General covariance, loops, and matter

Viqar Husain

Theoretical Physics Institute, University of Alberta
Edmonton, Alberta T6G 2J1, Canada.

Abstract

A four dimensional generally covariant field theory is presented which describes non-dynamical three geometries coupled to scalar fields.

The theory has an infinite number of physical observables (or constants of the motion) which are constructed from loops made from scalar field configurations. The Poisson algebra of these observables is closed and is the same as that for the 3+1 gravity loop variables in the Ashtekar formalism. The theory also has observables that give the areas of open surfaces and the volumes of finite regions.

Solutions to all the Hamilton-Jacobi equations for the theory and the Dirac quantization conditions in the coordinate representation are given. These solutions are holonomies based on matter loops. A brief discussion of the loop space representation for the quantum theory is also given together with some implications for the quantization of 3+1 gravity.

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I. INTRODUCTION

The Ashtekar Hamiltonian variables [1] for general relativity and the associated loop variable approach [2] to quantum gravity have inspired a number of toy models that have similar phase space variables and are amenable to quantization via the loop representation. The goal of these models is to try and mimic the basic features of the constraint structure of general relativity in the hope of gaining some insights into quantization. Most of the models considered so far have not included matter fields and it is not clear what role such fields will play in the loop variable methods. It is clearly important to include matter in this approach to quantum gravity and here we consider a model that attempts to do this.

A separate but essential reason for including matter fields is the issue of physical observables, which are phase space functions which have weakly vanishing poisson brackets with all the constraints. For the case of pure (spatially closed) gravity, no such observables are known but there are indications that inclusion of matter to define reference systems may help to determine physical observables [3–5].

These observables play the essential role in one approach to the quantization of generally covariant theories. In this method the goal is to define the quantum theory by finding a representation of the Poisson algebra of a complete set of observables. This has been applied successfully to 2+1 gravity [6] but it cannot even be started for 3+1 gravity since no such observables are known (for the compact case). There is some recent work indicating that the natural observables in general relativity are associated with functions that measure area or volume [7]. These are kinematical constructions and are not invariant under the dynamics generated via the Hamiltonian constraint. They are also not naturally spatial diffeomorphism invariant since auxiliary variables are used in some of the definitions. One can attempt to partly rectify this situation by the inclusion of matter fields, and use them in place of the auxiliary variables, to make truly diffeomorphism invariant observables. Attempts in this direction have been made recently using scalar fields [8] and anti-symmetric tensor fields [9] to define area observables, and their spectra given via a quantization based
on the loop representation. The present work is also a partial attempt in this direction but the emphasis is on constructing diffeomorphism invariant loop observables.

In the case of general relativity without matter fields, the Rovelli-Smolin loop observables, which are based on holonomies, play an important role in the construction of the loop representation. These variables are not diffeomorphism invariant (since they are functions of the auxiliary loops as well), but nevertheless a representation of their Poisson algebra provides an important step towards the solution of the diffeomorphism constraint.

In this paper we discuss a model that has spatially diffeomorphism invariant loop observables where the loops are made of matter. A four dimensional generally covariant model is discussed in the next section and its Hamiltonian version and physical observables are given. Section 3 is a discussion of 3+1 gravity with scalar fields, which is obtained from this model by including the Hamiltonian constraint on the phase space. There is also a discussion of the the Hamilton-Jacobi equations for the constraints, which are completely solved for the model and partially solved for 3+1 gravity by using a holonomy functional. Section 4 is a brief discussion of the Dirac and loop quantizations of the model followed by a summary and discussion in section 5.

II. GENERALLY COVARIENT MODEL AND OBSERVABLES

There is a type of generally covariant field theory that is not topological, has internal Yang-Mills symmetry, but no dynamics [10]. The lagrangian is a four form constructed from the curvature of an so(3) gauge field and a dreibein:

$$S_1 = \int_M d^4 x \epsilon^{\alpha\beta\gamma\delta} \epsilon_{ijk} e^i_{\alpha} e^j_{\beta} F^{k}_{\gamma\delta}$$  \hspace{1cm} (2.1)

The Hamiltonian formulation of this model is similar to that for general relativity in the Ashtekar formulation except that the Hamiltonian constraint is absent (or equivalently, vanishes strongly): the only first class constraints that appear on the phase space correspond to spatial diffeomorphisms and internal gauge transformations [10]. This theory therefore
has three local degrees of freedom and is effectively a three dimensional theory since the
dynamics is trivial. There are indications that it is related to a $c=0$ limit of general relativity
\[8\]. There is a similar model in three dimensions that may be viewed as arising from the
hamiltonian decomposition of a 2D de-Sitter group Chern-Simons theory \[11\].

