Unique determination of an inner product by adjointness relations in the algebra of quantum observables

Alan D. Rendall *
Physics Department
Syracuse University
Syracuse NY 13244-1130
USA

Abstract

It is shown that if a representation of a *-algebra on a vector space $V$ is an irreducible *-representation with respect to some inner product on $V$ then under appropriate technical conditions this property determines the inner product uniquely up to a constant factor. Ashtekar has suggested using the condition that a given representation of the algebra of quantum observables is a *-representation to fix the inner product on the space of physical states. This idea is of particular interest for the quantisation of gravity where an obvious prescription for defining an inner product is lacking. The results of this paper show rigorously that Ashtekar’s criterion does suffice to determine the inner product in very general circumstances. Two versions of the result are proved: a simpler one which only applies to representations by bounded operators and a more general one which allows for unbounded operators. Some concrete examples are worked out in order to illustrate the meaning and range of applicability of the general theorems.

Short title: Determination of inner product by adjointness relations
PACS number: 0463

---

* Supported in part by the NSF grant PHY90-16733 and research funds provided by Syracuse University

Permanent address: Max-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 8046 Garching bei München, Germany
1. Introduction

There exist many approaches to the issue of giving a general definition of what it means to quantise a classical system. Many of these involve defining an abstract algebra which incorporates various algebraic properties of classical observables and then studying representations of this algebra on a Hilbert space. Recently Ashtekar ([1], Chapter 10) has put forward a quantisation programme which is designed to be applicable to canonical quantum gravity. This has been investigated further in [2]. Since the Hamiltonian formulation of general relativity involves constraints the programme must be able to handle quantisation of constrained systems and in fact it is an extension of the Dirac procedure for quantising such systems. However the Dirac method does not give a prescription for finding an inner product on the space of physical states. In many situations there is some ‘background’ structure which can be used to fix this inner product. For instance, in the case of a field theory in flat space there is the background Minkowski metric and the associated notion of Poincaré invariance can be used to restrict the inner product. On the other hand the absence of any background structure is a characteristic feature of general relativity. To overcome this problem, Ashtekar gives a general criterion for choosing an inner product which is the following. Suppose that for a given classical system we have defined an abstract algebra \( \mathcal{A} \) (the algebra of quantum observables) and that by some means we have found a representation of \( \mathcal{A} \) on a complex vector space \( V \). In general there is at this stage no preferred inner product on \( V \). Suppose further that the structure of the classical system gives rise to a natural involution on the algebra, \( a \mapsto a^* \). Then the inner product is to be restricted by the requirement that this involution goes over into the operation of taking the adjoint of an operator. To be more precise, any inner product on \( V \) gives rise to a definition of the adjoint of a linear operator defined on \( V \). If \( \rho \) denotes the representation of \( \mathcal{A} \) on \( V \) then for any \( a \in \mathcal{A} \) the adjoint of the linear operator \( \rho(a) \) is required to be equal to \( \rho(a^*) \).

The purpose of this paper is to show that the above proposal determines the inner product up to a constant factor under rather general circumstances. The first task which has to be accomplished in doing this is to find an appropriate mathematical formulation of the problem. There are two constraints involved in doing this. On the one hand, the desired result will fail if the technical hypotheses are not strong enough. On the other hand, the result will only be of interest from the point of view of the original motivation if the hypotheses hold (and can reasonably be verified) for a large class of systems which one might wish to quantise. The ambiguities in the mathematical formulation will now be discussed. The first point is that in the end the representation of the algebra \( \mathcal{A} \) should be a representation on a Hilbert space. In general this Hilbert space will not be \( V \) but rather the completion of \( V \) with respect to the inner product which is to be determined. Moreover, the elements of \( \mathcal{A} \) may be represented by unbounded operators on this Hilbert space which leads, as is usual with unbounded operators, to technical difficulties with the domains of definition of the operators. For this reason, the first theorem in this paper will be concerned only with the case where all the elements of \( \mathcal{A} \) can be represented by bounded operators. This theorem will frequently not be applicable to the algebra of quantum operators in the form in which it most naturally arises in a given physical problem. The second theorem does allow unbounded operators but requires more technical assumptions.
These technical assumptions are such that it should often be practicable to check them in interesting examples. Nevertheless it may be easier to change the algebra of quantum observables itself so as to allow the theorem on representations by bounded operators to be applied. The standard example of such a modification of the algebra of observables for mathematical convenience is the replacement of the algebra of the canonical commutation relations by the Weyl algebra (the ‘integrated form’ of the commutation relations).

