Global structure of a black-hole cosmos and its extremes

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Abstract. We analyze the global structure of a family of Einstein-Maxwell solutions parametrized by mass, charge and cosmological constant. In a qualitative classification there are: (i) generic black-hole solutions, describing a Wheeler wormhole in a closed cosmos of spatial topology $S^2 \times S^1$; (ii) generic naked-singularity solutions, describing a pair of “point” charges in a closed cosmos; (iii) extreme black-hole solutions, describing a pair of “horned” particles in an otherwise closed cosmos; (iv) extreme naked-singularity solutions, in which a pair of point charges forms and then evaporates, in a way which is not even weakly censored; and (v) an ultra-extreme solution. We discuss the properties of the solutions and of various coordinate systems, and compare with the Kastor-Traschen multi-black-hole solutions.

I. Introduction

In classical gravity there is a family of black-hole solutions characterized by mass $M$, charge $Q$ and cosmological constant $\Lambda$ (Carter 1973). Such solutions are often considered most realistic, for example for astrophysical applications, when the black hole is of a generic type. However, today renewed interest is attached to various types of extreme objects (see for example Banks et al 1992a). Additionally, typical inflationary cosmologies have an appreciable $\Lambda$ during some epoch. For such reasons, it is of interest to examine cosmological black-hole spacetimes† for their global properties and to identify the various extreme objects that can be obtained by suitable choice of the parameters $(M, Q, \Lambda)$.

† A spacetime with future null infinity $\mathcal{I}^+$ contains black holes if the causal past of $\mathcal{I}^+$ is not the whole spacetime. We call a black-hole spacetime cosmological if it has spatial sections that are compact and bounded only by the black holes’ horizon, namely the boundary of the causal past of $\mathcal{I}^+$. The spacetimes we consider are time-reversal-symmetric, so that each “hole” has both black and white horizons, but we follow convention in discriminating against the white holes.
Furthermore, Kastor and Traschen (1992) have recently given the first cosmological Einstein-Maxwell solution containing many black holes, but in coordinates that do not cover the spacetime completely. The solutions are parametrized by the cosmological constant $\Lambda$, $n$ masses $M_i$ and positions $(x_i, y_i, z_i)$, with charges $|Q_i| = M_i$. In the case $\Lambda = 0$, they reduce to the Majumdar-Papapetrou multi-black-hole solutions (Hartle and Hawking 1972). Understanding the global structure of the single-mass case, and the “cosmological” coordinates used, is a first step to understanding the global structure of the general Kastor-Traschen solutions.

In Section II we recall briefly the conformal treatment of infinity and a procedure for analytic continuation of a large class of time-independent metrics. In Section III we use this procedure to classify the $(M, Q, \Lambda)$ solutions according to their qualitative global structure. In Section IV we discuss the properties of the spacetimes, and in Section V we consider alternative coordinates. The Conclusion includes an outline of the corresponding results for the Kastor-Traschen solutions.

II. Conformal infinity and analytic continuation

The $(M, Q, \Lambda)$ spacetimes have infinite regions whose physical significance requires careful analysis. One such region is the late time unbounded inflationary era, if $\Lambda > 0$. Also, in some of these spacetimes the black holes are spatially infinite “horns”. Null geodesics may or may not cross such infinite regions in finite affine parameter intervals (Carter 1973). A convenient way to analyze such infinities and their global structure is to change the metric by an unbounded conformal factor (Penrose 1963, 1965). Surfaces at infinity become conformally finite, and the global structure of certain simple spacetimes can be summarized in conformal diagrams (e.g. Hawking and Ellis 1973). Apart from conformal infinity, the other key causal features are the various horizons that occur in such spacetimes, such as black-hole event horizons and cosmological horizons.

