Static solutions of the spherically symmetric Vlasov-Einstein system

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Abstract
We consider the Vlasov-Einstein system in a spherically symmetric setting and prove the existence of static solutions which are asymptotically flat and have finite total mass and finite extension of the matter. Among these there are smooth, singularity-free solutions, which have a regular center and may have isotropic or anisotropic pressure, and also solutions, which have a Schwarzschild-singularity at the center. The paper is an extension of previous work, where only smooth, globally defined solutions with regular center and isotropic pressure were considered, cf. [8]

1 Introduction
It is well known that the only static, spherically symmetric vacuum solutions of Einstein’s field equations are the Schwarzschild solutions which possess a spacetime singularity (or are identically flat). In the present note we couple Einstein’s equations with the Vlasov or Liouville equation for a static, spherically symmetric distribution function \( f \) on phase space, describing an ensemble of identical particles such as stars in a galaxy, galaxies in a galaxy cluster etc.. This results in the following system of equations:

\[
\frac{v}{\sqrt{1 + v^2}} \cdot \partial_x f - \sqrt{1 + v^2} \frac{x}{r} \cdot \partial_v f = 0, \quad x, v \in \mathbb{R}^3, \quad r := |x|, \\
e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi r^2 \rho, \\
e^{-2\lambda}(2r\mu' + 1) - 1 = 8\pi r^2 p,
\]
where

\[ \rho(x) = \rho(r) := \int_{\mathbb{R}^3} \frac{1}{\sqrt{1 + v^2}} f(x, v) \, dv, \]

\[ p(x) = p(r) := \int_{\mathbb{R}^3} \left( \frac{x \cdot v}{r} \right)^2 f(x, v) \frac{dv}{\sqrt{1 + v^2}}. \]

Here the prime denotes derivative with respect to \( r \), and spherical symmetry of \( f \) means that \( f(Ax, Av) = f(x, v) \) for every orthogonal matrix \( A \) and \( x, v \in \mathbb{R}^3 \). If we let \( x = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) then the spacetime metric is given by

\[ ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]

Asymptotic flatness is guaranteed by the boundary condition

\[ \lim_{r \to \infty} \lambda(r) = \lim_{r \to \infty} \mu(r) = 0, \]

a regular center at \( r = 0 \) is guaranteed by the boundary condition

\[ \lambda(0) = 0. \]

We refer to [6] on the particular choice of coordinates on phase space leading to the above formulation of the system. It should be pointed out that the above equation for \( f \) is only equivalent to the Vlasov equation if \( f \) is spherically symmetric.

A brief overview of the literature related to the present investigation seems in order. In [6] the initial value problem for the corresponding time dependent system is investigated and the existence of singularity-free solutions is established for small initial data. In [7] it is shown that the classical Vlasov-Poisson system is the Newtonian limit of the Vlasov-Einstein system. The existence of globally defined, smooth, static solutions with a regular center, finite radius, and finite total mass is investigated in [8]; that investigation is restricted to solutions with isotropic pressure. For the Vlasov-Poisson system the existence of spherically symmetric steady states is established in [9], and we briefly recall the general approach followed there: Taking the distribution function \( f \) as a function of the local energy and the modulus of the angular momentum, which are conserved along characteristics, the problem is reduced to solving a nonlinear Poisson problem. For certain cases corresponding to the so-called polytropes it can then be shown that this leads to solutions with finite radius and finite total mass, cf. also [10]. Cylindrically symmetric, static solutions of the Vlasov-Poisson system are investigated in
A system in some sense intermediate between the Newtonian and the general-relativistic setting is the relativistic Vlasov-Poisson system, where similar results as for the nonrelativistic version hold, cf. [1]. In [11] it is shown that coupling Einstein’s equations with a Yang-Mills field can lead to static, singularity-free solutions with finite total mass. Static solutions of general-relativistic elasticity have been studied in [5]. For further references, especially on the astrophysical literature, we refer to [10].

The present investigation proceeds as follows: In the next section we give two conserved quantities, which correspond to the local energy and angular momentum in the Newtonian limit, and reduce the existence problem for static solutions of the Vlasov-Einstein system to solving a nonlinear system of ordinary differential equations for \( \lambda \) and \( \mu \). The fact that the distribution function \( f \) indeed has to be a function of the local energy and angular momentum—usually referred to as Jeans’ Theorem—is rigorously established for the Vlasov-Poisson system in [3] but not known in the present situation. In other words, it is not clear whether our ansatz for \( f \) covers all possible spherically symmetric, static solutions. The existence of solutions to the remaining system for \( \lambda \) and \( \mu \) is investigated in the third section. There we dispose of the restriction that \( f \) depends on the local energy only, which was used in [8] and lead to solutions with necessarily isotropic pressure. Nevertheless, the arguments used in the present, more general situation are considerably simpler than the ones used in [8]; a key element in the proof that the solutions exist globally in \( r \) is the Tolman-Oppenheimer-Volkov equation. In the fourth section we show that—for an appropriate ansatz for \( f \), corresponding to the polytropes in the classical case—the solutions obtained by our approach have regular center, finite mass, and finite radius. This is done by treating the Vlasov-Einstein system as a perturbation of the Vlasov-Poisson system and using the criteria for a finite radius from the classical Lane-Emden-Fowler equation. In the last section we prove the existence of solutions with a Schwarzschild-singularity at the center, surrounded by Vlasov-matter with finite radius and finite total mass. Surprisingly, the latter features of these solutions can be obtained easily and without the perturbation argument used in the nonsingular case mentioned above.
2 Conserved quantities and reduction of the problem

