ABSTRACT

We show that the gravitational field equations derived from an action composed of i) an arbitrary function of the scalar curvature and other scalar fields plus ii) connection-independent kinetic and source terms, are identical whether one chooses nonmetricity to vanish and have non-zero torsion or vice versa.
Scalar-Tensor theories of gravitation have been around for decades. These theories, commonly based on actions representing coupling between the gravitational metric tensor and other scalar fields have been of interest for various reasons since the birth of General Relativity (GR).\textsuperscript{1} The most well known example is the Brans-Dicke theory\textsuperscript{2} which was proposed to incorporate Mach’s principle in gravitational interactions. Dirac\textsuperscript{3} based his large-number hypothesis on a scalar-tensor theory of gravity. Other theorists favored such theories simply because of the presence of a scalar field which seems to be an inevitable bi-product of modern physics. More recent examples of such models include conformal-invariant theories of gravity, low-energy limit of superstring theories, Einstein-Cartan type theories coupled to scalar fields and many modern inflationary models based on scalar-tensor theories.

In deriving the field equations of a theory of gravity from an action functional one is faced with choices. The most common approach is to consider the metric tensor as the only independent field describing the geometry of space-time and to restrict the affine connections to be the well known Christoffel symbols. This guarantees the metricity of the theory. Thus Riemannian structure and local Lorentz structure are preserved and the metric is the solution to the ten metric field equations derived from the variation of the action. Another choice is to follow the Palatini formalism which is to consider the metric and the connections as independent fields. This choice allows the geometry to have a general affine structure. The space-time associated with this type of theory is usually called $\left( L_4, g \right)$. Here there are in general 10 equations for the metric tensor and 64 for the connections. In many cases this increase in the number of equations is compensated by the reduction in order of the metric field equations. For example in quadratic gravity the Palatini variation yields two sets of first order equations for the metric and the connections whereas the usual metric formalism yields fourth order equations for the metric. In the case of general relativity the two formalisms produce identical field equations.

The advantage of deriving the field equations using the Palatini method is that the geometry of space-time is less restrictive. In general the Palatini variation allows for the existence of torsion and non-metricity. The possible importance of such
exotic fields was not realized until the construction of a gauge theory of gravitation was attempted. In the more recent years the inclusion of such fields in gravitational interactions and the study of their properties have become more common as attempts to unify all fundamental interactions with standard GR have failed in one way or another. The simplest theory incorporating torsion and nonmetricity is the Einstein-Cartan-Sciama-Kibble (ECSK) theory in which torsion is coupled to the canonical spin tensor via the matter part of the lagrangian. In this theory torsion does not propagate and also vanishes in vacuum. The Brans-Dicke version of the ECSK theory (ECSK minimally coupled to a scalar field), with non-zero torsion and vanishing non-metricity, which we denote by BDT, was discussed by Rauch, German, Kim and others. It was shown that the scalar field can act as a source of propagating torsion even in vacuum. Furthermore Smalley showed that the simplest non-metric version of the Brans-Dicke model, with vanishing torsion, which we denote by BDN here, is equivalent to the BDT theory via a conformal transformation.

In this paper we generalize Smalley’s result to include a much larger class of scalar-tensor theories. Furthermore, we do this by considering the so-called projective transformation of the connections rather than by extended conformal transformations. This can be done because we allow nonmetricity to exist. We prove that given any scalar-tensor action composed of any arbitrary function of the scalar curvature and other scalar fields plus any kinetic and/or matter terms, which are independent of the connections, the field equations are independent of whether the torsion is set to zero and non-metricity is nonzero or non-metricity is set to zero and the torsion is non-zero. We show that this equivalence is a result of the projective gauge invariance of the action. We prove our results by showing that: a) two distinct choices of the projective gauge vector correspond to the two cases mentioned above, and b) the two cases considered in a) result in identical field equations. Our results then imply that the equivalence between the BDT and BDN theories is a consequence of this gauge freedom. As a further example we apply our results to the case of another simple action which involves a quadratic term in the scalar
curvature.