Originally, analyses of global structure were performed separately for such examples as the Schwarzschild solution (Finkelstein 1958, Kruskal 1960), the Reissner-Nordström solution (Graves and Brill 1960, Carter 1966b) and the Kerr solution (Carter 1966a, Boyer and Lindquist 1967). Walker (1970) has given a general prescription which can be applied to any spacetime of the type $T \times S$, where $T$ is totally geodesic with time-independent metric of the form

$$ds^2 = -F \, dT^2 + F^{-1} \, dR^2, \quad F = F(R),$$

where the geometry of $S$ is non-singular and a smooth function of $R$, and where $F$ has one or more simple zeros, which correspond to Killing horizons (Carter 1969). Namely, if $R_1$ and $R_2$ are two successive zeros of $F$, then the conformal diagram of the region $(T, R) \in (-\infty, \infty) \times (R_1, R_2)$ is a “block” with four inner null boundaries corresponding to $(T, R) = (\pm \infty, R_i)$. Similarly, if $R_1$ is the highest zero of $F$, then the region $(T, R) \in (-\infty, \infty) \times (R_1, \infty)$ is a block with an outer conformal boundary. Blocks are glued together at the inner null boundaries so that $R$ becomes a smooth (or analytic) function on the total space, and $T$ has a logarithmic singularity at the boundary. Once two or more blocks are glued together, further extensions can also be generated by noting that the symmetry $T \mapsto -T$ of each block must extend to a symmetry of the analytic extension.
It can be shown that coordinate charts exist which cover the boundaries, as follows. Denote the simple zero by \( R = R_0 \neq 0 \), so that \( F(R_0) = 0, \quad F'(R_0) = 2\kappa \neq 0 \), with \( |\kappa| \) being the surface gravity of the Killing horizon. Then Kruskal-like coordinates
\[
u = -e^{\kappa(R^*-T)}, \quad \nu = e^{\kappa(R^*+T)}, \quad R^* = \int F^{-1}dR
\] (2)
can be introduced, in terms of which the line-element (1) becomes
\[
ds^2 = -\kappa^{-2}Fe^{-2\kappa R^*}du dv + R^2dS^2,
\] (3)
where \( R^* = (2\kappa)^{-1}\log(-uv) \), and \( R \) and \( F \) are implicitly determined as functions of \( uv \). Near \( R = R_0 \),
\[
F = 2\kappa(R-R_0) + O(R-R_0)^2, \quad R^* = (2\kappa)^{-1}\log(R-R_0) + O(R-R_0),
\] (4)
where the constant of integration has been fixed, and it follows that
\[
ds^2 = -2\kappa^{-1}du dv + R^2_0dS^2 + O(R-R_0),
\] (5)
which is manifestly non-singular. Explicit coordinate charts are given in Section V for the case of interest.

The case of coincident roots can be treated by a simple extension of this procedure: when two blocks are glued across a double root, which corresponds to a degenerate Killing horizon in the sense of Carter (1973), \( T \) has a simple pole at the null boundary, and the usual spatio-temporal inversion of \((T, R)\) does not occur. A double (or higher multiple) root of \( F \) also implies that the geodesic distance along a constant-\( T \) surface from any point within the block to the degenerate horizon is infinite. If \( T \) is a timelike coordinate this means that the horizon is at an infinite distance, and the spatial geometry has the shape of a “horn” rather then a “wormhole”.

III. Global structure

The \((M, Q, \Lambda)\) solutions (Carter 1973) have line-element
\[
ds^2 = -FdT^2 + F^{-1}dR^2 + R^2dS^2,
\] (6)
where
\[
F(R) = \frac{Q^2}{R^2} - \frac{2M}{R} + 1 - \frac{1}{3}\Lambda R^2
\] (7)
and \(dS^2\) refers to the unit 2-sphere. The case \(\Lambda = 0\) yields the Reissner-Nordström solution, which was analyzed by Graves and Brill (1960) and Carter (1966b). The case \(Q = 0\) corresponds, for \(9\Lambda M^2 < 1\), to a Schwarzschild black hole in a de Sitter cosmos, which was analyzed by Gibbons and Hawking (1977). Henceforth we assume that \(\Lambda > 0, \quad M > 0\) and \(Q \neq 0\). The cases \(\Lambda < 0\) and \(M < 0\) can be analyzed similarly, but are of less interest for either cosmology or black holes.
According to the procedure explained in the previous Section, the global structure is determined by: behaviour at $R = 0$, where there is a scalar curvature singularity; behaviour at $R = \infty$, which for $\Lambda > 0$ is both timelike infinity $i^{\pm}$ and null infinity $\mathcal{I}^{\pm}$; and behaviour at the Killing horizons $F(R) = 0$. It is easily checked that the quartic $R^2 F(R)$ has exactly one negative root, and generically either one or three positive roots, with special cases of double or triple roots. The possibilities for the positive roots are as follows.