Even when only bounded operators are considered, there is another potential ambiguity. The inner product can only be expected to be unique if some assumption of irreducibility is made on the representation. Without this it would be possible to choose the inner product independently in different superselection sectors. There are two candidates for the definition of irreducibility. These are algebraic and topological irreducibility. A representation of an algebra $A$ on a vector space is algebraically irreducible if there are no invariant subspaces; if the vector space has a topology (e.g. if it is a Hilbert space) then the representation is said to be topologically irreducible if there are no closed invariant subspaces. The first of these definitions seems attractive in the context of the present application since it only involves the vector space structure of $V$ while to define topological irreducibility it is necessary to make use of the inner product to be determined. However it turns out that algebraic irreducibility is an unreasonably strong assumption. To see this, consider the standard representation of the Weyl relations for one degree of freedom on $L^2(\mathbb{R})$. Thus $U(a)$ is represented by translation by $a$ and $V(b)$ is represented by multiplication by the function $e^{ibq}$. Examples of invariant subspaces are the Schwartz space of rapidly decreasing functions, the smooth functions of compact support and the space of functions whose Fourier transforms are smooth and have compact support. For this reason only topological irreducibility is considered in the following.

The paper is organised as follows. In section 2 a theorem on the uniqueness of inner products is proved in the case that the elements of $A$ can be represented by bounded operators. Section 3 contains a generalisation of this to unbounded operators whose proof makes use of the previous result. Some illustrative examples are presented in the final section.

2. The case of bounded operators

Recall that a complex *-algebra is an associative algebra over $\mathbb{C}$ together with a mapping $a \mapsto a^*$ which is antilinear and satisfies the conditions that $(ab)^* = b^*a^*$ for any $a, b \in A$ and that $(a^*)^* = a$ for any $a \in A$.

**Definition 1** Let $A$ be a complex *-algebra with identity and $\rho$ a representation of $A$ on a complex vector space $V$. An inner product $\langle \ , \ \rangle_1$ on $V$ is said to be strongly admissible if:

(i) $\rho$ is a *-representation with respect to this inner product i.e. $\langle \rho(a)x, y \rangle_1 = \langle x, \rho(a^*)y \rangle_1$ for all $a \in A$ and $x, y \in V$

(ii) for each $a \in A$ the operator $\rho(a)$ is bounded with respect to the norm $\| \ |_1$ defined by the given inner product so that $\rho$ extends uniquely to a representation $\hat{\rho}_1$ on the Hilbert space completion $\hat{V}_1$ of $V$ by bounded operators

(iii) $\hat{\rho}_1$ is topologically irreducible.

**Theorem 1** Let $\langle \ , \ \rangle_1$ and $\langle \ , \ \rangle_2$ be inner products on the complex vector space $V$ which
are strongly admissible with respect to a representation $\rho$ of a complex *-algebra $A$. Then $\langle \ , \rangle_1 = c\langle \ , \rangle_2$ for some positive real number $c$.

