P$-invariance we must demand that, after having applied a unitary transformation to eq. (39), $\psi(t; -x)$ obey, as $\psi(t; x)$, the original eq. (38). In order that this requirement be satisfied a matrix $U_P$ must exist such that:

$$(i\gamma^0 \partial_t + i \nabla \cdot \gamma - m\gamma^5 ) U_P = \pm (i\gamma^0 \partial_t - i \nabla \cdot \gamma - m\gamma^5 ).$$

In fact, this is equivalent to impose that the l.h.s. of the space-inverted equation (39), after the unitary transformation $U_P^{-1}$, become equal, apart from an unessential sign, to the l.h.s. of equation (38):

$$U_P^{-1} (i\gamma^0 \partial_t + i \nabla \cdot \gamma - m\gamma^5 ) U_P = \pm (i\gamma^0 \partial_t - i \nabla \cdot \gamma - m\gamma^5 ).$$

If we take in eq. (40) the plus sign before $U_P$, an operator satisfying the above constraint does not exist; on the contrary, following the other choice the searched matrix results to be $e^{i\alpha}\gamma^0\gamma^5$. Quantity $e^{i\alpha}$ is an unessential phase factor because its choice does not affect the unitary character of the operator. As a matter of fact, if $O \rightarrow e^{i\alpha}O$ from $OO^{-1} = 1$ it follows that $O^{-1} \rightarrow e^{-i\alpha}O^{-1}$; at the same time $O^\dagger \rightarrow e^{-i\alpha}O^\dagger$. Therefore, since both $O^{-1}$ and $O^\dagger$ suffer the same transformation, their equivalence, $O^{-1} = O^\dagger$, continues to hold.

As is well-known\cite{13}, for bradyons analogous reasonings lead to the Dirac parity operator that is formally different from (42), namely: $U_P = \gamma^0$. As for the bradyonic sector, also for the PT sector quantity $mU_P$ coincides with the “mass term” in the hamiltonian, $\hat{H} = -i\alpha \cdot \nabla + m\gamma^0\gamma^5$. 

We may choose $\alpha = 0$, so that:

$$U_P = \gamma^0\gamma^5.$$ (42)
As well as in the bradonic theory, for the PT Hamiltonian we have the parity invariance, so that the PT energy is a space-inversion scalar:

\[ U_P \tilde{H}(\mathbf{x}) U_P^{-1} = (\gamma^0 \gamma^5) [-i\alpha \cdot (-\nabla) + m\gamma^0 \gamma^5] (-\gamma^0 \gamma^5) = \tilde{H}(\mathbf{x}). \]  

(43)

Furthermore, we may easily see that in the PT lagrangian (1) both the terms are pseudoscalar quantities, even the mass term \( m \bar{\psi} \psi \), which instead in the Dirac theory actually constitutes a scalar term. Thus, having a definite parity, our lagrangian is parity-conserving and the consequent theory is invariant under space-inversion.

**CHARGE-CONJUGATION**

The PT Dirac equation in an external field \( A^\mu \) writes:

\[ [\gamma^\mu (i\partial_\mu - qA_\mu) + m\gamma^5] \psi = 0, \]

(44)

Performing the complex conjugation and reversing all the signs, we obtain

\[ [\gamma^{\mu*} (i\partial_\mu + qA_\mu) + m\gamma^{5*}] \psi^* = 0. \]

(45)

The \( C \)-transformed wave-function is, by definition, obtainable through application of a suitable unitary operator \( U_C \) on \( \psi^* \) : \( \psi_C \equiv U_C \psi^* \). For the \( C \)-invariance of the theory to hold we now have to demand that \( U_C \) be such that \( U_C \psi^* \) obey the original eq.(44), but with opposite charge \(+q\):

\[ [\gamma^{\mu*} (i\partial_\mu + qA_\mu) + m\gamma^{5*}] U_C \psi^* = \pm U_C [\gamma^\mu (i\partial_\mu + qA_\mu) - m\gamma^5]. \]

(46)

Since we have (at least in the usual representations)

\[ (\gamma^{0,1,3,5})^* = \gamma^{0,1,3,5}, \quad \gamma^{2*} = -\gamma^2, \]

(47)

the only possible choice (the minus sign before \( U_C \)) is the following, up to phase factors:

\[ U_C = i \gamma^2 \gamma^5. \]

(48)

As for the parity operator, we see that the PT charge-conjugation operator is obtained from the Dirac homologous operator times \( \gamma^5 \).