(i) The generic black-hole case: three simple roots $R_c > R_o > R_i$. There are three types of Killing horizon: cosmological horizons at $R = R_c$ and inner and outer black-hole horizons at $R = R_i$ and $R = R_o$ respectively.

(ii) The generic naked-singularity case: one simple root, giving only cosmological horizons.

(iii) The extreme (or “cold”) black-hole case: a double root $R_d$ and a simple root $R_c > R_d$, corresponding to a degenerate black-hole horizon at $R = R_d$ and a cosmological horizon at $R = R_c$.

(iv) The extreme (or marginal) naked-singularity case: a double root $R_d$ and a simple root $R_i < R_d$, corresponding to a degenerate cosmological horizon at $R = R_d$ and an inner horizon at $R = R_i$.

(v) The “ultra-extreme” case: triple root, with one doubly degenerate horizon.

Figure 1 shows the conformal diagrams as obtained by the method of Section II. Many properties of the solutions are clear from these diagrams, and we make additional comments in the next Section.

To make explicit which case occurs for each $(M, Q, \Lambda)$, we note that the double roots occur if and only if $M = M_{\pm}(Q, \Lambda)$, where

$$M_{\pm} = P_{\pm}(1 - \frac{2}{3}\Lambda P_{\pm}^2), \quad P_{\pm}^2 = \frac{1}{2\Lambda} \left(1 \pm \sqrt{1 - 4\Lambda Q^2}\right),$$

with the double root being at $R = P_{\pm}$ (Romans 1992). The ultra-extreme case occurs when $9M^2 = 8Q^2 = 2/\Lambda$, so that $P_+ = P_-$. Otherwise, the extreme black hole occurs if $M = M_-(Q, \Lambda)$, and the extreme naked singularity occurs if $M = M_+(Q, \Lambda)$. The generic black hole occurs if $M_-(Q, \Lambda) < M < M_+(Q, \Lambda)$, and the generic naked singularity otherwise. Figure 2 shows where the various cases lie in parameter space.

As an example of relevance to the Kastor-Traschen solutions, we note that for the case $|Q| = M$, a generic black hole occurs if $\Lambda M^2 < 3/16$, an extreme naked singularity if $\Lambda M^2 = 3/16$, and a generic naked singularity if $\Lambda M^2 > 3/16$. The extreme black hole does not occur in this class. If $\Lambda M^2 \ll 1$ the Killing horizons are located approximately at

$$R_i \sim M - HM^2, \quad R_o \sim M + HM^2, \quad R_c \sim H^{-1} - M,$$

where $H = \sqrt{\Lambda}/3$. Romans (1992) gives such expressions to a higher order of approximation.

IV. Properties

The generic black-hole solution, Figure 1(i), can be interpreted as a pair of oppositely charged Reissner-Nordström black holes at opposite poles of a de Sitter cosmos (Mellor and Moss 1989). The interiors of the holes lead either to other de Sitter regions, or to each
other, depending on the topological identifications made. We refer to the regions $R < R_0$ as the holes, and the regions $R > R_c$ as the cosmological regions. Any constant-$T$ surface is a surface of reflection symmetry. When $F < 0$ this surface is Lorentzian and corresponds to reflection about the “equator” of the de Sitter exterior, or about the “throat” of the holes. For $F > 0$, $R_o < R < R_c$, the surfaces are free from singularities and can be connected smoothly to form a surface of time symmetry. This surface divides the spacetime into a future containing a black hole ($R < R_o$) and an expanding cosmological region ($R > R_c$), and a past containing a white hole and a contracting cosmological region.