**Proof** Suppose that $\langle \ , \rangle_1$ and $\langle \ , \rangle_2$ are strongly admissible inner products. Define

$$\langle \ , \rangle = \langle \ , \rangle_1 + \langle \ , \rangle_2. \quad (1)$$

This defines a Hermitian form on $V$. Now (1) implies that $\langle x, x \rangle \geq 0$ for all $x$ and moreover $\langle x, x \rangle = 0$ implies that $\|x\|_1 = 0$ and hence that $x = 0$. Hence $\langle \ , \rangle$ defines an inner product and the associated norm satisfies

$$\|x\|_1 \leq \|x\|, \quad \|x\|_2 \leq \|x\|, \quad (2)$$

for all $x \in V$. Let $\hat{V}$ be the Hilbert space completion of $V$ with respect to $\langle \ , \rangle$. The boundedness of the operators $\rho(a)$ with respect to $\langle \ , \rangle_1$ and $\langle \ , \rangle_2$ implies that they are bounded with respect to $\langle \ , \rangle$. Hence $\rho$ extends uniquely to a representation $\hat{\rho}$ on $\hat{V}$ by bounded operators. Now

$$\langle x, y \rangle_1 \leq \|x\|_1 \|y\|_1 \leq \|x\| \|y\| \quad (3)$$

for all $x, y$ in $V$. Thus $\langle \ , \rangle_1$ extends uniquely to a continuous Hermitian form on $\hat{V}$. The extension, which will also be denoted by $\langle \ , \rangle_1$, still satisfies (3). By continuity the relation

$$\langle \rho(a)x, y \rangle_1 = \langle x, \rho(a^*)y \rangle_1 \quad (4)$$

continues to hold for the extension. Consider now the mapping $x \mapsto \langle x, y \rangle_1$. As a result of (3) this is a continuous linear functional on $\hat{V}$. By Riesz’ lemma there exists a $z \in \hat{V}$ with $\langle x, y \rangle_1 = \langle x, z \rangle$. Doing this for each $y$ gives a linear operator $L_1 : y \mapsto z$ which is bounded by (3). This operator satisfies $\langle x, y \rangle_1 = \langle x, L_1y \rangle$ and is self-adjoint. Now

$$\langle x, L_1\rho(a)y \rangle = \langle x, \rho(a)L_1y \rangle, \quad (5)$$

for all $x, y \in \hat{V}$. Here the fact has been used that $\langle \ , \rangle$ satisfies the condition (i) of a strongly admissible inner product, which follows directly from its definition. Hence $L_1$ commutes with $\rho(a)$ for all $a \in A$. Since $\|x\|_1 \leq \|x\|$ the identity mapping of $V$ extends uniquely to a continuous linear mapping $Q : V \to \hat{V}_1$. It has already been mentioned that the representation $\rho$ extends to both $\hat{V}$ and $\hat{V}_1$ in a natural way. Given $x \in \hat{V}$ choose a sequence $x_n$ in $V$ such that $\|x - x_n\| \to 0$. Then $\hat{\rho}(a)$ is defined by the condition that $\|\hat{\rho}(a)x - \rho(a)x_n\| \to 0$. Now $\|Q(x) - x_n\|_1 \to 0$ and so $\|\hat{\rho}_1(a)Q(x) - \rho(a)x_n\|_1 \to 0$. This implies that $\hat{\rho}_1(a) \circ Q = Q \circ \hat{\rho}(a)$.

The operator $L_1$ is positive, bounded and self-adjoint and so its spectrum must be contained in the interval $[0, \lambda_1]$ for some $\lambda_1 > 0$. Choose $\lambda_1$ to be minimal for this property. There are now two cases to be considered. Case (i) is that the spectrum of $L_1$ is equal to $\{\lambda_1\}$. Then $L_1 = \lambda_1 I$ and $\langle \ , \rangle_1 = \lambda_1 \langle \ , \rangle$. In case (ii) the infimum $\lambda'_1$ of the numbers in the spectrum of $L_1$ is strictly less than $\lambda_1$. Then there exists some $\mu$ with $\lambda'_1 < \mu < \lambda_1$. Let $\Pi = \int_{\lambda'_1}^{\lambda_1} dE(\lambda)$, where $dE(\lambda)$ is the spectral measure of $L_1$. (For information concerning spectral measures see [4], chapters 12 and 13.) This projection is not equal to zero or the identity. Hence its image, $K$ say, is a closed linear subspace of $\hat{V}$ different from $\{0\}$. 