As explained in the introduction, a key point in our investigation is to reduce the full system to a nonlinear system of ordinary differential equations for $\lambda$ and $\mu$. This is achieved by the ansatz that the distribution function depends only on certain integrals of the characteristic system. In the coordinates used above, the characteristic system corresponding to the above equation for $f$ reads

$$\dot{x} = \frac{v}{\sqrt{1+v^2}},$$
$$\dot{v} = -\sqrt{1+v^2} \mu'(r) \frac{x}{r};$$

note that these equations are not the geodesic equations. One immediately checks that the quantities

$$E := e^{\mu(r)} \sqrt{1+v^2}, \quad F := x^2 v^2 - (x \cdot v)^2 = |x \times v|^2$$

are conserved along characteristics; in Sect. 4 the relation of $E$ to the classical local energy will become apparent, $F$ can be interpreted as the modulus of the angular momentum. If we take $f$ to be of the form

$$f(x,v) = \Phi(E,F)$$

for some function $\Phi$, the Vlasov equation is automatically satisfied. Inserting this into the definitions of $\rho$ and $p$ we obtain the following nonlinear system for $\lambda$ and $\mu$:

$$e^{-2\lambda} (2r \lambda' - 1) + 1 = 8\pi r^2 G_\Phi(r,\mu),$$
$$e^{-2\lambda} (2r \mu' + 1) - 1 = 8\pi r^2 H_\Phi(r,\mu).$$

Here

$$G_\Phi(r,u) = \frac{2\pi}{r^2} \int_1^\infty \int_0^{r^2(\epsilon^2-1)} \Phi(e^{\mu(r)} \epsilon, F) \frac{\epsilon^2}{\epsilon^2 - 1 - F/\epsilon^2} dF d\epsilon,$$
$$H_\Phi(r,u) = \frac{2\pi}{r^2} \int_1^\infty \int_0^{r^2(\epsilon^2-1)} \Phi(e^{\mu(r)} \epsilon, F) \sqrt{\epsilon^2 - 1 - F/\epsilon^2} dF d\epsilon$$

for $r,u > 0$ which follows by a simple transformation of variables in the integrals defining $\rho$ and $p$. These latter quantities are then given by

$$\rho(r) = G_\Phi(r,\mu(r)), \quad p(r) = H_\Phi(r,\mu(r)).$$
Once a solution of the system (2.1), (2.2) is known to exist.

Of course one has to make some assumptions on Φ in order to obtain existence results for the above system and to investigate the properties of its solutions. The assumption used below is certainly not the weakest possible, but on the one hand it is sufficiently general to encompass a large class of examples, leading to quite different classes of solutions, and on the other hand the main ideas of the proofs are not buried under technicalities. We require Φ to satisfy the following

**Assumption on Φ:**

\[ Φ(E, F) = φ(E)(F - F_0)_+, \quad E > 0, \quad F > 0, \]

where \( F_0 \geq 0 \), \( l > -\frac{1}{2} \), and \( φ \in L^∞([0, ∞[) \) is nonnegative with \( φ(E) = 0, \quad E > E_0, \) for some \( E_0 > 0 \).

Under this assumption the functions \( G_Φ \) and \( H_Φ \) are easily seen to take the form

\[ G_Φ(r, u) = c_l r^{2l} e^{-(2l+4)u} g_φ \left( e^u \sqrt{1 + F_0/r^2} \right), \quad (2.3) \]

\[ H_Φ(r, u) = \frac{c_l}{2l+3} r^{2l} e^{-(2l+4)u} h_φ \left( e^u \sqrt{1 + F_0/r^2} \right), \quad (2.4) \]

where

\[ g_φ(t) := \int_t^∞ φ(E) E^2 (E^2 - t^2)^{l+\frac{1}{2}} dE, \quad (2.5) \]

\[ h_φ(t) := \int_t^∞ φ(E) (E^2 - t^2)^{l+\frac{3}{2}} dE \quad (2.6) \]

for \( t > 0 \), and

\[ c_l := 2π \int_0^1 \frac{s^l}{\sqrt{1-s}} ds. \]

### 3 Existence of solutions

As a first step we show that the functions \( g_φ \) and \( h_φ \) defined by (2.3) and (2.6) are well behaved for functions φ as considered in the assumption above. More precisely:
**Lemma 3.1** Eqns. (2.3) and (2.6) define decreasing functions \(g_\phi, h_\phi \in C^1(\mathbb{R}^+)\) and

\[
h'_\phi(t) = -(2l+3) t \int_t^\infty \phi(E)(E^2 - t^2)^{l+1/2} dE, \quad t > 0.
\]

**Proof:** Obviously, the integrals defining \(g_\phi\) and \(h_\phi\) exist. Lebesgue’s dominated convergence theorem implies that the functions \(g_\phi\) and \(h_\phi\) are continuous. For \(t > 0\) and \(\Delta t > 0\) such that \(t - \Delta t > 0\) we have

\[
\frac{1}{\Delta t}(h_\phi(t-\Delta t) - h_\phi(t)) = \frac{1}{\Delta t} \int_{t-\Delta t}^t \phi(E) \left( E^2 - (t-\Delta t)^2 \right)^{l+3/2} dE + \int_t^\infty \phi(E) \frac{1}{\Delta t} \left( (E^2 - (t-\Delta t)^2)^{l+3/2} - (E^2 - t^2)^{l+3/2} \right) dE
\]

=: \(I_1 + I_2\).