**TIME-INVERSION**

Always in analogy with the standard theory, we write down the “time-inverted” PT Dirac equation:

\[ [-i\gamma^0 \partial_t - i\gamma \cdot \nabla - m\gamma^5] U_T \psi^*(-t; \mathbf{x}) = 0, \]

(49)

or, performing the complex conjugation,

\[ [i\gamma^{0*} \partial_t + i\gamma^* \cdot \nabla - m\gamma^{5*}] U_T \psi^*(-t; \mathbf{x}) = 0. \]

(50)

In order that the time-inversion invariance holds we must require:

\[ [i\gamma^{0*} \partial_t + i\gamma^* \cdot \nabla - m\gamma^{5*}] U_T = \pm U_T [i\gamma^0 \partial_t - i\gamma \cdot \nabla - m\gamma^5]. \]

(51)

Taking the plus sign we shall have for the time-inversion matrix:

\[ U_T = i\gamma^1 \gamma^3, \]

(52)

which is identical to the homologous Dirac operator.
4-INVERSION AND PCT TRANSFORMATIONS

The request for symmetry under 4-inversion, i.e. after \((t,\mathbf{x}) \mapsto (-t; -\mathbf{x})\), is a consequence of the relativistic invariance because 4-inversion is equivalent to a suitable proper Lorentz transformation. The PT equation for the spacetime-inverted wave-function writes:

\[
(-i\gamma^\mu \partial_\mu - m\gamma^5) U_I \psi(-t; -\mathbf{x}) = 0. \tag{53}
\]

In order to have symmetry under 4-inversion we must find an unitary \(U_I\) such that:

\[
(-i\gamma^\mu \partial_\mu - m\gamma^5) U_I = \pm U_I (i\gamma^\mu \partial_\mu - m\gamma^5). \tag{54}
\]

The plus sign yields a PT 4-inversion operator identical to the bradyonic one:

\[
U_I = i\gamma^5. \tag{55}
\]

It is easy to see, by a direct evaluation, that we have:

\[
U_P U_C U_T = U_I, \quad \text{PCT} \psi(t; \mathbf{x}) = I \psi(t; \mathbf{x}) = i\gamma^5 \psi(-t; -\mathbf{x}). \tag{56}
\]

We have therefore found that the present theory is symmetric under spacetime inversion and that the PCT and 4-inversion transformations are equivalent symmetry operations. In such a way the invariance under PCT descends from the relativistic invariance itself.

RELATIVISTIC COVARIANCE

With regard to the relativistic covariance of the Dirac equation, we may read in ref.[14]: "...this form invariance of the Dirac equation expresses the Lorentz invariance of the underlying energy-impulse connection \(p_\mu p^\mu = m^2\)...". In the same way, the Lorentz (L) covariance of the PT Dirac equation (3) —which we are going to prove— expresses the relativistic invariance of the PT kinematical constraint \(p^2 = -m^2\).

The proper homogeneous \(L\) group generates the following transformations on spacetime coordinates:

\[
(x^\nu)' = a^\nu_\mu x^\mu, \tag{57}
\]

that, in a compact form, we may write as \(x' = ax\). The transformation tensor \(a\) reads:

\[
a^\nu_\mu = \eta^\nu_\mu + \Delta \omega^\nu_\mu, \tag{58}
\]

where the \(\Delta \omega^\nu_\mu\) are the six independent generators of the infinitesimal \(L\)-transformations. In the standard theory\[14\] the \(L\)-transformed Dirac equation writes

\[
[i\gamma^\mu \partial_\mu - m] S(a) \psi'(x') = 0, \tag{59}
\]

provided that the \(L\)-transformed wave-function is obtained out from the initial one through the application of the linear (in general not unitary) matrix \(S^{-1}(a)\)

\[
\psi(x) = S(a) \psi'(x'), \quad \psi'(x') = S^{-1}(a) \psi(x), \tag{60}
\]

10
with \( S(a^{-1}) = S^{-1}(a) \). As a function of \( x' \) only, eq. (59) becomes:

\[
[i\gamma^\mu a^\nu_{\mu} \frac{\partial}{\partial x'^\nu} - m] S(a) \psi'(x') = 0 ,
\]

(61)

If we want the \( L \)-covariance, we must impose that this transformed equation be put in the canonic form of the Dirac equation \((i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m)\psi(x') = 0\). Then, as before, we have to require:

\[
\gamma^\mu a^\nu_{\mu} S(a) = S(a) \gamma^\nu .
\]