4
and \(\hat{V}\). Since all the \(\rho(a)\) commute with \(L_1\) they also commute with \(\Pi\) and hence \(K\) is an invariant closed linear subspace for the representation \(\hat{\rho}\). If it was known that \(\hat{\rho}\) was irreducible this would give a contradiction and complete the proof. However it is not clear that the irreducibility of \(\hat{\rho}_1\) and \(\hat{\rho}_2\) implies that of \(\hat{\rho}\) and a less direct approach will be adopted.

Now \((L - \mu I)|_K\) is an operator with positive spectrum and hence a positive operator. Thus \(\langle L_1x, x\rangle \geq \mu \langle x, x\rangle\) or \(\|x\| \leq \mu^{-1/2}\|x\|_1\) on \(K\). It follows that the two norms \(\|\|\) and \(\|\|_1\) are equivalent when restricted to \(K\). This implies in particular that the restriction of \(Q\) to \(K\) is injective so that \(K\) can be regarded as a linear subspace of \(\hat{V}_1\). The subspace \(K\) is closed with respect to \(\|\|\) and hence complete. It follows that it is complete with respect to \(\|\|_1\) and hence closed in \(\hat{V}_1\). It is also invariant under \(\hat{\rho}_1\). Since \(\hat{\rho}_1\) is irreducible this is impossible unless \(K = \hat{V}_1\). Suppose that, for a given choice of \(\mu\), \(K = \hat{V}_1\). Let \(\tilde{\mu}\) be a number in the interval \((0, \mu)\), let \(\tilde{\Pi} = \int_{\tilde{\mu}}^{\lambda_1} dE(\lambda)\) and let \(\tilde{K}\) be the image of \(\tilde{\Pi}\). By the same argument as before \(Q|\tilde{K}\) is injective and so \(K = \hat{V}_1\) implies that \(K = \tilde{K}\). If \(\lambda_1\) were greater than zero then choosing \(\tilde{\mu} < \lambda_1\) would give a contradiction to the fact that \(K\) is not the whole of \(\hat{V}\). Hence \(\lambda_1 = 0\) and moreover, for all \(\varepsilon\) sufficiently small, \(\int_{\varepsilon}^{\lambda_1} dE(\lambda)\) is the orthogonal projection on the orthogonal complement of \(K\). However this means that the value of the spectral measure on the set \(\{0\}\) is also this projection. It can be concluded that the orthogonal complement of \(K\) is the kernel of \(L_1\). If \(x \in \ker L_1\) then \(\langle x, x\rangle_1 = 0\). The non-degeneracy of \(\langle , \rangle_1\) thus implies that the kernel of \(L_1\) is trivial. This shows that the assumption that case (ii) holds necessarily leads to a contradiction. It follows that only case (i) is in fact realised. The conclusion of all this is that \(\langle , \rangle_1 = \lambda_1\langle , \rangle\). Similarly \(\langle , \rangle_2 = \lambda_2\langle , \rangle\) for some \(\lambda_2 > 0\). So \(\langle , \rangle_1 = (\lambda_1/\lambda_2)\langle , \rangle_2\).

3. The case of unbounded operators

For applications it is useful to have a version of Theorem 1 where the assumption (ii) in the definition of a strongly admissible inner product is weakened to allow representations by unbounded operators. If we have a representation \(\rho\) of an algebra \(A\) on a vector space \(V\) as before and an inner product \(\langle , \rangle\) on \(V\) then \(\rho\) can be regarded as a representation of \(A\) on \(\hat{V}\) by unbounded operators with domain \(V\). For the sake of clarity this representation by unbounded operators will be denoted by \(\hat{\rho}\). The assumptions for a generalisation of Theorem 1 should include the irreducibility of this representation and so it is necessary at this point to introduce a definition of irreducibility of a representation by unbounded operators. In fact there is more than one way in which one might think of defining irreducibility for a representation by unbounded operators. The one used here agrees with that used in [5]. If two representations \(\rho_1\) and \(\rho_2\) are given on Hilbert spaces \(H_1\) and \(H_2\) with domains \(D_1\) and \(D_2\) respectively, then a representation on \(H_1 \oplus H_2\) with domain \(D_1 \oplus D_2\) is obtained by defining \((\rho_1 \oplus \rho_2)(a)(x_1, x_2) = (\rho_1(a)x_1, \rho_2(a)x_2)\). The representation \(\rho_1 \oplus \rho_2\) is called the direct sum of \(\rho_1\) and \(\rho_2\). A representation is called irreducible if it cannot be written in a non-trivial way as a direct sum of representations. Note that in the case of a representation by bounded operators this definition agrees with the notion of topological irreducibility used in the previous section.