Obviously, \(I_1 \to 0\) as \(\Delta t \to 0\). The term \(I_2\) has a limit as \(\Delta t \to 0\) by Lebesgue’s theorem. Thus, the function \(h_\phi\) is left-differentiable with

\[
\frac{d}{dt} h_\phi(t) = -(2l+3) t \int_t^\infty \phi(E)(E^2 - t^2)^{l+1/2} dE.
\]

Again by Lebesgue’s theorem this function is continuous, and a continuous and continuously left-differentiable function is continuously differentiable. Similarly, we see that

\[
\frac{d}{dt} g_\phi(t) = -(2l+1) t \int_t^\infty \phi(E)E^2(E^2 - t^2)^{l-1/2} dE,
\]

which is again continuous. The derivatives are negative, and thus \(g_\phi\) and \(h_\phi\) decrease. \(\square\)

The regularity of the functions \(g_\phi\) and \(h_\phi\) being established, we can now prove local existence and uniqueness of \(\lambda\) and \(\mu\). Note that in the last section we will need to pose initial data at some \(r_0 > 0\) which is why we include this case in the following results.

**Theorem 3.2** Let \(\Phi\) satisfy the general assumption and let \(G_\Phi\) and \(H_\Phi\) be defined by Eqns. (2.3), (2.4), (2.5), (2.6). Then for every \(r_0 \geq 0\) and \(\lambda_0, \mu_0 \in \mathbb{R}\) with \(\lambda_0 = 0\) if \(r_0 = 0\) there exists a unique maximal solution \(\lambda, \mu \in C^1([r_0, R[), \quad R > r_0\), of the Eqns. (2.4) and (2.5) with

\[
\lambda(r_0) = \lambda_0, \quad \mu(r_0) = \mu_0.
\]
Proof: Due to the regularity of the right hand sides in (2.1), (2.2)—cf. Lemma 3.1—the result is of course trivial in the case \( r_0 > 0 \). In the case \( r_0 = 0 \) one has to deal with the singularity of the Eqns. (2.1) and (2.2) at \( r = 0 \). Using the boundary condition at \( r = 0 \), Eqn. (2.1) can be integrated to give

\[
e^{-2\lambda(r)} = 1 - \frac{8\pi}{r} \int_0^r s^2 G_\Phi(s, \mu(s)) ds.
\]

(3.1)

If we insert this into Eqn. (2.2), we obtain an equation for \( \mu \) alone:

\[
\mu'(r) = \frac{4\pi}{1 - \frac{8\pi}{r} \int_0^r s^2 G_\Phi(s, \mu(s)) ds} \left( r H_\Phi(r, \mu(r)) + \frac{1}{r^2} \int_0^r s^2 G_\Phi(s, \mu(s)) ds \right).
\]

(3.2)

Integrating Eqn. (3.2) subject to the initial condition \( \mu(0) = \mu_0 \) we obtain the following fixed point problem for \( \mu \):

\[
\mu(r) = (T \mu)(r), \quad r \geq 0
\]

where

\[
(T \mu)(r) := \mu_0 + \int_0^r \frac{4\pi}{1 - \frac{8\pi}{s} \int_0^s \sigma^2 G_\Phi(\sigma, \mu(\sigma)) d\sigma} \left( s H_\Phi(s, \mu(s)) + \frac{1}{s^2} \int_0^s \sigma^2 G_\Phi(\sigma, \mu(\sigma)) d\sigma \right) ds.
\]

A lengthy but straightforward argument shows that \( T \) maps the set

\[
M := \left\{ \mu : [0, \delta] \rightarrow \mathbb{R} \mid \mu(0) = \mu_0, \ \mu_0 \leq \mu(r) \leq \mu_0 + 1, \right\}
\]

\[
\frac{8\pi}{r} \int_0^r s^2 G_\Phi(s, \mu(s)) ds \leq \frac{1}{2}, \quad r \in [0, \delta] \}
\]

into itself and acts as a contraction with respect to the norm \( \| \cdot \|_\infty \), provided \( \delta > 0 \) is chosen small enough. This gives the existence of a solution of Eqn. (3.2) on the interval \([0, \delta] \). On this interval we define \( \lambda \) by Eqn. (3.1) and obtain a local solution \( \lambda, \mu \in C^1([0, \delta]) \) of (2.1), (2.2). Obviously, the boundary conditions at \( r = 0 \) are satisfied, and the solution is unique. \( \square \)

In order to show that the above solutions actually extend to \( r = \infty \) we need a relation which is known as the Tolman-Oppenheimer-Volkov equation in the context of the general relativistic description of spherically symmetric fluid balls, cf. [12, Eqn. 6.2.19].
Lemma 3.3 Define

\[ p_T(r) := \frac{1}{2} \int_{\mathbb{R}^3} \frac{|x \times v|^2}{r} f(x, v) \frac{dv}{\sqrt{1 + v^2}} \]

and let \( \lambda, \mu \) be a solution of the system (2.1), (2.2) on the interval \([r_0, R]\). Then \( p_T \) has the form

\[ p_T(r) = (l + 1)p(r) + \frac{c_l}{2} F_0 r^{2l-2} e^{-(2l+2)\mu(r)} k_\phi \left( e^{\mu(r)} \sqrt{1 + F_0/r^2} \right), \]

where

\[ k_\phi(t) := \int_t^\infty \phi(E) (E^2 - t^2)^{l+1/2} dE, \quad t > 0, \]

defines a decreasing \( C^1 \)-function, and

\[ p'(r) = -\mu'(r) (p(r) + \rho(r)) - \frac{2}{r} (p(r) - p_T(r)), \quad r \in [r_0, R]. \]

Proof: The formula for \( p_T \) is obtained by a transformation of variables, and the regularity and monotonicity of \( k_\phi \) follow as in Lemma 3.1. Using (2.4), (2.6), and Lemma 3.1 one obtains the relation

\[ p'(r) = \frac{2l}{r} p(r) - (2l + 4) \mu'(r) p(r) \]

\[ -c_l \left( 1 + F_0/r^2 \right) r^{2l} e^{-(2l+2)\mu(r)} k_\phi \left( e^{\mu(r)} \sqrt{1 + F_0/r^2} \right) \mu'(r) \]

\[ + c_l^2 r^{2l-3} F_0 e^{-(2l+2)\mu(r)} k_\phi \left( e^{\mu(r)} \sqrt{1 + F_0/r^2} \right) \mu'(r) \]

which, after some further calculations, leads to the Tolman-Oppenheimer-Volkov equation. □

Remark: The quantity \( p_T \) is the tangential pressure generated by the phase space density \( f \), as opposed to the radial pressure \( p \). Note that if \( F_0 = 0 \) then \( p \) and \( p_T \) differ only by a multiplicative constant, and if \( F_0 = l = 0 \), i.e. the phase space distribution is independent of the angular momentum, then \( p = p_T \), i.e. the solution has isotropic pressure.