Here we consider theories of this type with coupling to a number of scalar fields. Since
the metric obtained from the dreibein $g_{\alpha\beta} = e^i_\alpha e^j_\beta$ is degenerate, the usual form of coupling
scalar fields is not possible. However there is a way to couple non-dynamical scalar fields
(which is in the spirit of the model (2.1)) using the vector density $n^\alpha \equiv \epsilon^{\alpha\beta\gamma\delta} e^i_\beta e^j_\gamma e^k_\delta \epsilon_{ijk}$. We
add to $S_1$ the term

$$S_2 = \int_M d^4x n^\alpha \pi_n \partial_\alpha \phi_n$$

where $\pi_n$ and $\phi_n$ are a pair of scalar fields for $n = 1, ..., N$. The equations of motion for
these scalars give $n^\alpha \partial_\alpha \phi_n = 0$ and $\partial_\alpha (\pi_n n^\alpha) = 0$ which effectively imply that the fields do
not evolve (since $n^\alpha$ is the degeneracy direction for $g_{\alpha\beta}$).

The Hamiltonian theory for the action $S = S_1 + S_2$ on $M = \Sigma \times R$ is easily obtained
since the action is in first order form. The 3+1 form of the action is

$$S = \int_R dt \int_\Sigma d^3x \left[ E^{ai} \dot{A}_a^i + \Pi_n \dot{\phi}_n + N^a (F_{ab} E^{bi} + \Pi_n \partial_a \phi_n) + A_0^i (D_a E^{ai}) \right]$$

where $N^a = e^ai e^j_0$ and $e^ai$ is the inverse of the projection of the dreibein onto the surface $\Sigma$
(where it is invertible). The canonical phase space variables are those of so(3) Yang-Mills
theory with conjugate variables $(A_a^i, E^{ai})$, together with the $n$ scalar field variables $(\phi_n, \Pi_n)$. $E^{ai} \equiv \epsilon^{abc} \epsilon^{ij} e^j_b e^k_c$ and $\Pi_n \equiv n^0 \pi_n$. The first class constraints on the phase space are the
Gauss law and spatial diffeomorphisms obtained by varying $S$ with respect to $A_0^i$ and $N^a$.

$$G^i = D_a E^{ai} = 0,$$

$$C_a = F_{ab} E^{bi} + \Pi_n \partial_a \phi_n = 0$$

where $D_a$ and $F_{ab}$ are the covariant derivative and curvature of $A$. Since there is no Hamilto-
nian constraint, the ‘dynamics’ generated by a linear combination of the constraints is pure
gauge. This completes the description of the Hamiltonian theory. (The basic reason that the Hamiltonian constraint is absent in the theory is a result of the presence of a degeneracy direction given by the vector field density \( n^\alpha \). When converted into a vector field by means of an auxiliary foliation of \( M \), it Lie derives the metric \([10]\)).

This Hamiltonian theory for one scalar field has been considered in a context related to extracting area observables as constants of the motion and their quantization \([8]\). Here we give a discussion of loop observables in this theory, and point out that the Gauss law invariant loop observables introduced on the Ashtekar phase space of general relativity can be made diffeomorphism invariant as well when scalar fields are present.

To define loops on the 3D surfaces \( \Sigma \) we need two scalar fields, and a loop \( \gamma[\phi_1, \phi_2] \) is defined as the intersection of two surfaces \( \phi_1 = c_1, \phi_2 = c_2 \). A vector density tangent to the loop is

\[
\gamma^a = \epsilon^{abc} \partial_b \phi_1 \partial_c \phi_2 \bigg|_{\phi_1 = c_1, \phi_2 = c_2} \tag{2.6}
\]

This form is not necessary for any computations, for which we need only the variation of \( \phi_n = c_n: \delta \phi_n + \delta \gamma^a \partial_a \phi_n = 0 \). (Regarding this method we note that in some recent work \([15]\) Newman and Rovelli have used scalar fields in a similar way to solve (classically) the Gauss law constraints in Yang-Mills theory. In particular, for the Abelian theory they set the electric field \( E^a := \epsilon^{abc} \partial_b u \partial_c v \) for two scalar fields \((u, v)\), which solves the Gauss law. The electric field lines in this solution are tangent to the loops defined by \( u = c_1, v = c_2 \).