Another concept which will be required in the following is that of a closed representation. It is a generalisation of the concept of a closed operator. If \(L\) is an operator on the
Hilbert space $H$ with domain $D$ then its graph is the subset of $H \times H$ consisting of pairs of the form $(x, Lx)$ with $x \in D$. The operator is called closed if its graph is a closed subset of $H \times H$. More generally it may be that if the graph of an operator is not closed the closure of its graph is the graph of an operator $\bar{L}$. This operator is uniquely determined and is called the closure of $L$. It is an extension of $L$. Now suppose that $A$ is a $\ast$-algebra and $\rho$ a $\ast$-representation of $A$ on a Hilbert space $H$ with domain $D$. It can be shown that each operator $\rho(a)$ has a closure. Let $\bar{D}$ be the intersection of the domains of the closures of the operators $\rho(a)$ as $a$ runs over $A$. This is a subspace of $H$ containing $D$. If in fact $D = \bar{D}$ then the representation is called closed. Further details can be found in [5].

**Definition 2** Let $A$, $\rho$ and $V$ be as in Definition 1. Let $S$ be a set of elements of $A$ which satisfy $a^* = a$ and which generate $A$. An inner product $\langle , \rangle_1$ on is said to be admissible if:

(i) $\rho$ is a $\ast$-representation with respect to this inner product
(ii) for each $a \in S$ the operator $\hat{\rho}_1(a)$ is essentially self-adjoint
(iii) $\hat{\rho}_1$ is irreducible
(iv) $\hat{\rho}_1$ is closed

**Remarks** 1. The algebra generated by a set of elements of a given algebra means here and in the following the smallest subalgebra containing these elements and the identity.
2. As implied by the terminology, a representation which is strongly admissible is admissible. For in that case $S$ can be taken to be the set of all $a \in A$ satisfying $a^* = a$.

**Theorem 2** Let $\langle , \rangle_1$ and $\langle , \rangle_2$ be inner products on the complex vector space $V$ which are admissible with respect to a representation $\rho$ of a complex $\ast$-algebra $A$. Then $\langle , \rangle_1 = c \langle , \rangle_2$ for some positive real number $c$.

**Proof** If $a \in S$ define $F_1(a)$ to be the closure of $\hat{\rho}_1(a)$. By assumption $F_1(a)$ is self-adjoint and so it is possible to define $G_1(a) = (i + F_1(a))/(i - F_1(a))$. The operator $G_1(a)$ satisfies the equation

$$
(i - F_1(a))G_1(a) = i + F_1(a).
$$

An operator $G_2(a)$ can be defined similarly using the inner product $\langle , \rangle_2$. Now the restrictions of $F_1(a)$ and $F_2(a)$ to $V$ are both equal to $\rho(a)$. Hence (6) and the analogous equation satisfied by $F_2(a)$ and $G_2(a)$ imply that $(i - \rho(a))(G_1(a) - G_2(a))|_V = 0$. Since $\rho(a)$ is essentially self-adjoint its deficiency indices must be zero and so $i - \rho(a)$ has trivial kernel. It follows that the restrictions of $G_1(a)$ and $G_2(a)$ to $V$ are equal. Let $b(a) = G_1(a)|_V$. Now define a new algebra $B$ to consist of the algebra of linear mappings generated by the $b(a)$ as $a$ runs through the set $S$. Every element of $B$ possesses an adjoint in the algebra and this defines a star operation in $B$. Consider now the defining representation of $B$ on $V$. It extends to $\ast$-representations of $B$ on $\hat{V}_1$ and $\hat{V}_2$ by bounded operators. Thus the representation of $B$ satisfies conditions (i) and (ii) of Definition 1.