Theorem 3.4 Let \( \Phi \) satisfy the general assumption and let \( G_\Phi \) and \( H_\Phi \) be defined by Eqns. (2.3), (2.4), (2.5), (2.6). Then for every \( r_0 \geq 0 \) and \( \lambda_0 \geq 0, \mu_0 \in \mathbb{R} \) with \( \lambda_0 = 0 \) if \( r_0 = 0 \) there exists a unique solution \( \lambda, \mu \in C^1([r_0, \infty[) \) of the Eqns. (2.7) and (2.8) with

\[ \lambda(r_0) = \lambda_0, \mu(r_0) = \mu_0. \]
Proof: Let $\lambda, \mu \in C^1([r_0, R])$ be the maximal solution to (2.1), (2.2) which exists by Thm. 3.2. We have the following equations on $[r_0, R]$:

$$e^{-2\lambda}(2r\lambda'-1) + 1 = 8\pi r^2 \rho(r), \quad (3.3)$$

$$e^{-2\mu}(2r\mu'+1) - 1 = 8\pi r^2 p(r), \quad (3.4)$$

$$p'(r) = -\mu'(r)(p(r) + \rho(r)) - \frac{2}{r}(p(r) - p_T(r)). \quad (3.5)$$

Note that—as opposed to the case considered in [8]—the functions $\rho$ and $p$ are not necessarily strictly decreasing on their support so that we cannot write $\rho$ as a function of $p$, and we cannot use the analysis in [9] directly. Nevertheless, the proof given here is somewhat simpler than the one given in [8]. If we integrate Eqn. (3.3) we obtain

$$e^{-2\lambda(r)} = 1 - \frac{8\pi}{r} \int_{r_0}^r s^2 \rho(s) \, ds - \frac{1}{r}r_0(1 - e^{-2\lambda_0}). \quad (3.6)$$

Inserting this into Eqn. (3.4) yields

$$\mu'(r) = 4\pi e^{2\lambda(r)}(p(r) + w(r)) \quad (3.7)$$

where

$$w(r) := \frac{1}{r^3} \left( \int_{r_0}^r s^2 \rho(s) \, ds + \frac{r_0(1 - e^{-2\lambda_0})}{8\pi} \right). \quad (3.8)$$

Finally, by adding Eqns. (3.3) and (3.4) we have

$$(\lambda' + \mu')(r) = 4\pi r e^{2\lambda(r)}(p(r) + \rho(r)). \quad (3.9)$$

We now wish to establish a differential inequality for the quantity $e^{\lambda+\mu}(p+w)$, which will allow us to bound $\lambda$ and $\mu$ and conclude that $R=\infty$. Using Eqns. (3.3), (3.5), (3.7), and (3.8) we see that

$$\left(e^{\lambda+\mu}(p+w)\right)'(r) = e^{\lambda+\mu} \left( -\frac{2p}{r} + \frac{2p_T}{r} + 3\frac{w}{r} + \rho \right) \leq e^{\lambda+\mu} \left( \frac{2p_T}{r} + \frac{\rho}{r} \right);$$

note that the terms which were dropped are indeed negative by the assumption $\lambda_0 \geq 0$. Now assume that $R<\infty$. Without loss of generality, we may assume that there exists $r_1 \in ]r_0, R]$ and $C > 0$ such that $w(r) \geq C$ for
$r \in [r_1, R[$, since otherwise the solution is trivial. The fact that $\mu$ is increasing and the functions $g_\phi$ and $k_\phi$ are decreasing implies that $\rho$ and $p_T$ and thus also $p/r$ and $p_T/r$ are bounded on the interval $[r_1, R[$. Thus, we can continue the above estimate to obtain
\[
\left(e^{\lambda+\mu}(p+w)\right)'(r) \leq C e^{\lambda+\mu} \leq Ce^{\lambda+\mu} (p(r)+w(r)).
\]
This implies that
\[
e^{\lambda(r)+\mu(r)} \leq \frac{C}{p(r)+w(r)} \leq C, \quad r \in [r_1, R[.
\]
Since $\lambda(r) \geq 0$ by (3.6) and $\mu(r) \geq \mu_0$ by monotonicity, this implies that $\lambda$ and $\mu$ are bounded on $[r_0, R]$ which is a contradiction to the maximality of $R$. Thus $R = \infty$, and the proof is complete. $\square$

**Remarks:**

1. If $r_0 > 0$ then $\lambda, \mu \in C^2([r_0, \infty[)$ and $\rho, p \in C^1([r_0, \infty[)$. 

2. If $r_0 = F_0 = 0$ then $\lambda, \mu \in C^1([0, \infty[) \cap C^2([0, \infty[)$ with $\lambda'(0) = \mu'(0) = 0$ and $\rho, p \in C^1([0, \infty[)$. If in addition $l = 0$ or $l > 1/2$ then $\lambda, \mu \in C^2([0, \infty[) \cap C^2(\mathbb{R}^3)$ and $\rho, p \in C^1([0, \infty[) \cap C^1(\mathbb{R}^3)$ with $\rho'(0) = p'(0) = 0$ where functions in $r$ are identified with the corresponding radially symmetric functions on $\mathbb{R}^3$. Thus we see that in the case $r_0 = 0$ the solutions are as regular at the center as anywhere else if various parameters are chosen appropriately.