The spatial geometry on such constant-$T$ surfaces is given by

$$F^{-1}dR^2 + R^2dS^2,$$

which can be embedded in four-dimensional flat space

$$dZ^2 + dR^2 + R^2dS^2$$

by

$$Z = \int \sqrt{F^{-1} - 1} dR,$$

as depicted in Figure 3. The surface has maximal radius $R_c$ at the cosmological horizon, and minimal radius $R_o$ at the throats of the holes. Isometric copies of the surface can be smoothly joined at the throats, producing a periodic $S^2 \times R^1$ spatial topology, in which the interiors of the black holes lead to further de Sitter regions. Alternatively, one may simply identify the throats with each other, giving an $S^2 \times S^1$ spatial topology. The latter identification provides a classic example of a charged wormhole in the sense of Wheeler (1962): the lines of electric force converge on one side of the wormhole and diverge from the other, so that the appearance of two opposite charges is generated by non-trivial topology rather than by charged sources. This appears to be the only explicitly known example of a Wheeler wormhole spacetime without “external” fields.

The generic naked-singularity solution is shown in Figure 1(ii). A constant-$T$ spatial surface has topology $S^2 \times R^1$, which can be visualized as an $S^3$ punctured at opposite poles, with the two singularities being at finite affine distance. The lines of electric force diverge from one pole and converge on the other, so that the singularities have the appearance of point charges.

We now turn to the extreme cases. The extreme black-hole solution, Figure 1(iii), can be interpreted as a pair of oppositely charged extreme Reissner-Nordström black holes at opposite poles of a de Sitter cosmos. Note that for $\Lambda > 0$ the extreme case occurs for $M < |Q|$. That is, in the cosmological context, the maximal charge on a black hole—beyond which there is a naked singularity instead—is larger than in asymptotically flat space (Romans 1992, cf Banks et al 1992b). The topology of a constant-$T$ spatial surface is $S^2 \times R^1$. The two minimal-radius spheres, which also represent the black holes’ horizons, are located at infinite affine distance on this spatial surface, preventing a wormhole identification, so that these black holes are horns rather than wormholes.

The extreme naked-singularity case, Figure 1(iv), has a novel structure that suggests the creation and subsequent annihilation of a pair of point charges. A constant-$T$ spatial
surface has topology $S^2 \times R^1$, which again can be visualized as an $S^3$ punctured at opposite poles, with the two singularities being at finite affine distance, and the lines of electric force diverging from one pole and converging on the other. Unlike the generic naked singularities, or that of the negative-mass Schwarzschild solution, these singularities do not exist for all time, but develop from an initially regular state, i.e. there are partial Cauchy surfaces, of topology $S^2 \times R^1$ or $S^2 \times S^1$, depending on the identifications made. The singularities are not even weakly censored, in the sense that any observer whose future life is long enough will see them: any path from $i^-$ to $i^+$ passes through the causal future (and the causal past) of the singularities. This is in distinction to the generic black hole, or the Reissner-Nordström black hole, whose singularities remain unseen by wary observers. Another novel feature is that the singularities subsequently dissolve, with the process being visible from $\mathcal{I}^+$.

Finally, we note that the ultra-extreme case is similar to the generic naked-singularity case, except that the singularities are at infinite affine distance.

V. Other charts

We now consider alternative coordinate systems, for three main reasons. Firstly, the static coordinates $(T, R)$ break down at the Killing horizons, and while it was shown in Section II that coordinate charts do exist there, it is useful to find such charts explicitly. Secondly, the Kastor-Traschen solutions are given in different coordinates, and understanding the nature of these “cosmological coordinates” is important for the physical interpretation. Thirdly, different coordinates emphasize different features, which may or may not persist in the general Kastor-Traschen solutions.

As a preliminary example, we consider the de Sitter cosmos (Hawking and Ellis 1973). It is completely covered by the usual “hyperbolic” de Sitter coordinates,

$$ds^2 = 3\Lambda^{-1}(-d\theta^2 + \cosh^2 \theta d\Sigma^2),$$

where $d\Sigma^2 = d\chi^2 + \sin^2 \chi dS^2$ refers to the unit 3-sphere. The conformal diagram, Figure 4(i), shows conformal infinity $\mathcal{I} \ (\theta = \pm \infty)$ and the (opposite but otherwise arbitrary) poles of the 3-sphere, $\chi = 0$ and $\chi = \pi$. In such diagrams one can also show other coordinates and the extent to which they cover the complete spacetime. Figure 4(ii) shows this for the “cosmological” (or “steady-state-universe”) coordinates $(t_+, r)$ or $(t_-, r)$, in terms of which the de Sitter metric is