To examine whether it also satisfies condition (iii), suppose that $K$ is a closed linear subspace left invariant by the extensions to $\hat{V}_1$ of all elements of $B$. In particular, if $a \in S$ then $G_1(a)$ commutes with the projection $\pi_K$ on $K$. This means that $\pi_K$ commutes with all spectral projections of $G_1(a)$, which in turn means that it commutes with all spectral projections of $F_1(a)$. It follows that $\pi_K$ maps the domain of $F_1(a)$ into itself and that for any $x$ in this domain $\pi_KF_1(a)x = F_1(a)\pi_Kx$. Hence condition (iv) implies that $\pi_K$ maps
V into itself. Since elements of S generate A it can be concluded that \( \pi_K \rho(a) x = \rho(a) \pi_K x \) for all \( a \in A \) and \( x \in V \). Using the fact that \( \hat{\rho}_1 \) is irreducible and applying Lemma 8.3.5 of [5] shows that \( K \) must be the trivial subspace or the whole of \( \hat{V}_1 \). It has now been shown that \( \langle , \rangle_1 \) is strongly admissible for the representation of \( B \) and of course the same argument can be applied to \( \langle , \rangle_2 \). With this information the conclusion of Theorem 2 for the algebra \( A \) is obtained by applying Theorem 1 to \( B \).

4. Examples

The first example to be considered is one which is rather simple from the point of view of the general theory developed above. This is the case where the vector space \( V \) is finite dimensional. In this case there are several simplifications. When \( V \) is finite dimensional it is complete with respect to any inner product so that \( V = \hat{V}_1 = \hat{V}_2 \). Moreover all linear operators are bounded and so in this case Theorem 2 is unnecessary. Only properties (i) and (iii) need to be checked in order to apply Theorem 1. The distinction between algebraic and topological irreducibility also becomes superfluous since in finite dimensions all linear subspaces are closed. Finally, it should be remarked that Theorem 1 is much easier to prove when \( V \) is finite dimensional; the result is then a rather straightforward application of the classical Schur lemma. One of the constrained systems whose quantisation is discussed in [2] consists of two harmonic oscillators which are required to have a fixed total energy. This is a situation where Theorem 1 can be applied to a finite dimensional representation.

The remaining examples are intended to illustrate how the hypotheses of the above theorems can be checked in practice. They also give some idea of the relative merits of Theorems 1 and 2 in applications. The first is the Weyl algebra for one degree of freedom and the second the algebra of the canonical commutation relations. Of course these are just two alternative algebras of quantum operators for a single classical system. Then the case of more (possibly infinitely many) degrees of freedom is examined.

The Weyl algebra for one degree of freedom is the algebra generated by elements \( U(a) \) and \( V(b) \) with \( a, b \in \mathbb{R} \) subject to the relations

\[
U(a + b) = U(a)U(b), \quad V(a + b) = V(a)V(b),
\]

\[
U(a)V(b) = e^{iab}V(b)U(a).
\] (7)