Our index conventions are the same as those of Held et. al. The connections are defined such that when a vector $A^\lambda$ is parallel transported, it undergoes an infinitesimal change given by:

$$dA^\lambda = -\Gamma^\lambda_{\mu\nu}(x) A^\nu dx^\mu.$$  

**THE FIELD EQUATIONS**

Let us consider the action:

$$A = \int [f(\phi, R) \sqrt{-g} + L_{\text{matter}} + L_{\text{Kinetic}}] d^4 x, \quad (1)$$

where $L_{\text{matter}}$ and $L_{\text{Kinetic}}$ represent the matter part of the lagrangian and the kinetic energy terms associated with the fields respectively. These are assumed to be independent of the connection fields. The only term depending on the connections is the function $f(\phi, R)$ through its dependence on the scalar curvature $R(g, \Gamma) = g^{\mu\nu} R_{\nu\mu}(\Gamma)$ where $g$ and $\Gamma$ denote the metric and the connection fields. In this paper we are primarily interested in the field equations for the connections. Therefore the analytic form of the kinetic and the matter terms and their dependence on other fields are of no importance here.

Variation of the action Eq. (1), with respect to the connections $\Gamma^\alpha_{\mu\nu}$ gives:

$$S_{\alpha\gamma}^\beta + 2\delta^\beta_{[\alpha} S_{\gamma]} + \delta^\beta_{[\alpha} Q_{\mu]} \gamma^\mu - 2\delta^\beta_{[\alpha} Q_{\gamma]} = -\delta^\beta_{[\alpha} \partial_{\gamma]} \ln f' \equiv P_{\alpha\gamma}^\beta, \quad (2)$$

where

$$S_{\alpha\gamma}^\beta \equiv \Gamma^\beta_{[\alpha\gamma]} \equiv \frac{1}{2}(\Gamma^\beta_{\alpha\gamma} - \Gamma^\beta_{\gamma\alpha}) \quad (3)$$

is the torsion tensor and

$$Q_{\alpha} \gamma^\beta = \nabla_{\alpha} g^\gamma\beta, \quad (4)$$
is the non-metricity tensor. The square brackets denote anti-symmetrization as in Eq. (3) and the prime on \( f \) denotes partial differentiation with respect to \( R \). Also in Eq. (2) \( Q_\gamma \equiv \frac{1}{4} Q_{\gamma\beta}^\beta \).

Multiplying Eq. (2) by \( g^\alpha\gamma \) we get:

\[
\delta^\beta_{[\gamma} Q_{\mu]}^{\mu\alpha} = 0,
\]

which implies that four of the sixty four equations do not contain any information. This implies that the connections can only be determined to within an arbitrary four-vector \( V^\alpha \). We will show that the freedom of choosing \( V^\alpha \) corresponds to the well known projective invariance of the action discussed by other authors\(^{10}\).

In order to construct a unique theory of scalar-tensor gravity one is usually forced to choose this vector. In torsion theories of gravity, the choice of setting the non-metricity equal to zero corresponds to taking the projective freedom away and Eq. (5) is trivially satisfied. In this way the connections can be uniquely determined in terms of the Christoffel symbols and the torsion tensor.

Let us define the vector

\[
V^\alpha = Q_\beta^{\beta\alpha}.
\]

Substitution of Eq. (6) into the field equations Eq. (2) gives:

\[
S_{\alpha\beta\gamma} - \frac{1}{2} Q_{\alpha\beta\gamma} = P_{\alpha\beta\gamma} + \frac{1}{2} g_{\gamma\alpha} V_\beta - g_{\gamma\beta} V_\alpha - g_{\gamma\alpha} S_\beta + g_{\gamma\beta} S_\alpha,
\]

where \( S_\alpha \equiv S_{\alpha\beta}^\beta \) and \( P_{\alpha\beta\gamma} = -g_{\gamma[\alpha} \partial_{\beta]} \ln f' \).

The connections are given by:\(^9\)

\[
\Gamma^\sigma_{\alpha\beta} \equiv g^{\sigma\mu} \Delta^{\eta\lambda\tau}_{\beta\alpha\mu} \left( \frac{1}{2} \partial_\eta g_{\lambda\tau} - S_{\eta\lambda\tau} + \frac{1}{2} Q_{\eta\lambda\tau} \right),
\]

where \( \Delta^{\eta\lambda\tau}_{\beta\alpha\mu} \) is the permutation tensor given by:

\[
\Delta^{\eta\lambda\tau}_{\beta\alpha\mu} = \delta^\eta_{\beta} \delta^\lambda_{\alpha} \delta^\tau_{\mu} + \delta^\eta_{\alpha} \delta^\lambda_{\mu} \delta^\tau_{\beta} - \delta^\eta_{\mu} \delta^\lambda_{\beta} \delta^\tau_{\alpha}.
\]