(62)

This equation is satisfied by the following linear operator, where \( \sigma_{\mu\nu} \equiv \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \) denotes the antisymmetric “spin tensor”,

\[
S = 1 - \frac{i}{4} \sigma_{\mu\nu} \Delta \omega^{\mu\nu} .
\]

(63)

Repeating now these considerations for the PT sector, we have for the PT \( L \)-transformed equation:

\[
[i\gamma^\mu a^\nu_{\mu} \frac{\partial}{\partial x'^\nu} - m\gamma^5] \bar{S}(a) \psi'(x') = 0 .
\]

(64)

Demanding the relativistic covariance now leads to the two contemporary conditions for the \( \bar{S} \) matrix:

\[
\gamma^\mu a^\nu_{\mu} \bar{S}(a) = \bar{S}(a) \gamma^\nu , \quad \gamma^5 \bar{S}(a) = \bar{S}(a) \gamma^5 .
\]

(65)

For the commutation of the spin tensor with \( \gamma^5 \), we are induced to conclude that such an operator coincides with the standard one given by (63), \( S = \bar{S} \), and that lagrangian (1), which originates our theory, is relativistically invariant. Thus, also in the present PT theory, the matrices \( \gamma^\mu \) transform as components of a 4-vector, \( \gamma^5 \) transforms as a pseudoscalar quantity, the bilinear \( \bar{\psi} \gamma^\mu \psi \) transforms as a (true) 4-vector, the bilinear \( \bar{\psi} \gamma^5 \gamma^\mu \psi \) transforms as a (pseudo)vector, and so on.

## 4 Duality between momentum and velocity

Starting from the PT hamiltonian (8) we can write the Heisenberg equation for the velocity operator:

\[
\hat{v} \equiv \hat{\dot{x}} = i [\hat{H}, \, \hat{x}] ;
\]

(66)

the solution of this operatorial equation is identical to Dirac’s:

\[
\hat{v} = \alpha .
\]

(67)

As implied in the denomination itself, classical tachyons\footnote{[12]} perform superluminal motions, so that they behave as “true” tachyons. By contrast, we are going to show that quantum spinning PT’s perform subluminal mean motions, just as bradyons do. For this reason we have proposed to call them Pseudotachyons. Let us now calculate explicitly \( \bar{v} \equiv \frac{\int \psi^\dagger \alpha \psi \, dV}{\int \psi^\dagger \psi \, dV} \), and, in so doing,
let us continue to confront the present PT theory with the bradyonic one. Remind that, like it occurs in the Dirac theory, also PT hamiltonian (8) do not commute with the velocity operator $\alpha$, so that $v$, differently from $p$, is not determined and conserved. For simplicity we consider plane waves, normalized, as anticipated in the second section, “to one particle in the the considered volume $V$”:

$$\int \bar{\psi} \psi \, dV = 1.$$  

(68)

The impulse-helicity eigenstates for ordinary Dirac particles write

$$\psi_{p\lambda}^{(+)} = \frac{1}{\sqrt{2\varepsilon V}} u_{p\lambda} e^{-ipx}, \quad \psi_{p\lambda}^{(-)} = \frac{1}{\sqrt{2\varepsilon V}} v_{p\lambda} e^{ipx},$$  

(69)

where the amplitudes $u_{p\lambda}, v_{p\lambda}$ are given by eqs. (30). For the average velocity, through eqs. (30), we obtain the well-known momentum-velocity relation of classical mechanics ($v_{cl}$ denoting the classical expression of the velocity)

$$\bar{v} = \int \bar{\psi}_{p\lambda} \alpha \psi_{p\lambda} \, dV \equiv V \bar{\psi}_{p\lambda} \alpha \psi_{p\lambda} = \frac{p}{\varepsilon} = v_{cl},$$  

(70)

for both the positive-energy and the negative-energy states. Analogous considerations and evaluations can be made for the PT sector. We have

$$\psi_{p\lambda}^{(+)} = \frac{1}{\sqrt{2kV}} u_{p\lambda} e^{-ipx}, \quad \psi_{p\lambda}^{(-)} = \frac{1}{\sqrt{2kV}} v_{p\lambda} e^{ipx},$$  