3. The phase space density $f := \Phi(E, F)$ is a solution of the Vlasov equation in the sense that it is constant along characteristics. If $\Phi$ is continuous or continuously differentiable then in addition $f$ has the same regularity, and in the latter case satisfies Vlasov’s equation classically.

4 **Singularity-free solutions with finite mass and finite radius**

In this section we are interested in smooth, singularity-free solutions with with a regular center. Thus, we consider the boundary condition $\lambda(0) = 0$. Also, we set $F_0 = 0$ in this section, the case $F_0 > 0$ will play its role in the next section.
In order to decide whether a solution obtained in the previous section has finite total mass or whether \( \rho(r) \) vanishes for \( r \) large, one has to have rather detailed information on the behaviour of the function \( \mu \). Due to the complexity of Eqn. (3.2) we have not been able to decide these questions directly, even for simple examples of \( \Phi \) and without dependence on \( F \). However, it is possible to show that the solutions obtained above converge to solutions of the corresponding Newtonian problem as the speed of light tends to infinity. It is then possible to use the results on finite mass and finite radius which are known in the Newtonian case for the so-called polytropes to obtain solutions with the same properties for the Vlasov-Einstein system.

To carry out this program we introduce the parameter \( \gamma := \frac{1}{c^2} \) where \( c \) denotes the speed of light, define \( \nu := \frac{1}{\gamma} \mu \), and recall from [7] that the Vlasov-Einstein system with \( \gamma \) inserted in the appropriate places reads

\[
\frac{v}{\sqrt{1+\gamma v^2}} \partial_x f - \sqrt{1+\gamma v^2} \nu \frac{x}{r} \partial_v f = 0,
\]

\[
e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi \gamma r^2 \rho,
\]

\[
e^{-2\lambda}(2r\nu' + 1/\gamma) - 1/\gamma = 8\pi \gamma r^2 \rho,
\]

where

\[
\rho(x) := \int_{\mathbb{R}^3} \sqrt{1+\gamma v^2} f(x,v) dv,
\]

\[
p(x) := \int_{\mathbb{R}^3} \left( \frac{x \cdot v}{r} \right)^2 f(x,v) \frac{dv}{\sqrt{1+\gamma v^2}}.
\]

The conserved quantity \( E \) now becomes

\[
\sqrt{1+\gamma v^2} e^{\gamma \nu(x)},
\]

and \( F \) remains unchanged. In order to obtain the correct limit as \( \gamma \to 0 \), we have to rewrite our ansatz for the distribution function in the following form:

\[
f(x,v) = \phi \left( \frac{1}{\gamma} \sqrt{1+\gamma v^2} e^{\gamma \nu(r)} - \frac{1}{\gamma} \right) F^l. \tag{4.1}
\]

As above the Vlasov-Einstein system can then be reduced to a single equation for \( \nu \) alone, namely

\[
\nu'(r) = \frac{4\pi}{1 - \frac{8\pi}{r} \gamma \int_0^r s^{2l+2} g_{\phi,\gamma}(\nu(s)) ds} \left( \gamma r^{2l+1} h_{\phi,\gamma}(\nu(r)) + \frac{1}{r^2} \int_0^r s^{2l+2} g_{\phi,\gamma}(\nu(s)) ds \right), \tag{4.2}
\]
where
\[ g_{\phi, \gamma}(u) := c_t e^{-2(l+2)\gamma u} \int_{\gamma u-1}^{\infty} \phi(E) \left( \frac{1}{\gamma} (1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma u} \right)^{l+1/2} dE, \]
\[ h_{\phi, \gamma}(u) := \frac{c_t}{2l+3} e^{-2(l+2)\gamma u} \int_{\gamma u-1}^{\infty} \phi(E) \left( \frac{1}{\gamma} (1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma u} \right)^{l+3/2} dE, \]
so that
\[ \rho(r) = r^{2l} g_{\phi, \gamma}(\nu(r)), \quad p(r) = r^{2l} h_{\phi, \gamma}(\nu(r)). \]

For the Newtonian case the corresponding ansatz
\[ f(x, v) = \phi \left( \frac{v^2}{2} + U(r) \right) \]
reduces the Vlasov-Poisson system to the equation
\[ U'(r) = \frac{4\pi}{r^2} \int_{0}^{r} s^{2l+2} g_0(U(s)) ds, \]
where
\[ g_0(u) := c_t \int_{u}^{\infty} \phi(E) (2(E - u))^{l+1/2} dE. \]

Assume that \( \phi \in L^\infty(\mathbb{R}) \) and \( E_0 = 0 \), and fix \( \nu_0 < 0 \). Clearly, the results of the previous section apply so that for every \( \gamma > 0 \) Eqn. \( (4.2) \) has a unique, nontrivial, global solution with \( \nu(0) = \nu_0 \). Let \( U \in C^1([0, \infty[) \) be the global solution of \( (4.6) \) with \( U(0) = \nu_0 \). We shall prove that \( \nu \) converges to \( U \) as \( \gamma \to 0 \), more precisely:

**Lemma 4.1** For every \( R > 0 \) there exist constants \( C > 0 \) and \( \gamma_0 > 0 \) such that for every \( \gamma \in [0, \gamma_0] \) the solution \( \nu \) of \( (4.2) \) with \( \nu(0) = \nu_0 \) satisfies the estimate
\[ |\nu(r) - U(r)| \leq C \gamma \min(l+1/2, 1), \quad r \in [0, R], \]
where \( U \) is the solution of \( (4.6) \) with \( U(0) = \nu_0 \).