Substitution of Eq. (7) in Eq. (8a) gives:
\[ \Gamma^\sigma_{\alpha\beta} = \left\{ \frac{\sigma}{\alpha\beta} \right\} - g^{\sigma\mu}[P_{\beta\alpha\mu} + P_{\alpha\mu\beta} - P_{\mu\alpha\beta}] - g^{\sigma\mu} \left\{ -\frac{3}{2}g_{\alpha\beta}V_{\mu} - \frac{1}{2}g_{\beta\mu}V_{\alpha} - \frac{3}{2}g_{\alpha\mu}V_{\beta} \right\} + 2\left[ g_{\alpha\beta}S_{\mu} - g_{\alpha\mu}S_{\beta} \right] \],

(8b)

where \( \left\{ \frac{\sigma}{\alpha\beta} \right\} \) are the Christoffel symbols.

The torsion tensor can now be found by antisymmetrizing the connections given by Eq. (8b). We get:

\[ S_{\alpha\beta\gamma} = \Gamma^\gamma_{[\alpha\beta]} = \frac{1}{2}\delta^\gamma_{[\alpha} \partial_{\beta]} \ln f' - \frac{1}{2}\delta^\gamma_{[\alpha} V_{\beta]}. \]

Substituting this back into Eq. (8b) we get:

\[ \Gamma^\sigma_{\alpha\beta} = \left\{ \frac{\sigma}{\alpha\beta} \right\} + g^{\sigma\mu}g_{\alpha[\beta}\partial_{\mu]} \ln f' - \frac{1}{2}\delta^\sigma_{\beta\alpha\gamma}V_{\alpha}. \]

(10)

The non-metricity is found by taking the covariant derivative of the metric using the full connections Eq. (10). After some algebra we get:

\[ \nabla_\alpha g^{\beta\gamma} = Q^\beta_{\alpha\gamma} = V_\alpha g^{\beta\gamma}. \]

(11)

Note that Eq. (10) is not an explicit solution for the connections but rather an implicit equation since the connections are present in the function \( f \) itself.

The Ricci tensor can now be calculated using the standard definition. We have:

\[ R_{\beta\gamma} = R^\sigma_{\beta\gamma\sigma} = \Gamma^\sigma_{\beta\gamma,\sigma} - \Gamma^\sigma_{\sigma\gamma,\beta} + \Gamma^\sigma_{\sigma\lambda} \Gamma^\lambda_{\beta\gamma} - \Gamma^\sigma_{\beta\lambda} \Gamma^\lambda_{\sigma\gamma} = R_{\beta\gamma} (\{\}) + \frac{1}{2}g_{\beta\gamma}D_\sigma \partial^\sigma \ln f' + D_\beta \partial_\gamma \ln f' - \frac{1}{2}g_{\beta\gamma} (\partial_\gamma \ln f') (\partial^\sigma \ln f') + \frac{1}{2} \left( \partial_\beta \ln f' \right) (\partial_\gamma \ln f') + D_{[\beta}V_{\gamma]}, \]

(12)

where \( R_{\beta\gamma} (\{\}) \) and \( D_\alpha \) are the Ricci tensor and the covariant derivative derived from the Christoffel symbols.

Inspection of Eq. (12) reveals the independence of the scalar curvature and thus the action on the vector \( V_\alpha \) since \( D_{[\beta}V_{\gamma]}g^{\beta\gamma} = 0 \) for a symmetric metric. The vector
$V_\alpha$ is the projective gauge vector. Furthermore we note that if $V_\alpha$ is the derivative of a scalar one can show that the Riemann-Cartan tensor, the Ricci tensor itself and therefore the metric field equations are also independent of $V_\alpha$. Two cases are of interest here:

**Case A:** $V_\alpha = 0$; Here non-metricity vanishes via Eq. (9). and the torsion field is given by:

$$S_{\alpha\beta\gamma} = \frac{1}{2} \delta_{\gamma}[^{\alpha} \partial_{\beta}] \ln f'$$

and

$$\begin{align*}
R_{\beta\gamma} &= R_{\beta\gamma} (\{\}) + \frac{1}{2} g_{\beta\gamma} D_{\sigma} \partial^{\sigma} \ln f' + D_{\beta} \partial_{\gamma} \ln f' - \frac{1}{2} g_{\beta\gamma} (\partial_{\sigma} \ln f') (\partial^{\sigma} \ln f') \\
&\quad + \frac{1}{2} (\partial_{\beta} \ln f') (\partial_{\gamma} \ln f').
\end{align*}$$