(71)

with the amplitudes $u_{p\lambda}, v_{p\lambda}$ given by eqs. (22). For the average velocity, through eqs. (22), we now find that the average velocity is equal to the so-called “dual velocity$^{[12]}$ of the classical velocity $v_{cl}^4$:

$$\bar{v} = \int \bar{\psi}_{p\lambda} \alpha \psi_{p\lambda} \, dV \equiv V \bar{\psi}_{p\lambda} \alpha \psi_{p\lambda} = \frac{\varepsilon p}{k^2}, \quad |\bar{v}| = \frac{1}{|v_{cl}|}.$$  

(72)

for both the positive-energy and the negative-energy states. Since for PT’s it is $|v_{cl}| > 1$, we find that for PT’s the average speed is smaller than the light speed.

Starting from the above solutions of the standard Dirac and PT Dirac equations, we now work out the constraints linking 4-impulse with mass, with mean 4-velocity $\bar{\pi}^\mu$ and with mean spin 4-vector (or polarization 4-vector), $\pi^\mu$, for bradyons and PT’s, respectively.

Let us introduce the 4-vector $\bar{p}^\mu$, which may be defined as dual of the 4-vector $p^\mu \equiv (\varepsilon; \varepsilon p)$:

$$\bar{p}^\mu = \left(k; \frac{\varepsilon p}{k}\right),$$  

(73)

$^{[12]}$Let us recall that, in Extended Relativity, a theory not invoked in the present work, a Lorentz transformation associated to a boost $V > 1$ is obtained by the composition of a Lorentz transformation with boost $v = \frac{1}{V}$ dual of $V$, and of a trascendent Lorentz transformation, associated to an infinite-velocity “boost”. In other words, in Extended Relativity, the relative velocity of two bodies endowed with reciprocally dual velocities is infinite, and viceversa.
with $p\bar{\rho} = 0$. Quantity $p^\mu$ is a timelike 4-vector for bradyons and a spacelike 4-vector for PT’s, respectively. By contrast, quantity $\bar{p}^\mu$ is a spacelike 4-vector for bradyons and a timelike 4-vector for PT’s: in fact $\bar{p}^2 = k^2 - \varepsilon^2 = -p^2$. As a consequence, $k$ is the time component of a timelike 4-vector [eq. (73)] for PT’s, while $\varepsilon$ is the time component of a timelike 4-vector ($p^\mu$) for bradyons. As is well-known, the product $\varepsilon V$ constitutes a Lorentz-scalar quantity for ordinary particles. Analogously, for the just said considerations, the product $kV$ actually constitutes a Lorentz-invariant quantity for PT’s.

Because $\mathbf{v} = \int \psi^\dagger \alpha \psi \, dV \equiv \int \bar{\psi} \gamma^\dagger \psi \, dV$ and $\mathbf{s} = \int \psi^\dagger \alpha \gamma^5 \psi \, dV \equiv \int \bar{\psi} \gamma \gamma^5 \psi \, dV$, we may define, as usual, the following 4-vectorial quantities referring to bradyons:

$$\mathbf{v}^\mu \equiv \frac{\varepsilon}{m} \int \bar{\psi} \gamma^\mu \psi \, dV = \frac{\varepsilon V}{m} \bar{\psi} \gamma^\mu \psi = \frac{1}{m} (\varepsilon; \, p) = \frac{p^\mu}{m},$$

(74)

and

$$\mathbf{s}^\mu \equiv \frac{\varepsilon}{m} \int \bar{\psi} \gamma^\mu \gamma^5 \psi \, dV = \frac{\varepsilon V}{m} \bar{\psi} \gamma^\mu \gamma^5 \psi = \frac{\lambda}{m} \left( k; \frac{\varepsilon p}{k} \right) = \frac{\lambda \bar{p}^\mu}{m},$$

(75)

[as it is found employing eqs. (30, 69)]. We have $\mathbf{v}^2 = 1$ and $\mathbf{s}^2 = -1$, and in the “quiet frame”, $\mathbf{v} = 0$, $k = 0$, $\mathbf{s}^0 = 0$, $|\mathbf{s}| = 1$, as expected.