The first few loop observables are defined as

\[
T^0[\phi_1, \phi_2, A](c_1, c_2) = Tr P e x p \left[ \int_{\gamma[\phi_1, \phi_2]} ds \gamma^a(s) A_a(\gamma(s)) \right] \tag{2.7}
\]

\[
T^1[A, E, \phi_1, \phi_2](c_1, c_2) = \int_{\gamma[\phi_1, \phi_2]} dw_a(s) Tr \left[ E^a(\gamma(s)) U_\gamma(s, s) \right] \tag{2.8}
\]

\[
T^2[A, E, \phi_1, \phi_2](c_1, c_2) = \int_{\gamma[\phi_1, \phi_2]} ds \int_{\gamma[\phi_1, \phi_2]} dt w_a(s) w_b(t) \left[ Tr \left[ E^a(\gamma(s)) U_\gamma(s, t) E^b(\gamma(t)) U_\gamma(t, s) \right] \right] \tag{2.9}
\]

where the 1-form density

\[
w_a \equiv \epsilon_{abc} \frac{\delta \gamma^c}{\delta \phi_1}. \]
These are functionals of the fields \((A, E, \phi_1, \phi_2)\) and functions of the two parameters \((c_1, c_2)\). All the other observables \(T^N\), with \(N\) \(E\)-insertions in the holonomies are constructed in a similar way. These observables are modeled after the (Gauss law invariant) Rovelli-Smolin loop variables and they have essentially the same closed Poisson algebra since they are independent of the scalar field momenta \(\Pi_n\). The novelty here is that the elements of this \(T\) algebra are *the constants of the motion* associated with the action \(S\). It is clear by inspection that these observables are diffeomorphism invariant (and this may be checked by explicitly computing Poisson brackets with \(C_a\)).

There are an infinite number of loops determined by the configurations of the two scalar fields and the constants \(c_1, c_2\). It is also easy to visualize how multiloops may arise by considering, for example, the case where one of the scalar fields defines a plane and the other defines a nearly parallel plane but with a large number of ‘bumps’ that intersect the first plane.

These \(T^N\) physical observables are however not the complete set since they do not involve the scalar field momenta \(\Pi_n\). The diffeomorphism invariant observables involving these are, for \(n = 1, 2\), \(P_n[\Pi_n] = \int_\Sigma d^3x \Pi_n\). These have non-trivial Poisson brackets with the \(T^N\). For example

\[
\{T^0, P_n\} = \int_\Sigma d^3x \frac{\delta T^0}{\delta \phi_n} = \int_{\gamma[\phi_1, \phi_2]} ds \epsilon^{abc} w_c(s) Tr[F_{ab}(\gamma(s))U_\gamma(s, s)]. \tag{2.10}
\]

In general \(\{T^N, P_n\} = \int_\Sigma d^3x (\delta T^N/\delta \phi_n)(x)\), and the functional derivative with respect to \(\phi_n\) acts effectively to shift the loop \(\gamma^a[\phi_1, \phi_2](c_1, c_2)\) by shifting \(\phi_1, \phi_2\) but leaving \(c_1, c_2\) fixed: \(\delta/\delta \phi_1 = (\delta \gamma^a/\delta \phi_1)\delta/\delta \gamma^a\). The Poisson brackets of any observable involving the momenta \(\Pi_n\) will therefore not close with the \(T^N\). To obtain closure it appears that the set of \(T^N\) will have to be extended to include all the additional loop observables that involve higher functional derivatives of the form \(\delta w/\delta \phi_n\) in the integrands of the \(T^N\). Such terms result from calculating the Poisson brackets. This extension can be done but the resulting algebra loses the elegance of just the \(T^N\) observables.
We note that there are other diffeomorphism invariant observables that may be constructed from scalar fields. These are the area observables discussed by Rovelli [8] which are defined using one scalar field:

\[ A = \int_{S[\phi]} d^2\sigma \sqrt{h} \equiv \int_{S[\phi]} d^2\sigma \sqrt{E_{bi}^a n_a n_b} \quad (2.11) \]

where \( n_a = \epsilon_{abc} (\partial_x^b / \partial \sigma^1)(\partial x^c / \partial \sigma^2) \). This observable commutes with \( T^0 \):

\[
\{T^0[\phi_1, \phi_2, A], A[\phi_1, E]\} = \int_{\gamma[\phi_1, \phi_2]} ds \int_{S[\phi_1]} d^2\sigma \delta^3(\gamma(s), \sigma) \frac{\dot{\gamma}^a Tr[E^b U](\gamma(s)) n_a n_b}{\sqrt{h}(\sigma)} = 0 \quad (2.12)
\]

since the normals \( n_a \) to the surfaces \( \phi_1 = C \) are perpendicular to the tangent vector to the loop. A further observation regarding \( A \) is that, since we have two scalar fields, it may be defined for open surfaces where the boundary of the surface is determined by the loop \( \gamma[\phi_1, \phi_2] \). This a different way of constructing observables associated with open surfaces than the one given by Smolin [9] where an Abelian gauge field is used to specify the boundary of a surface specified by an antisymmetric tensor field.

Another diffeomorphism invariant observable is

\[ Q[E, \phi] = \int_{\Sigma} d^3x \sqrt{E_{ai}^a E_{bi}^b \partial_a \phi_1 \partial_b \phi_1} . \quad (2.13) \]

This functional is essentially the same as the one used to define the ‘weaves’, which are distributional dreibeins taking non-zero values only on given configurations of loops. It is shown in ref. [7] that this functional may be converted into a well defined operator in the loop space representation, and that its eigenstates are distributional dreibeins which take values on sets of loops. Whereas in [7] this observable is defined using an auxiliary 1-form \( \omega_a \) on \( \Sigma \), we see that with a scalar field one can define it using \( \partial_a \phi \) thereby converting it into a diffeomorphism invariant phase space functional. (We also note that the integrand is the square root of the scalar field contribution to the Hamiltonian constraint for gravity [14]. See eqn. (3.1) below). As for the area observable, \( Q \) commutes with \( T^0 \) for the same reason, namely \( \dot{\gamma}^a \partial_a \phi_1 = 0 \) on the loop.
\[
\{Q[A, \phi_1], T^0[A, \phi_1, \phi_2]\} = \int_{\gamma[\phi_1, \phi_2]} ds \frac{s^a \partial_a \phi_1 \partial_b \phi_1 Tr[E^b U]}{\sqrt{E^{a_i} E^{b_j} \partial_a \phi_1 \partial_b \phi_1}} = 0 \tag{2.14}
\]

In fact this result generalizes: all the \(T^N\) defined above commute with \(A\) and \(Q\). Thus it is possible to specify sets of three mutually commuting observables that are associated with loops, areas and metrics: any one of the \(T^N\), \(A\) and \(Q\). It may be possible to find a representation where the corresponding operators have simultaneous eigenstates.

A final diffeomorphism invariant observable is one that measures the spatial volume:

\[
V = \int d^3x \sqrt{detE(x)}. \tag{2.15}
\]

Since its definition requires no auxiliary fields it is diffeomorphism invariant without the scalar fields. However, the two scalar fields allow one to define diffeomorphism invariant boundaries of a spatial region, and so it becomes possible to limit the range of volume integration to regions bounded by the surfaces \(\phi_1 = c_1, \phi_2 = c_2\). This is analogous to defining the areas of surfaces bounded by loops as discussed above. \(V\) does not commute with the \(T^N\) but does (trivially) with \(Q\) and \(A\).

### III. 3+1 GRAVITY

The models discussed in the previous section may be converted into 3+1 general relativity with massless scalar fields by the addition of the Hamiltonian constraint. This constraint is [14]

\[
H = \epsilon^{ijk} F_{ab} E^{ai} E^{bj} + E^{ai} E^{bi} \partial_a \phi_n \partial_b \phi_n - \Pi_n^2 \tag{3.1}
\]

Such a generalization also involves complexifying \(A_a\), together with the accompanying reality conditions.