$$ds^2 = -dt_\pm^2 + e^{\pm 2Ht}(dr^2 + r^2 dS^2),$$

where the coordinate transformation is

$$\pm Ht_\pm = \log(\sinh \theta + \cosh \theta \cos \chi), \quad \pm Hr = \frac{\cosh \theta \sin \chi}{\sinh \theta + \cosh \theta \cos \chi},$$

and $H = \sqrt{\Lambda/3}$. We see that most of de Sitter spacetime is covered by two charts, one “expanding” $(t_+, r)$ chart and one “contracting” $(t_-, r)$ chart, but that both fail to cover the cosmological horizon $t_\pm = \mp \infty$. By the reflection $\theta \mapsto -\theta$ one obtains similar
coordinates that cover the previous cosmological horizon but break down at the reflected cosmological horizon.

The line-element may also be written in static \((T, R)\) coordinates,

\[
ds^2 = -F \, dT^2 + F^{-1} \, dR^2 + R^2 \, dS^2, \quad F = 1 - \frac{1}{3} \Lambda R^2, \tag{16}
\]

which are related to the cosmological coordinates by

\[
R = e^{\pm H t \pm r}, \quad t_{\pm} = T \pm \int H R F^{-1} dR, \tag{17}
\]

and are shown in Figure 4(iii). The static coordinates divide de Sitter spacetime into four regions, and break down at both cosmological horizons. The two regions that do not include \(\mathcal{I}^\pm\) are static in the usual sense. The regions attached to \(\mathcal{I}^+\) and \(\mathcal{I}^-\) are expanding and contracting respectively, with constant-\(T\) surfaces being spatial homogeneous cylinders. Thus the cosmological coordinates have an advantage over the static coordinates, namely that they can be used to cover more of the manifold, but have the disadvantage that by breaking the time symmetry they can give a somewhat misleading picture of the spacetime.‡

The \((M, Q, \Lambda)\) solutions \((6)-(7)\) can likewise be written in cosmological (or isotropic) coordinates \((t_{\pm}, r)\) or \((t_{-}, r)\),

\[
ds^2 = -V^{-2} \, dt_{\pm}^2 + U^2 e^{\pm 2 H t_{\pm}} (dr^2 + r^2 dS^2), \tag{18}
\]

where

\[
U(\rho) = 1 + M \rho^{-1} + \frac{1}{4} (M^2 - Q^2) \rho^{-2}, \quad V(\rho) = \frac{U(\rho)}{1 - \frac{1}{4} (M^2 - Q^2) \rho^{-2}} \tag{19}
\]

and

\[
\rho = e^{\pm H t \pm r}, \tag{20}
\]

with \(H = \sqrt{\Lambda/3}\) as before. Here the relation between the coordinates is

\[
R = U \rho, \quad t_{\pm} = T \pm \int \frac{HR dR}{F \sqrt{F + H^2 R^2}}. \tag{21}
\]

These coordinates yield various charts (Figure 5) which cover several of the static blocks for \(R > R_+\) or \(R < R_-\), but with a gap for \(R_- \leq R \leq R_+\), where

\[
R_{\pm} = M \pm \sqrt{M^2 - Q^2}. \tag{22}
\]

‡ The Schwarzschild solution is a similar example. The static Schwarzschild coordinates break down at the horizons, which was originally taken as indicating a singularity or boundary of the spacetime. Finkelstein (1958) showed that the black-hole horizon could be covered by different coordinates, but concluded that this indicated a time asymmetry of the spacetime. Kruskal (1960) showed how to cover the whole manifold with a different chart.
Because $\rho$ is not a single-valued function of $R$, successive constant-$t$ surfaces intersect along a caustic at $R_{\pm}$, and the constant-$r$ surfaces have cusps there, with the two branches corresponding to different charts related by $T \mapsto -T$. The black-hole horizon occurs at $\pm HVU \rho = 1$.