**Proof:** We have the estimate
\[ |\nu'(r) - U'(r)| \leq I_1 + I_2 + I_3 + I_4, \]

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where

\[ I_1 := \frac{4\pi}{1 - \frac{8\pi}{r^2}} \int_0^r s^{2l+2} g_{\phi,\gamma}(\nu(s)) \, ds \gamma^{2l+1} h_{\phi,\gamma}(\nu(r)), \]

\[ I_2 := \left| \frac{1}{1 - \frac{8\pi}{r^2}} \right| \int_0^r s^{2l+2} g_{\phi,\gamma}(\nu(s)) \, ds - 1 \left| \frac{4\pi}{r^2} \int_0^r s^{2l+2} g_{\phi,\gamma}(\nu(s)) \, ds \right|, \]

\[ I_3 := \frac{4\pi}{r^2} \int_0^r s^{2l+2} |g_{\phi,\gamma}(\nu(s)) - g_0(\nu(s))| \, ds, \]

\[ I_4 := \frac{4\pi}{r^2} \int_0^r s^{2l+2} |g_0(\nu(s)) - g_0(U(s))| \, ds. \]

Since for \( r \geq 0 \) we have \( \nu(r) \geq \nu_0 \) and since \( \phi \) is bounded with \( \phi(E) = 0 \) for \( E > 0 \), it is easily seen that

\[ g_{\phi,\gamma}(\nu(r)) \leq C, \ h_{\phi,\gamma}(\nu(r)) \leq C; \ r \geq 0, \ \gamma \in [0,1], \]

where the constant \( C \) depends only on \( \phi \) and \( \nu_0 \). Thus, for \( R > 0 \) fixed and \( r \in [0,R] \) we obtain the estimate

\[ I_1 \leq \frac{1}{1 - CR^{2l+2} \gamma_0} CR^{2l+1} \gamma \leq C \gamma \]

where \( \gamma \in [0,\gamma_0] \), and \( \gamma_0 \in [0,1] \) is chosen such that \( 1 - CR^{2l+2} \gamma_0 > 0 \). Similarly,

\[ I_2 \leq C \gamma \]

for \( \gamma \in [0,\gamma_0] \). Next we estimate the difference \( |g_{\gamma,\phi}(\nu(r)) - g_0(\nu(r))| \). We can restrict ourselves to the case \( \nu(r) < 0 \) since otherwise this difference is zero. Thus,

\[ e^{\gamma \nu(r)} - \frac{1}{\gamma} \geq \nu(r), \]

which implies that we can split the difference in the following way:

\[ |g_{\gamma,\phi}(\nu(r)) - g_0(\nu(r))| \leq C(J_1 + J_2 + J_3), \]

where

\[ J_1 := |e^{-2(l+2)\gamma \nu} - 1| \int_{\nu_{\nu-1}}^\infty \phi(E)(1 + \gamma E)^2 \left( \frac{1}{\gamma}(1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma \nu} \right)^{l+1/2} dE, \]

\[ J_2 := \int_{\nu_{\nu-1}}^{e^{\gamma \nu - 1}} \phi(E)(2(2 - \nu))^{l+1/2} dE, \]
\[ J_3 := \int_{e^{\gamma \nu - \frac{1}{2}}}^{\infty} \phi(E) \left| (1 + \gamma E)^2 \left( \frac{1}{\gamma} (1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma \nu} \right)^{t+1/2} - (2(E - \nu))^{t+1/2} \right| dE. \]

Now
\[ J_1 \leq C \gamma \frac{e^{-2(l+2)\gamma \nu(r)} - 1}{\gamma} \leq C \gamma, \]
\[ J_2 \leq C \left( \frac{e^{\gamma \nu(r)} - 1}{\gamma} - \nu(r) \right) \leq C \gamma, \]

and
\[ (1 + \gamma E)^2 \left( \frac{1}{\gamma} (1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma \nu} \right)^{t+1/2} - (2(E - \nu))^{t+1/2} \]
\[ \leq C \left| (1 + \gamma E)^2 - 1 \right| + C \left| (1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma \nu} \right|^{t+1/2} - (2(E - \nu))^{t+1/2} \]
\[ \leq C \left| (1 + \gamma E)^2 - 1 \right| + C \left| (1 + \gamma E)^2 - \frac{1}{\gamma} e^{2\gamma \nu} \right|^{t+1/2} - (2(E - \nu))^{t+1/2} \]
\[ \leq C \gamma + C \left| \frac{1 + 2\gamma \nu(r) - e^{2\gamma \nu(r)}}{\gamma} + \gamma E^2 \right|^{\min\{t+1/2,1\}} \]
\[ \leq C \gamma + C \gamma^{\min\{t+1/2,1\}}. \]

Finally, using the method of Lemma 3.1 it is easily seen that \( g_0 \in C^1(\mathbb{R}) \) and \( |g_0'(u)| \leq C \) for \( u \geq \nu_0 \). Thus
\[ I_4 \leq \frac{C}{r^2} \int_0^r s^{2l+2} |\nu(s) - U(s)| ds. \]

If we combine all the above estimates we see that
\[ |\nu'(r) - U'(r)| \leq C \gamma^{\min\{t+1/2,1\}} + C \int_0^r s^{2l+2} |\nu(s) - U(s)| ds \]
for \( r \in [0,R] \) and \( \gamma \in [0,\gamma_0] \). The assertion of the lemma now follows by the usual Gronwall argument. \( \Box \)