The star relations are \( U(a)^* = U(-a) \) and \( V(b)^* = V(-b) \). The standard representation is that on \( L^2(\mathbb{R}) \) given by taking \( U(a) \) to be translation by \( a \) and \( V(b) \) to be multiplication by \( e^{ibq} \). It is well known that this representation is irreducible and in fact it is the unique weakly continuous irreducible representation up to unitary equivalence (von Neumann uniqueness theorem). In order to demonstrate what is needed to check the hypotheses of Theorem 1 a proof of the irreducibility will be now be given. (The other hypotheses are easily verified.) The method of proof is useful in more general situations. To show that the representation is irreducible it is sufficient to show that any projection \( \Pi \) which commutes with all \( U(a) \) and \( V(b) \) is either equal to zero or the identity. Suppose then that \( \Pi \) is a projection which commutes with all elements of the Weyl algebra. It is useful at this point to bear in mind the following vague but helpful principle: any operator on \( L^2(X) \), where \( X \) is some measure space, which commutes with sufficiently many multiplication operators
is itself a multiplication operator. Denote by $M_f$ the multiplication operator associated to $f$, i.e., $M_f(g) = fg$. If $f, g$ are two smooth functions of compact support on $\mathbb{R}$, choose $R$ so that the supports of both are contained in the interval $[-R, R]$. The function $f$ can be altered outside this interval to give a function $fg$ which is periodic with period $2R$. This can be uniformly approximated by a sequence of functions $f_{R,n}$, each of which is a finite linear combination of the functions $e^{ibq}$. For instance this could be the sequence of partial sums of the Fourier series of $f_R$. Now the projection $\Pi$ commutes with $M_{f_{R,n}}$ for each $n$. Moreover $M_{f_{R,n}}$ converges to $M_{f_R}$ in the norm topology as $n \to \infty$. Thus

$$M_{f_R} \Pi g = \Pi M_{f_R} g = \Pi M_f g.$$

If $R$ is allowed to tend to infinity then $M_{f_R}(\Pi g)$ will converge to $M_f(\Pi g)$ in $L^2(\mathbb{R})$. It follows that $M_f \Pi g = \Pi M_f g$. Since $g$ was an arbitrary smooth function of compact support this implies that $M_f$ commutes with $\Pi$. Now let $h = e^{x^2} \Pi(e^{-x^2})$. Then

$$M_h(e^{-x^2} f) = e^{-x^2} f e^{x^2} \Pi(e^{-x^2}) = M_f \Pi(e^{-x^2}) = \Pi M_f(e^{-x^2}) = \Pi(e^{-x^2} f)$$

Since this is true for all smooth $f$ of compact support it follows that $\Pi = M_h$. A multiplication operator which is a projection must be that associated with the characteristic function of some measurable set, $h = \chi_E$. But now, using the fact that $\Pi$ commutes with all $U(a)$, we see that the set $E$ is invariant under all translations. This means that $\Pi$ can only be zero or the identity and the proof of irreducibility is complete.

Next the canonical commutation relations for one degree of freedom will be examined. Here the algebra is generated by two elements $Q, P$ satisfying $[Q, P] = i$. The star relations are $Q^* = Q$ and $P^* = P$. In the standard Schrödinger representation $Q$ is represented by multiplication by $q$ and $P$ by $-id/dq$, considered as unbounded operators on $L^2(\mathbb{R})$. It will be shown that, if a common domain for these operators is chosen appropriately, the inner product of $L^2(\mathbb{R})$ is admissible with respect to this representation. A dense subspace of $L^2(\mathbb{R})$ where the operators representing all elements of the algebra are defined is $V_0 = C_c(\mathbb{R})$, the smooth functions of compact support. To check condition (iv) in the definition of an admissible inner product it is necessary to find a domain for which the representation is closed. It will be shown that the Schwartz space $S$ is such a domain. If $L$ is an operator on a Hilbert space $H$ with domain $D$ the domain of its closure consists of all elements $f$ of $H$ with the following property: there exists an element $g \in H$ and a sequence $\{f_n\}$ in $D$ such that $f_n \to f$ and $L f_n \to g$. If $f \in S$ there exists a sequence $\{f_n\}$ in $V_0$ such that $f_n \to f$ in the topology of $S$. In particular, if $L = Q^k P^l$ then $Q^k P^l f_n \to Q^k P^l f$ in $L^2(\mathbb{R})$ so that $f$ belongs to the closure of $L$. Conversely, suppose that $f$ belongs to the closure of the operator $Q^k P^l$ with domain $V_0$ for all $k, l$. Thus for any given $k, l$ there exists a sequence $\{f_n\}$ in $V_0$ such that $f_n \to f$ and $q^k d^l f_n/dq^l \to g$ in $L^2(\mathbb{R})$. From the first of these properties of the sequence $\{f_n\}$ it can be concluded that $q^k d^l f_n/dq^l \to q^k d^l f/dq^l$ in the sense of distributions. Thus $g = q^k d^l f/dq^l$, where $d^l f/dq^l$ is to be interpreted as a distributional derivative. It can now be seen that for fixed $k, l$ all derivatives of $q^k d^l f/dq^l$ belong to $L^2(\mathbb{R})$. By the Sobolev embedding theorem it follows that $q^k d^l f/dq^l$ is continuous and bounded. Hence $f$ is in $S$. 