For the case $\lvert Q \rvert = M$, the gap between $R_{\pm}$ disappears and the solution (18)-(20) coincides with the single-mass Kastor-Traschen solution in cosmological coordinates. A further transformation to

$$\tau_{\pm} = \pm H^{-1} e^{\pm H t_{\pm}}$$

(23)

puts the line-element in the form

$$ds^2 = -(Mr^{-1} \pm H \tau_{\pm})^{-2} d\tau_{\pm}^2 + (Mr^{-1} \pm H \tau_{\pm})^2 (dr^2 + r^2 dS^2).$$

(24)

Now a single $(\tau, r)$ chart covers, for $\tau < 0$ and $\tau > 0$, two $(t, r)$ charts, whereas the latter break down at $\tau = 0$, corresponding to $R = M$. Figure 6 shows the new cosmological coordinates for the $\lvert Q \rvert = M$ solution. A single $(\tau, r)$ chart covers four $(T, R)$ charts, including the three Killing horizons between them, namely the cosmological, event and inner horizons. For $(\tau_{+}, r)$, this includes the expanding cosmological region and part of the white hole, while for $(\tau_{-}, r)$, the contracting cosmological region and part of the black hole are covered. The boundaries of the $(\tau_{+}, r)$ chart are the singularity $H r \tau_{+} = -M$, the other event horizon $(\tau_{+}, r) = (\infty, 0)$, the other cosmological horizon $(\tau_{+}, r) = (0, \infty)$, the other inner horizon $(\tau_{+}, r) = (-\infty, 0)$, and $\mathbb{R}^+$ $(\tau_{+} = \infty, r$ finite). By using the time-inversion symmetry $T \mapsto -T$ of the static frame, similar cosmological charts can be constructed to cover the entire manifold, except for the isolated 2-surfaces where the Killing horizons cross. These are regular since the constant-$T$ surfaces are smooth and totally geodesic (Section IV).

Thus we have verified the global structure derived in Section III. The cosmological coordinates are useful for covering the horizons, but give highly distorted charts which do not respect the time symmetry. The time symmetry can be restored by yet another transformation, to double-null coordinates $(u, v)$. First note that the $\lvert Q \rvert = M$ case can be written in $(t_{\pm}, \rho)$ coordinates as

$$ds^2 = -U^{-2} dt_{\pm}^2 + U^2 (d\rho \mp H \rho dt_{\pm})^2 + U^2 \rho^2 dS^2,$$

(25)

where $U(\rho) = 1 + M \rho^{-1}$. Then the transformation

$$u = t_{\pm} + \int \frac{d\rho}{U^{-2} \mp H \rho}, \quad v = t_{\pm} - \int \frac{d\rho}{U^{-2} \pm H \rho}$$

(26)

puts the line-element in the form

$$ds^2 = -(U^{-2} - \frac{1}{2} \Delta U^2 \rho^2) du \, dv + U^2 \rho^2 dS^2,$$

(27)

where $\rho$ is implicitly determined as a function of $u - v$ by (26). This double-null form has the manifest time-inversion symmetry

$$(u, v) \mapsto (-v, -u),$$

(28)

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which allows the recovery of all the different \((t, r)\) or \((\tau, r)\) charts.

Although the cosmological coordinates cover some horizons of the spacetimes discussed here, others occur at \(\tau = 0\) and \(\tau = \infty\) in these coordinates. Some cosmological models take a metric like (18) or (24) to be valid only for a finite range of \(\tau\). The global structure of such models can be quite different from that shown in our Figures. For example, if the universe started out as an expanding \(k = 0\) Friedmann model with \(\Lambda = 0\), and \(\Lambda\) was “turned on” at some finite \(\tau\), then \(\mathcal{Z}^-\) in Figure 6 would be replaced by the big bang singularity.

VI. Conclusion

We have seen that the metrics that appear to describe a single charged black hole with cosmological constant may be maximally extended to a spacetime that generically has an (approximately de Sitter) cosmological region and two black hole regions for each \(\mathcal{Z}^+\). In the alternative extension with \(S^2 \times S^1\) spatial topology, there is only one \(\mathcal{Z}^+\) and one black hole, with the black hole having two horizons at opposite poles of the cosmos, constituting a Wheeler wormhole. In addition, there is a novel type of extreme case where a pair of point charges forms and subsequently dissolves.