In order to exploit the above result, we have to restrict our investigation to such cases where finiteness of the radius and total mass is known in the Newtonian situation. Thus we consider
\[ \Phi(E, F) := (-E)^{\frac{4}{3}} F^l, E \in \mathbb{R}, \ F > 0, \quad (4.7) \]
where \(k \geq 0\) and \(l > -\frac{1}{2}\) are such that \(k < 3l + 7/2\). Then we know from [3, 5.4] that the Newtonian potential \(U\) has a zero, which is also the radius where the density vanishes. Since by the assumption \(\nu_0 < 0\) the solution is nontrivial, \(U\) has to strictly increase so that there exists \(R > 0\) such that \(U(R) > 0\). The above lemma then tells us that for all \(\gamma > 0\) sufficiently small, \(\nu(R) > 0\) which by the definition of \(\Phi\) and Eqn. (4.5) implies that \(\rho(r) = p(r) = 0\) for \(r \geq R\). Thus we have proved the following theorem:

**Theorem 4.2** Let \(\Phi\) be defined by Eqn. (4.7) and take \(\nu_0 < 0\). Then for all \(\gamma > 0\) sufficiently small the corresponding solution \(\nu\) has a zero, the density \(\rho\) and radial pressure \(p\) as defined by Eqns. (4.3), (4.3), and (4.4) have finite support, and

\[
0 < 4\pi \int_0^\infty s^2 \rho(s) \, ds < \infty
\]

i.e. the solution is nontrivial with finite total mass.

**Remarks:**

1. In this section we have obtained static solutions of the spherically symmetric Vlasov-Einstein system with finite radius and finite mass, provided \(\gamma > 0\) is sufficiently small. However, if \(f, \lambda, \nu\) is such a solution then \(\gamma^{-3/2} f(\gamma^{-1/2}, \gamma^{-1/2}), \lambda(\gamma^{-1/2}), \gamma \nu(\gamma^{-1/2})\) is a solution of the system with \(\gamma = 1\), which again has finite radius and finite total mass.

2. If \(\Phi\) is defined by Eqn. (4.7) then at least \(f \in C^1(\text{supp} f)\), and \(f \in C(\mathbb{R}^6)\) or \(f \in C^1(\mathbb{R}^6)\) if \(k > 0, l \geq 0\) or \(k > 1, l \geq 1\) respectively. The Vlasov equation then holds classically on \(\text{supp} f\) or on \(\mathbb{R}^6\) respectively.

3. The finiteness of the total mass implies that \(\lambda(r) \to 0\) as \(r \to 0\) and \(\nu\) and \(\mu\) have a finite limit as \(r \to \infty\). This means that the spacetime is asymptotically flat; the fact that the limit of \(\mu\) is not zero only corresponds to a rescaling of the time coordinate \(t\).

4. Our solutions are not only global in the coordinates which we used, but are singularity-free in the sense that the corresponding spacetime is timelike and null geodesically complete.
5 Solutions with a Schwarzschild singularity at the center

In this last section we take \( \Phi \) as in Sect. 1, with \( F_0 > 0, E_0 = 1 \), and \( \phi(E) > 0 \) for \( E < 1 \). This implies that

\[
G_\Phi(r,u) = H_\Phi(r,u) = 0 \quad \text{if} \quad e^u \sqrt{1 + F_0/r^2} \geq 1,
\]
\[
G_\Phi(r,u), H_\Phi(r,u) > 0 \quad \text{if} \quad e^u \sqrt{1 + F_0/r^2} < 1.
\]

Consider the field equations (2.1), (2.2). It is well known that in the vacuum case, i.e. if the right hand sides are identically zero, the asymptotically flat solutions are given by

\[
e^{2\mu(r)} = 1 - \frac{2M_0}{r}, \quad e^{2\lambda(r)} = \left(1 - \frac{2M_0}{r}\right)^{-1}, \quad r > 2M_0,
\]

the so-called Schwarzschild metric; \( M_0 \geq 0 \). This metric is investigated in probably every textbook on general relativity, it represents a spacetime singularity which is hidden inside of an event horizon, a black hole. In passing we mention that the apparent singularity at \( r = 2M_0 \) is only a coordinate singularity, i.e. in other coordinates the spacetime can be extended beyond this radius, which is also called Schwarzschild radius, cf. [12, Ch. 6]. In the present section we wish to construct static solutions of the spherically symmetric Vlasov-Einstein system which have such a black hole at the center.

Let us first see if or where the Schwarzschild metric solves our system (2.1), (2.2). This is obviously the case if

\[
\sqrt{1 - \frac{2M_0}{r}} \sqrt{1 + \frac{F_0}{r^2}} \geq 1,
\]

i.e. for \( r \in [r_-, r_+] \), where

\[
r_\pm := \frac{F_0 \pm \sqrt{F_0^2 - 16M_0^2 F_0}}{4F_0}.
\]

If we take \( F_0 > 16M_0^2 > 0 \) we obtain \( 2M_0 < r_- < r_+ \), and we may set

\[
e^{2\mu(r)} = 1 - \frac{2M_0}{r}, \quad e^{2\lambda(r)} = \left(1 - \frac{2M_0}{r}\right)^{-1}
\]
and

$$\rho(r) = p(r) = f(x,v) = 0$$

for $$2M_0 < r \leq r_+.$$ At $$r_0 := r_+$$ we now pose initial conditions

$$\mu(r_0) = \mu_0 := \sqrt{1 - \frac{2M_0}{r_0}}, \lambda(r_0) = \lambda_0 := \sqrt{1 - \frac{2M_0}{r_0}}^{-1},$$

and solve (2.1), (2.2) to the right of $$r_0$$ according to Thm. 3.4; note that $$\lambda_0 > 0.$$ In this way we obviously obtain a static solution of the spherically symmetric Vlasov-Einstein system, which coincides with the Schwarzschild solution for $$r \leq r_0.$$ The phase space density $$f$$ is no longer given by one function of $$E$$ and $$F$$ because the vacuum solution is not consistent with our ansatz $$f = \Phi(E,F)$$ in the region $$]2M_0, r_-[,$$ but $$f$$ still solves the Vlasov equation everywhere in the region $$]2M_0, \infty[$$ since no characteristics which carry mass can cross the region $$]r_-, r_0[.$$