8
Take $V = S$. Then properties (i) and (iv) of an admissible inner product are satisfied. To verify the other properties choose $S = \{Q, P\}$. The essential self-adjointness of $Q$ and $P$ can be checked by looking at their deficiency indices. The equations $Qf = \pm if$ and $Pf = \pm if$ have no solutions in $L^2(\mathbb{R})$. This means that the deficiency indices of $Q$ and $P$ are zero so that they are indeed essentially self-adjoint. It remains to show that the representation is irreducible. This means showing that the only projections $\Pi$ which map $S$ into itself and which commute with $Q$ and $P$ on that domain are zero and the identity. If $\Pi$ commutes with $Q$ then it commutes with any spectral projection of $Q$ ([4], Theorem 13.33). This in turn implies that $\Pi$ commutes with every bounded measurable function of $Q$ ([4], Theorem 12.21). In other words, it commutes with the multiplication operators $M_f$ for any $f \in L^\infty(\mathbb{R})$. It has been shown above that any projection which commutes with $M_f$ for all $f$ in the more restricted class of smooth functions of compact support must be the multiplication operator defined by the characteristic function of a measurable set $E$. Thus $\Pi = M_h$ where $h$ is of the form $\chi_E$ for some measurable set $E$.

The operator $\Pi$ maps $S$ into itself. It will now be shown that this can only be true if $E = \emptyset$ or $E = \mathbb{R}$. For otherwise the set $E$ must have a boundary point $q_0$. Let $f$ be a function in the Schwartz space which has the value one in a neighbourhood $U$ of $q_0$. On that neighbourhood $\Pi f = \chi_E$ and since $q_0$ is a boundary point of $E$ this means that $\Pi f$, like $\chi_E$, must be discontinuous there. Hence $\Pi f$ is not in the Schwartz space. It can now be concluded that $\Pi$ is zero or the identity and that the representation is irreducible. It follows that the inner product of $L^2(\mathbb{R})$ is admissible with respect to the Schrödinger representation of the algebra of the canonical commutation relations for one degree of freedom.

The above arguments extend without difficulty to show that the inner product of $L^2(\mathbb{R}^n)$ is strongly admissible with respect to the Schrödinger representation of the Weyl algebra for $n$ degrees of freedom and admissible for the corresponding representation of the canonical commutation relations. In the case of infinitely many degrees of freedom there still exists a Schrödinger representation of the Weyl algebra ([3], section 6.4). However there are many other unitarily inequivalent weakly continuous irreducible representations. It should be stressed that the question of interest in this paper, namely the unique determination of an inner product by a given representation, is logically independent of the question of whether the representation is unique under certain hypotheses. It is assumed throughout this paper that not only the algebra $A$ but also the vector space $V$ and the representation $\rho$ are given. The results then show that there is at most one admissible inner product for the given representation. No statement is made concerning the question whether, for a given representation, any admissible inner product exists.

It turns out that Theorem 1 can be applied to the Schrödinger representation of the Weyl algebra in the case of infinitely many degrees of freedom. The above arguments for irreducibility do not obviously generalise to the infinite dimensional case since techniques have been used (Fourier series and distribution theory) which are usually only considered for finite dimensional spaces. However there exist other arguments to prove the irreducibility of the Schrödinger representation for infinitely many degrees of freedom. It is, for instance, possible to use the equivalence of this representation with the Fock representation ([3], section 7.3). It follows that the hypotheses of Theorem 1 are still satisfied in
the infinite dimensional case.

Acknowledgements
I am grateful to Abhay Ashtekar and Jorma Louko for helpful discussions.

References