The observables given in the last section do not Poisson commute with this constraint. However we note that the functional \(T^0\) is an approximate solution to the Hamilton-Jacobi
equations for gravity determined from the constraints by replacing the momenta $E$ and $\Pi$ by $\delta S[A, \phi]/\delta A$ and $\delta S[A, \phi]/\delta \phi$. With $S[A, \phi] \equiv T^0$ we find

$$D_a \frac{\delta S[A, \phi]}{\delta A^i_a} = 0 \quad (3.2)$$

$$F_{ab} \frac{\delta S[A, \phi]}{\delta A^i_b} + \partial_a \phi_n \frac{\delta S[A, \phi]}{\delta \phi_n} = 0 \quad (3.3)$$

Equations (3.2-3.3) show that the Hamilton-Jacobi equations for the model discussed here can be solved. If we now include the Hamiltonian constraint $H$ however, we have

$$\epsilon^{ijk} F_{ab} \frac{\delta S[A, \phi]}{\delta A^j_a} \frac{\delta S[A, \phi]}{\delta A^k_b} = 0, \quad (3.4)$$

$$\partial_a \phi_n \partial_b \phi_n \frac{\delta S[A, \phi]}{\delta A^i_a} \frac{\delta S[A, \phi]}{\delta A^i_b} = 0 \quad (3.5)$$

but

$$\frac{\delta S[A, \phi]}{\delta \phi_n} = \int_{\gamma[\phi_1, \phi_2]} ds \gamma^a(s) \frac{\delta \gamma^b}{\delta \phi_n}(s) Tr[F_{ab}(\gamma(s))U(s, s)]. \quad (3.6)$$

((3.3) follows since the $T^0$ is diffeomorphism invariant. For $H$, we note that each functional derivative ($\delta S/\delta A$) brings down a term proportional to the tangent vector to the loop, which by definition of the loop is orthogonal to $\partial_a \phi_n$. For the first term in $H$, the two tangent vectors contracted with $F_{ab}$ give zero).

The fact that the holonomy is a solution of the Hamilton-Jacobi equation associated with the Hamiltonian constraint without matter (3.4) has already been noted [12]. Here we see that when two scalar fields are included, $T^0$ is a solution of all the Hamilton-Jacobi equations if the momentum term $\Pi^2$ in $H$ is ignored. It may be possible to develop a perturbation series that allows the construction of $S$ to better approximations. A perturbation series in powers of the gradient of the scalar field for the ADM variables and scalar fields has been considered in [13] and the methods there may be useful in the present context as well.

For the purpose of addressing the integrability of the model discussed here, the solution of the Hamilton-Jacobi equations presented above are not sufficient since there are an insufficient number of integration momenta in the solution. A more general method of solving
the H-J equations for the diffeomorphism and Gauss constraints without matter has been discussed by Newmann and Rovelli [15], and in the following we note their results together with an extension to include the matter fields: the Gauss law equation (3.2) is solved by

\[ S[A; U^{(a)}, V^{(a)}] = \sum_{(a)} \int du^{(a)} \int dv^{(a)} T^0[A, U^{(a)}, V^{(a)}](u^{(a)}, v^{(a)}) \]  

(3.7)

where \( U^{(a)}, V^{(a)} \) are three pairs of scalar fields \( (a = 1, 2, 3) \) parametrizing the solution, and the \( T^0 \) is constructed as in (2.7), but with loops obtained from the intersection of the surfaces \( U^{(a)}(x) = u^{(a)}, V^{(a)}(x) = v^{(a)} \). The part of the diffeomorphism constraint involving the gravitational variables \( (A, E) \) can be written in an intuitively understood form involving the \( U^{(a)}(a) \) and \( V^{(a)}(a) \) and the coordinates \( Q_U^{(a)}, Q_V^{(a)} \) conjugate to the reduced momenta, which are determined by the three congruences associated with \( U^{(a)}, V^{(a)} \) [15]. Using this, the reduced diffeomorphism constraint becomes

\[ C_a = \Pi_n \partial_a \phi_n + Q_U^{(a)} \partial_a U^{(a)} + Q_V^{(a)} \partial_a V^{(a)} \]  

(3.8)