The cosmological or isotropic coordinates are simplest in the case \(|Q| = M\), where isometric copies of a single coordinate chart suffice to cover most of the maximal extension. The “Hubble constant” \(\pm H\) is positive in some of these charts and negative in others, though the adjectives “expanding” and “contracting” can really be justified only for the blocks attached to \(\mathcal{Z}^+\) and \(\mathcal{Z}^-\) respectively. Each cosmological chart breaks down at a cosmological horizon given by a finite value of the coordinate \(\rho\) of (20), namely the largest solution of \(\pm HVU\rho = 1\), approximately \(\rho \sim \pm H^{-1} - 2M\). This horizon is covered by a different cosmological chart obtained using the time-inversion symmetry (28).

The Kastor-Traschen metric is similar to (24), with \(Mr^{-1}\) replaced by \(\sum_i m_i r_i^{-1}\), where \(r_i\) are Euclidean distances of the field point \(r\) from \(n\) fixed centers, and \(m_i\) are mass parameters associated with these centers. At large \(\tau\) the metric approaches that of (24), so that \(\mathcal{Z}^\pm\) exist as for the single-mass case. The incompleteness at large \(r\) is also similar to that of the single-mass case, and the extension across the horizon at \((\tau, r) = (0, \infty)\) leads to a symmetrically related spacetime that contains the opposite charge. In the Kastor-Traschen metric this region is described by an identical expression but with the opposite sign of \(H\). Thus in a background de Sitter space, we have a set of black holes with one sign of charge joined across the horizon to a corresponding set with the opposite charge. Details of this geometry as well as the continuation across the black-hole horizons will be discussed in a separate paper.

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Figure captions

1. Penrose-Carter conformal diagrams of the $(M, Q, \Lambda)$ solutions, showing
   (i) the generic black-hole case,
   (ii) the generic naked-singularity case,
   (iii) the extreme black-hole case,
   (iv) the extreme naked-singularity case.
   The conformal diagram for the ultra-extreme case is identical to (ii). Wavy lines represent $R = 0$ singularities, diagonal lines represent $F(R) = 0$ Killing horizons, and horizontal lines represent conformal infinity $\mathbb{I}^+$ or $\mathbb{I}^-(R = \infty)$. The curves represent surfaces of constant $R$. The maximal analytic extension is obtained by identifying isometric copies along the notched horizons.
   To be inserted in Section III.

2. Parameter space $(M, Q)$ for fixed $\Lambda > 0$. Extreme black holes lie along the curve $M = M_-(Q)$, and extreme naked singularities along $M = M_+(Q)$, with the two curves meeting in a cusp, corresponding to the ultra-extreme case. Generic black holes lie inside the double curve, and generic naked singularities outside.
   To be inserted in Section III.

3. Geometry of a constant-$T$ surface of time symmetry for the generic black hole, embedded in flat space, with a polar angle suppressed. Here $HM = 0.05$, $(HQ)^2 = 0.001$. The curves represent lines of electric force. The two wormhole throats can be smoothly identified to form a Wheeler wormhole universe at a moment of time symmetry.
   To be inserted in Section IV.

4. Conformal diagrams of the de Sitter cosmos, showing conformal infinity $\mathbb{I}^\pm$ and the regular poles $O$. The curves represent 3-surfaces of constant (i) hyperbolic $(\theta, \chi)$, (ii) cosmological $(t, r)$ and (iii) static $(T, R)$.
   To be inserted in Section V.

5. Conformal diagram as in Figure 1(i), showing the cosmological, or isotropic, coordinates for the case $|Q| < M$. The curves represent 3-surfaces of constant $\tau_+$ and of constant $r$. (Here $\tau_+$ is the coordinate defined in (23) and can take negative values.) Charts are shown for $R > R_+$ and $R < R_-$, with the region $R_- \leq R \leq R_+$ not being covered by such charts. The dashed curves show the continuation of one curve of constant $\tau$, and one of constant $r$, beyond the caustic at $R = R_+$.
   To be inserted in Section V.

6. A “cosmological” $(\tau_+, r)$ chart for the $|Q| = M$ solution, for (i) a generic black hole, (ii) the extreme naked singularity, and (iii) a generic naked singularity. The curves represent surfaces of constant $\tau_+$ and $r$.
   To be inserted in Section V.