For PT’s, analogously:

$$\mathbf{v}^\mu \equiv \frac{k}{m} \int \bar{\psi} \gamma^\mu \psi \, dV = \frac{k V}{m} \bar{\psi} \gamma^\mu \psi = \frac{1}{m} \left( k; \frac{\varepsilon p}{k} \right) = \frac{k^\mu}{m},$$

(76)

and

$$\mathbf{s}^\mu \equiv \frac{k}{m} \int \bar{\psi} \gamma^\mu \gamma^5 \psi \, dV = \frac{k V}{m} \bar{\psi} \gamma^\mu \gamma^5 \psi = \frac{\lambda}{m} \left( \varepsilon; \, p \right) = \frac{\lambda \bar{p}^\mu}{m},$$

(77)

[as it is found employing eqs. (22, 71)]. As before we have a timelike 4-velocity ($\mathbf{v}^2 = 1$) and a spacelike 4-polarization ($\mathbf{s}^2 = -1$), and in the “quiet frame”, $\mathbf{v} = 0$, $\varepsilon = 0$, $\mathbf{s}^0 = 0$, $|\mathbf{s}| = 1$.

From the above equations we obtain the following constraints, which are constitutive of the bradyonic kinematics:

$$\begin{cases}
  \varepsilon^2 = m^2 \\
  p^\mu \mathbf{v}^\mu = m \\
  p^\mu \mathbf{s}^\mu = 0.
\end{cases}$$

(78)

The PT kinematics is instead governed by the following rules:

$$\begin{cases}
  p^2 = -m^2 \\
  p^\mu \mathbf{v}^\mu = 0 \\
  p^\mu \mathbf{s}^\mu = -m \lambda.
\end{cases}$$

(79)

Notice the orthogonality between 4-impulse and 4-velocity which is a typical property also of the kinematics of massless spinning particles.
Momentum and velocity are independent variables, as expected for spinning particles which, having an intrinsic angular momentum, are endowed with “internal” degrees of freedom. In quantum theories for spin-$\frac{1}{2}$ particles, in fact, the momentum operator, $-i\nabla$, which conserves, and the velocity operator, $\alpha$, which does not (and whose components $\alpha_i$ do not commute), are independent operators. For Dirac particles the independence between the momentum and velocity operators is at the origin—for packets composed by opposite energies, generally describing spatially-localized systems—of the so-called zitterbewegung.\footnote{The average velocity in the Dirac theory, $\int \psi^\dagger \alpha \psi \, dV$, for the most general wave-packet shows,\cite{13,15} beyond the classical-like translational term $p/\varepsilon$, the so-called zitterbewegung term involving a rapidly varying motion, with frequency $\omega = 2\varepsilon \geq 2m$. The zitterbewegung term has several observable consequences, of which the most famous is perhaps the Darwin term in the non-relativistic approximation of the Dirac Hamiltonian. In a recent paper of the author\cite{16} it is suggested that even the so-called “quantum potential”, appearing in the hydrodynamical formulations of the Schrödinger and Pauli equations for electron, may be a very zitterbewegung effect.} For PT’s, as already seen, we have the fundamental consequence of the momentum-velocity duality.

The duality between the dispersion velocity $\partial\varepsilon/\partial p$ and the travelling velocity of momentum and energy was found, even if under very particular conditions, by Maccarrone and Recami\cite{17} for classical, non-quantum tachyons. Naranan\cite{18} also obtained that for tachyons the observed velocity is dual of the classical one, always inside a non-quantum context, by recourse to a simple generalization of the ordinary Lorentz transformations by using complex variables. In ref.\cite{19} Gott III showed that the motion of a Schwarzschild black hole, associated with a spacelike world-line, does not involve transport of energy or signals at speeds larger than the light speed. Bathia and Pande\cite{20} proved that for a degenerate gas of classical tachyons the sound velocity is always subluminal. In ref.\cite{21} Robinett found that the group speed of a spinless quantum tachyon does not exceed the light speed, so that the Klein–Gordon field propagates at subluminal velocities; analogous results had been precedently found by Fox\textit{ et al.}\cite{22}, Ecker\cite{9} and Bers\textit{ et al.}\cite{23}, and have been subsequently re-obtained by Ziolkowski\cite{24}.\textsuperscript{\#5} Let us notice, incidentally, that also for the electromagnetic radiation we may find a kind of duality between superluminal group velocity and subluminal transmission velocity of signals or energy-impulse. See in particular Strnad and Kodre\cite{25} who studied the transmission of electromagnetic waves across an optically less-dense layer, that is, the propagation in a medium with anomalous refraction index. Let us recall that the spontaneous breaking of the gauge symmetries endowes Higgs bosons (which are spinless particles with $m^2 < 0$) with a positive mass squared. In such a way they may be eventually observed only like ordinary bradyonic scalars.