What remains to be seen is that our solution is not vacuum everywhere for $$r > 2M_0,$$ i.e. we really get Vlasov matter for $$r > r_0,$$ and that this matter outside $$r_0$$ again has finite radius and finite total mass. First note that

$$\left( e^{\mu(r)} \sqrt{1 + \frac{F_0}{r^2}} \right)' |_{r=r_0} < 0$$

which implies that $$\rho(r) > 0$$ and $$p(r) > 0$$ in a right neighborhood of $$r_0.$$ To show that for large enough values of $$r$$ the density vanishes again, it is obviously sufficient to show that $$\lim_{r \to \infty} \mu(r) > 0.$$ To see the latter we first recall that by integrating Eqn. (2.1) we obtain

$$e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r} - \frac{1}{r} r_0(1 - e^{-2\lambda_0}) = 1 - \frac{2M(r)}{r}, \quad r > 2M_0,$$

where

$$M(r) := M_0 + m(r) := M_0 + 4\pi \int_{r_0}^{r} s^2 \rho(s) ds.$$

Inserting this into Eqn. (2.2) we obtain

$$\mu'(r) = \frac{1}{1 - \frac{2M(r)}{r}} \left( \frac{M(r)}{r^2} + 4\pi r p(r) \right)$$

$$\geq \frac{M_0}{r(r - 2M_0)} + \frac{m(r)}{r^2},$$

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and integrating this yields
\[
\mu(r) \geq \mu_0 + M_0 \int_{r_0}^{r} \frac{ds}{s(s - 2M_0)} + \int_{r_0}^{r} \frac{m(s)}{s^2} ds
\]
\[
= \mu_0 + \ln \sqrt{1 - \frac{2M_0}{r}} - \ln \sqrt{1 - \frac{2M_0}{r_0}} + \int_{r_0}^{r} \frac{m(s)}{s^2} ds
\]
\[
= \ln \sqrt{1 - \frac{2M_0}{r}} + \int_{r_0}^{r} \frac{m(s)}{s^2} ds.
\]
Therefore
\[
\lim_{r \to \infty} \mu(r) \geq \int_{r_0}^{\infty} \frac{m(s)}{s^2} ds > 0,
\]
which proves that the density vanishes for \( r \) large enough. Note also that the inner boundary of the Vlasov matter satisfies the estimate
\[
r_0 > \frac{F_0}{4M_0} > 4M_0.
\]

We collect these results in the following theorem:

**Theorem 5.1** Let \( \Phi \) satisfy the general assumption in Sect. 1 with \( E_0 = 1 \) and \( \phi(E) > 0 \) for \( E > 1 \), and let \( F_0 > 16M_0^2 > 0 \). Then there exists a static solution \((f, \lambda, \mu)\) of the spherically symmetric Vlasov-Einstein system such that
\[
e^{2\mu (r)} = 1 - \frac{2M_0}{r}, \quad e^{2\lambda (r)} = \left( 1 - \frac{2M_0}{r} \right)^{-1}, \quad 2M_0 < r < r_0,
\]
\[
\rho (r) = p (r) = f(x,v) = 0, \quad 2M_0 < r < r_0 \text{ or } r > R,
\]
and
\[
0 < 4\pi \int_{r_0}^{R} \rho (r) dr < \infty,
\]
where
\[
r_0 := \frac{F_0 + \sqrt{F_0^2 - 16M_0^2 F_0}}{4F_0} > 4M_0
\]
and \( R > r_0 \). Furthermore, \( \lambda, \mu \in C^2([2M_0, \infty]), \rho, p \in C^1([2M_0, \infty]) \), and the spacetime in asymptotically flat.

**Remarks:**
1. As pointed out above, the phase space density $f$ is not given as a function of $E$ and $F$ globally for $r > 2M_0$.

2. If we start at the center with a smooth static solution as obtained in Sect. 4 instead of the Schwarzschild singularity we can use the method of the present section to obtain static solutions of the following kind: For $0 \leq r \leq r_0$ the nontrivial matter distribution is given by $f(x,v) = \Phi_0(E)$, for $r_0 < r < r_1$ we have vacuum, for $r_1 \leq r \leq r_2$ the matter distribution is again nonzero and given by $f(x,v) = \Phi_1(E,F)$, and this procedure can be continued. The resulting solution is smooth, geodesically complete, asymptotically flat, with finite mass and finite radius, and consists of rings of Vlasov matter separated by vacuum. Contrary to our expectation it is not clear that such a solution violates Jeans’ Theorem, which in spite of 1) might still be valid for smooth solutions. Note however that as the radius increases, the pressure can change from isotropic to unisotropic.

3. In the Newtonian case one can construct analogous solutions, where the role of the Schwarzschild singularity is played by a point mass $M_0$ situated at the center $r = 0$. It then turns out that the resulting solution does not violate Jeans’ Theorem—in the above notation, $r_+ = 0$ in the Newtonian case. Thus one can at least say that the range of validity of Jeans’ Theorem is larger in the classical case than in the general relativistic. One can also construct solutions of the type described in 2) for the Newtonian case.

References