The H-J equation associated with this constraint is obtained by setting \( \Pi_n = (\delta S/\delta \phi_n) \) and \( Q_U^{(a)} = \delta S/\delta U^{(a)} , Q_V^{(a)} = \delta S/\delta V^{(a)} \). This has a solution

\[ S[U^{(a)}, V^{(a)}, \phi_n; P^{(a)}, p_n] = \sum_{(a)} \int d^3x \tilde{V} U^{(a)}(x) P^{(a)}(V^{(a)}(x)) \phi_n(x) p_n(V^{(a)}(x)) \]  

(3.9)

where the \( P^{(a)}, p_n \) are \( n+3 \) integration momenta parametrizing the solution and the density \( \tilde{V} = \epsilon^{abc} \partial_a V^{(1)} \partial_b V^{(2)} \partial_c V^{(3)} \). \( S \) is clearly diffeomorphism invariant since its integrand is a density constructed solely from the configuration variables \( U^{(a)}, V^{(a)}, \phi_n \). This solution is the generalization of the results of [15] to include matter. An extension of this result to general relativity would require rewriting the Hamiltonian constraint in terms of the reduced momenta \( P^{(a)}, p_n \) and their conjugate coordinates, and then solving the associated H-J equation. To address the question of integrability, such a solution should involve \( n+2 \) integration momenta.

For general relativity there is also an issue as to what phase space functionals should be called physical observables since the Hamiltonian constraint generates both time
reparametrizations and evolution of the phase space variables from one spatial surface to another. In the sense of physical observables as those that commute with all the constraints but $H$, the $T^N$ defined above form a kinematically gauge invariant set whose classical evolution via $H$ may be studied.

IV. QUANTIZATION

We can study the quantization of this model in the connection/scalar field representation or the loop representation. These are discussed and compared in this section.

(i) Configuration representation. For Dirac quantization, we can write down a diffeomorphism and gauge invariant wavefunction. This is the Wilson loop $\Psi[A, \phi_1, \phi_2](c_1, c_2) = T^0$. These Wilson loop states are therefore physical states of the quantum theory for the action $S$. These are a two parameter $(c_1, c_2)$ family of states and so most likely do not make up the full state space. (This point is discussed further below).

For gravity with scalar fields, $\Psi[A, \phi_1, \phi_2](c_1, c_2)$ is not a full solution to the Hamiltonian constraint. It is annihilated by all terms in the Hamiltonian constraint except for the momentum squared terms of the scalar field. (The reasons are the same as those discussed above for the Hamilton-Jacobi equations). Thus one may perhaps view these states as solutions to the entire set of Dirac quantization conditions in the approximation that the scalar field momenta are constants. Since the states $\Psi$ are diffeomorphism invariant in the $A, \phi$ representation one can attempt to complete the quantization in this representation by some type of perturbation expansion to incorporate the momenta of the scalar fields.

(ii) Loop representation. For pure gravity or the model given by (2.1), an alternative representation for the quantum theory can be obtained by converting the closed Poisson algebra of the (Gauss law invariant) loop observables into an operator algebra on functions of loops [3]. In the limit $\hbar \to 0$, this operator algebra reduces to the classical Poisson algebra. The motivation for this representation (for pure gravity) is that, unlike the configuration representation, it allows the solution of the diffeomorphism constraints in a natural way: The
diffeomorphism invariant information in a loop are its knot invariants and so the ‘quantum numbers’ labelling the physical states are the knot invariants.

In the present context with matter fields, although some solutions of the diffeomorphism constraint can be obtained in the configuration representation, one might attempt the same with the observables $T^N$, but there is a potential problem: If the scalar field momenta are included as part of the observables to be represented as operators in the quantum theory, then the Poisson algebra of the $T^N$ doesn’t close unless it is extended as discussed previously. Such an extension is inelegant but can be done, and one can then attempt to construct a loop representation analogous to the one for pure gravity. This is under study.

One can on the other hand proceed with a ‘reduced’ quantization of the model where the scalar field momenta are not represented as operators, in which case the same loop representation as that for gravity may be used. This is necessarily limiting since all the basic variables of the classical theory will not be realized as operators in the quantum theory. One can nonetheless see what can be learned of the quantum theory from this (restricted) quantization. (In this regard Van Hove’s result [17] is worth pointing out: It is not possible to convert all the fundamental variables (satisfying some closed Poisson algebra) into operators such that the correspondence is maintained between all the Poisson brackets and commutators). Proceeding in this manner would give a loop representation on which there would be no need to impose the diffeomorphism constraints since the observables are already invariant. The physical states would therefore no longer be labelled by knot classes, but still by the same two parameter set.