From all these considerations we see that the momentum-velocity duality has already been met in literature for scalars particles. By contrast, as far as we know, this property is quite a novelty for spin-$\frac{1}{2}$ particles. Furthermore, for quantum scalar tachyons we have to consider the signals transmission speed for very particular wave-packets, or in the presence of special boundary conditions. For PT’s, instead, the momentum-velocity duality is intrinsic in the theory, in that it is due to the presence of spin, and to the consequent independence reciprocal
between the momentum and velocity operators. Anyway, either for spinless or spinning particles, we might assert that Nature is always able to associate subluminal behaviours to negative masses squared, in such a way hindering the observation of charges and fields travelling at speeds greater than \( c \).

The negative mass squared suggests that the present theory might actually describe (massive) neutrinos. From many recent experiments neutrinos seem to be endowed with a small non-zero mass.\(^6\) Furthermore, in many laboratory experiments measuring direct kinematical limits, the measured 4-impulse squared of neutrinos appears to be negative. The experimental data involving \( p^2 < 0 \) for electron \([c]\) and muon \([a] \) and \([b]\) neutrinos has been deduced out from: a) observation of pion decay,[28] b) precision measurements of \( \pi^- \) mass,[29] and c) kinematical analysis of tritium beta-decay.[30–35] Let us mention the experimental results for \( m^2 \) (eV)\(^2 \) found in c): \((−65±85±65)[30]\); \((−24±48±61)[31]\); \((−147±68±41)[32]\); \((−39±34±15)[33]\); \((−72±41±30)[34]\).

The idea that neutrinos might be tachyons (in the “classical” sense of the word) dates back the end of sixties, and has been sometimes re-proposed.\(^2,[36]\) Therefore, this hypothesis is not an absolute novelty. Nevertheless, whenever the mentioned authors spoke of “tachyonic neutrinos”, they always asserted or supposed that the condition \( p^2 < 0 \) implied faster-than-light motions. For such particles, as well as it occurs for classical tachyons, it would result necessary the extension of the standard relativity to superluminal Lorentz transformations (i.e. to boosts \( > c \)), with recourse to the already mentioned Extended Relativity. All that is not necessary for PT’s, which, in our theory, are always subluminal bodies. Thus, we may restrict ourselves to employ just the ordinary transformations allowed in special relativity, only bearing in mind that the classical dispersion law \( v = p/\epsilon \) does not hold anymore. For the previous computations, the speed which is foreseen for neutrinos will be given by:

\[
v = \frac{\epsilon}{k} = \frac{\epsilon}{\sqrt{\epsilon^2 + m^2}}.
\] (80)

This expression does differ from the one holding for bradyons

\[
u = \frac{k}{\epsilon} = \frac{\sqrt{\epsilon^2 - m^2}}{\epsilon},
\] (81)

which is smaller than \( c \) too, and from the one holding for classical tachyons

\[
w = \frac{k}{\epsilon} = \frac{\sqrt{\epsilon^2 + m^2}}{\epsilon},
\] (82)

which is instead greater than the light speed. The curves corresponding to these three dispersion laws are reported in figure for a direct comparison. Anyway, for very light particles as neutrinos,

\(^6\) As pointed out in ref. [26], many experimental indications of a non-zero neutrino mass may be found mostly in atmospheric physics and astrophysics. Evidences come in particular from the solar neutrino deficit, (cf., e.g., the GALLEX experiment) and from the observation, by means of the Cosmic Background Explorer (COBE), of density fluctuations in the cosmic microwave background accounting for the existence of hot components of the “mixed” dark matter. Other evidences come from the detection of antineutrinos cooling the neutron star associated with the supernova SN 1987A[27], and from the recent Liquid Scintillator Neutrino Detector experiment at the Los Alamos National Laboratories.
Figure 1: Energy-speed dispersion plots for bradyons ($u$), PTs ($v$), and classical tachyons ($w$)

since $m^2 \ll \varepsilon^2$, each one of the three above expressions entails speeds experimentally indistinguishable from $c$. Notice that for $m \rightarrow 0$ the plots referring to the bradyonic and PT speeds overlap, and that for $v \rightarrow 0$ the PT dispersion law shows a newtonian-like trend: $v \sim \varepsilon/m$.

Suited, highly precise experiments, looking into the small-energy region of the dispersion law, might clearly show if neutrinos are, or not, PT's.