**Case B:** $V_\alpha = \partial_{\alpha} \ln f'$; In this case torsion vanishes via Eq. (10) and non-metricity is given by:

$$Q_{\alpha}^{\mu\nu} = g^{\mu\nu} \partial_{\alpha} \ln f',$$

and $R_{\mu\nu}$ remains the same as in Eq. (14) because $V_\alpha$ is a derivative of a scalar. Therefore the field equations in the two cases are the same.

For the action corresponding to the Brans-Dicke theory, $f(\phi, R) = \phi R$. Case A corresponds to the BDT theory discussed by Refs. (5-7). Case B corresponds to the BDN theory discussed by Smalley. It is clear that the two theories are equivalent. It is interesting to note that an extended conformal transformation which would gauge away the torsion field for the BDT theory was found by German which corresponds to the choice of $V_\alpha$ made in case B. However, the freedom of having nonmetricity here makes it unnecessary to rescale the metric, or even the scalar field, to make the action invariant.

**EXAMPLE**
As a further example we consider another class of actions, which has regained its popularity in recent years, because of its emergence in the low energy limit of superstrings. The simplest such action is given in terms of the Hilbert-Einstein term plus a quadratic scalar term, \( f(R) = R + \alpha R^2 \). Recently the non-metric version of this theory has been discussed by Shahid-Saless. It was shown that, assuming no torsion, the Palatini variation of this action yields a non-metric theory. A more general version of this type of theory which assumes vanishing non-metricity but includes torsion was discussed by many authors. In particular Minkevich derived and examined the cosmological field equations based on an action which included all the possible quadratic combinations of curvature. Given the formalism developed here it is clear that the two theories yield the same field equations in the limit that the action considered by Minkevich corresponds to that examined by Shahid-Saless; that is \( f(R) = R + \alpha R^2 \). In the case of vanishing torsion Eqs. (10) and (9) imply:

\[
Q_{\beta \mu \nu} = g^{\mu \nu} \frac{2 \alpha R_{\beta \beta}}{1 + 2 \alpha R},
\]

and the connections are:

\[
\Gamma^\sigma_{\alpha \beta} = \left\{ \begin{array}{l} \sigma \\ \alpha \beta \end{array} \right\} + \frac{\alpha}{1 + 2 \alpha R} (\delta^\sigma_{\beta} R_{,\alpha} + \delta^\sigma_{\alpha} R_{,\beta} - g^{\sigma \mu} g_{\alpha \beta} R_{,\mu}),
\]

which agree with the results given by Shahid-Saless. However since \( V_\alpha \) is a total derivative, it does not contribute to the the Ricci tensor. Thus Eq. (12) will agree with the expression for the Ricci tensor used by Ref. (11). Therefore even if we had set \( V_\alpha = 0 \), we would have the same expression for the Ricci tensor. This case would however correspond to the theory considered by Minkevich which is a metric theory with torsion. The equivalence of his theory with that considered by Shahid-Saless can be inspected easily by setting Minkevich’s \( f_6 \) by \(-\alpha\). The cosmological equations discussed by these authors have also been checked for their equivalence.

**CONCLUSION**
We have proven that for all actions composed of i) an arbitrary function of the scalar curvature and other scalar fields, plus ii) any other kinetic and matter terms which are independent of the connections, the field equations are the same whether the torsion is set to zero and the theory is non-metric or non-metricity is set to zero but torsion is non-vanishing.

Traditionally non-metricity has been viewed as an unwanted bi-product of some extensions of GR because of its volume non-preserving property. On the other hand it is generally argued that the existence of torsion in nature does not pose a problem for fundamental physics. Our results show that within the class of scalar-tensor theories considered here the two fields result in identical field equations and therefore imply the same physics. One conclusion that could be made is that perhaps one needs a deeper understanding of the meaning of these fields and their inter-relationships in all aspects of measurement before discarding them as physically unreasonable mathematical artifacts.

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10. See for example V. D. Sandberg, Phys. Rev. D 12, 3013 (1975) and references therein.