An alternative approach to the loop representation when there is matter is to not work with the $T^N$ made from matter loops, but rather to have auxiliary loops parametrizing the gravitational loop observables. This would lead to the usual loop space representation for the gravitational degrees of freedom, with the physical states labelled by knot invariants. The matter would be incorporated separately resulting in product states of the form $|\text{knotclass} > |\text{matter} >$.

Comparison. We have seen that there appear to be two different approaches to incorpo-
rating matter in the loop representation: (1) Work with a (suitably completed or reduced) set of diffeomorphism invariant matter-loop observables and find a representation of their Poisson algebra to obtain the quantum theory, or (2) treat loops as auxilliary and attempt to obtain suitably defined product states for the matter gravity variables. The latter would give a larger set of physical states because a general knot is not obtainable from the configurations of two scalar fields discussed here. This shows that the two parameter set of states given above are most likely not all the physical states. (In this regard it is worth emphasizing that in general, the chosen representation determines the size of the Dirac quantization state space, and in the absence of an innerproduct the sizes of the state spaces in different representations may not be the same \[16\]).

The representation of the gravitational part of the functionals \( T^0, Q, V, A \) as operators on the loop representation space has been studied in ref. \([7]\). Since the definitions of these observables here differ only in the replacement of auxiliary variables (loops and surfaces) by configurations of the two scalar fields, the gravitational part of the representation needs to be extended to include a representation for the scalar fields. One such extension has been discussed in \([8]\) and a similar procedure may be applied here. The essential results \([7]\) concerning the quantization of the spectra of \( V \) and \( A \) remains unaffected. The new features are that the scalar fields allow diffeomorphism invariant definitions for the areas of surfaces with boundary and the volumes of particular regions bounded by surfaces.

V. DISCUSSION

We have given a four dimensional generally covariant theory that has an infinite number of observables (which are the constants of motion). In particular since the Hamilton-Jacobi equations can be solved, the theory may be completely integrable if the functionals relating the new and old phase space variables can be inverted.

The Dirac quantization conditions for the theory can also be solved exactly to give a class of physical states in the coordinate representation - the physical states are traces of
holonomies, with the loops determined by scalar fields. It is possible to construct a reduced loop space representation for the observable algebra by choosing not to represent the scalar field momentum. On the other hand one can also attempt to complete the classical loop variables by extending the algebra of the $T^N$, and then seeking a loop space representation. The model also has a number of other observables which are known to have well-defined operator versions on the loop representation space $\mathcal{H}$.

The inclusion of the Hamiltonian constraint on the phase space of the model gives general relativity with scalar fields and so the above results partially carry over to this case. The observables are now gauge invariant only under transformations generated by the kinematical constraints, and one obtains approximate solutions of the Dirac quantization conditions and the additional Hamilton-Jacobi equation.

The role of the scalar fields is similar in a way to choosing coordinates in general relativity that are based on the presence of fluids whereby the fluid particles mark space points and their clocks specify the time foliation. Such ‘reference fluids’ [5] have been used recently to attempt to solve the problem of time in quantum gravity. In this approach coordinate conditions are chosen in the action via lagrange multipliers thereby breaking diffeomorphism invariance, and then the theory is re-parametrized to restore this invariance. The resulting theory gives an effective source term for the Einstein equations with the source determined by the coordinate conditions. In this spirit the scalar fields here may be viewed as the sources that specify matter loops on which the diffeomorphism invariant loop observables are based. The loop observables do not commute with the Hamiltonian constraint but we still have an interesting set of spatial diffeomorphism invariant variables that form a closed algebra whose dynamics may be worth studying. The physical picture seems rather interesting since as the scalar fields evolve one can envision the matter loops joining and breaking.

An extension of the results may be to see if diffeomorphism invariant loop observables can be found by coupling Yang-Mills and spinor fields to the model given here, and to general relativity. Such observables, with loops constructed out of the different matter fields, may provide one way to include general matter fields into the loop representation.
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