In a work in preparation[10] we carefully study the PT current for the general wave-packet and try to perform the second quantization of the theory. In so doing, we find that the PT field obeys the same fermionic anticommutation rules of the Dirac field.

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Appendix

Solutions of the PT Dirac equation in spinorial (Weyl’s) representation

In the so-called **spinorial representation** (also said the **Weyl’s representation**) the matrices $\gamma^\mu$, $\gamma^5$ write:

$$
\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma \equiv \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}, \quad \gamma^5 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(A.1)

In this representation the bispinorial amplitude in $\frac{1}{\sqrt{2kV}}w e^{-ipx}$ may be put in the following form:

$$
w \equiv \begin{pmatrix} \xi \\ \eta \end{pmatrix},
$$

(A.2)

where $\xi, \eta$ are two-component **Weyl spinors** (“not pointed” and “pointed” respectively). The interrelation between the spinors in the present representation and the ones in standard representation is:

$$
\varphi \equiv \frac{\xi + \eta}{\sqrt{2}}, \quad \chi \equiv \frac{\xi - \eta}{\sqrt{2}}.
$$

(A.3)

The PT Dirac equation in spinorial representation writes:

$$
\begin{cases}
(\varepsilon + \mathbf{p} \cdot \sigma) \eta = m \xi \\
(\varepsilon - \mathbf{p} \cdot \sigma) \xi = -m \eta
\end{cases}
$$

for the positive-energy states, and

$$
\begin{cases}
(-\varepsilon - \mathbf{p} \cdot \sigma) \eta = m \xi \\
(-\varepsilon + \mathbf{p} \cdot \sigma) \xi = -m \eta
\end{cases}
$$

(A.5)

for the negative-energy ones. The relative solutions may be written as follows:

$$
u_p = \begin{pmatrix} \xi \\ \mathbf{p} \sigma - \varepsilon \xi \end{pmatrix}, \quad v_p = \begin{pmatrix} -\mathbf{p} \sigma + \varepsilon \\ m \eta \end{pmatrix}.
$$

(A.6)

Bearing in mind the PT equality $m = \sqrt{(k + \varepsilon\lambda)(k - \varepsilon\lambda)}$, the $\lambda$-eigenfunctions are:

$$
u_{p\lambda} = \sqrt{k + \varepsilon\lambda} \begin{pmatrix} \theta_\lambda \\ \frac{\lambda m}{k + \varepsilon\lambda} \theta_\lambda \end{pmatrix}, \quad v_{p\lambda} = \sqrt{k + \varepsilon\lambda} \begin{pmatrix} m \\ k \lambda + \varepsilon \end{pmatrix} \theta_{-\lambda}.
$$

(A.7)
These solutions are normalized, like in eqs. (22), with $\psi^\dagger \psi = 1/V$, $u^\dagger_{p\lambda} u_{p\lambda} = v^\dagger_{p\lambda} v_{p\lambda} = 2k$. As expected, also in spinorial representation a short calculation yields $\int \psi^\dagger \alpha \psi \, dV = \frac{eP}{k^2}$.

As previously made in standard representation, we write down, for comparison, the solutions of the bradyonic Dirac equation in the Weyl representation. We have for the bispinorial amplitudes:

$$u_p = \begin{pmatrix} \xi \\ \varepsilon - \frac{p \sigma}{m} \xi \end{pmatrix}, \quad v_p = \begin{pmatrix} -\frac{\varepsilon + p \sigma}{m} \eta \\ \eta \end{pmatrix}. \quad (A.8)$$

By using the bradyonic equality $m = \sqrt{(\varepsilon + k\lambda)(\varepsilon - k\lambda)}$, we have for the helicity eigenfunctions (normalized, as in the standard representation, with $\psi^\dagger \psi = 1/V$, $u^\dagger_{p\lambda} u_{p\lambda} = v^\dagger_{p\lambda} v_{p\lambda} = 2\varepsilon$):

$$u_{p\lambda} = \sqrt{\varepsilon + k\lambda} \begin{pmatrix} \theta_\lambda \\ \frac{m}{\varepsilon + k\lambda} \theta_\lambda \end{pmatrix}, \quad v_{p\lambda} = \sqrt{\varepsilon + k\lambda} \begin{pmatrix} -\frac{m}{\varepsilon + k\lambda} \theta_{-\lambda} \\ \theta_{-\lambda} \end{pmatrix}. \quad (A.9)$$

References


