Non-Perturbative Canonical Quantization of Minisuperspace Models:  
Bianchi Types I and II

Nenad Manojlović

and

Guillermo A. Mena Marugán *

Physics Department, Syracuse University,  
Syracuse, NY 13244-1130, USA.

Abstract

We carry out the quantization of the full type I and II Bianchi models following the non-perturbative canonical quantization program. These homogeneous minisuperspaces are completely soluble, i.e., it is possible to obtain the general solution to their classical equations of motion in an explicit form. We determine the sectors of solutions that correspond to different spacetime geometries, and prove that the parameters employed to describe the different physical solutions define a good set of coordinates in the phase space of these models. Performing a transformation from the Ashtekar variables to this set of phase space coordinates, we endow the reduced phase space of each of these systems with a symplectic structure. The symplectic forms obtained for the type I and II Bianchi models are then identified as those of the cotangent bundles over \( \mathcal{L}^+_{(+,+)} \times S^2 \times S^1 \) (modulo some identification of points) and \( \mathcal{L}^+_{(+,+)} \times S^1 \), respectively, with \( \mathcal{L}^+_{(+,+)} \) the positive quadrant of the future light-cone. We construct a closed \(*\)-algebra of Dirac observables in each of these reduced phase spaces, and complete the quantization program by finding unitary irreducible representations of these algebras. The real Dirac observables are represented in this way by self-adjoint operators, and the spaces of quantum physical states are provided with a Hilbert structure.

* On leave from Instituto de Matemáticas y Física Fundamental, C.S.I.C., Serrano 121, 28006 Madrid, Spain.
I. Introduction

Quantization of gravity is one of the main obstacles that Modern Physics has to face in order to obtain an unified treatment of all physical interactions. To achieve this goal, a new and promising approach, the non-perturbative canonical quantization program, has been developed systematically over the last years [1-3]. Even though this approach has succeeded in solving a variety of physical problems [3], the implementation of this quantization program for the full theory of gravity remains incomplete.

To apply the canonical quantization program to a given theory, one first selects an over-complete set of complex classical variables in the phase space of the system that is closed under the Poisson-brackets structure [4]. This set is promoted to an abstract algebra of quantum elementary operators, with the complex conjugation relations between classical variables translated into relations in this algebra. The next step consists of choosing a complex vector space, and finding on it a representation of the algebra of elementary operators. The kernel of all the operators that represent the constraints of the system provides us with the space of quantum solutions to all physical constraints in the considered representation, i.e., with the space of quantum states. At this stage of the quantization program, one should determine a set of “real” Dirac observables for the theory, that is, a set of operators which correspond to real classical variables and such that they commute weakly with all the quantum constraints [4]. If this set is sufficiently large, one can fix uniquely the inner product in the quantum physical space by requiring that the “real” Dirac observables are promoted to self-adjoint operators (reality conditions) [2,3,5]. The space of quantum states is endowed in this way with a Hilbert structure. Finally, to extract predictions from the quantum theory so constructed, one has to supply the obtained mathematical framework with a physical interpretation.

Owing to the great complexity of the general theory of gravitation, both the complete space of physical states and a satisfactory set of Dirac observables for gravity are still to be determined in the new variables formalism. The analysis and quantization of minisuperspace gravitational models, following the non-perturbative canonical program, can be at this point a helpful way to develop some insight into the kind of problems, methods, and techniques that are involved in the quantization of the full
theory of gravity. In addition to the lessons one can learn by applying the quantization program to simple models, it is clear that obtaining consistent quantum theories for minisuperspace models of cosmological interest is physically relevant by itself, as it enables us to address cosmological problems quantum mechanically. It is therefore not surprising that the recent literature contains a considerable number of works on canonical quantization of minisuperspace models [6-11].

A special attention has been paid, in particular, to the quantization of the Bianchi models [7-10]. These are spatially homogeneous spacetimes which admit an isometry group that acts transitively on each leaf of the homogeneous foliation [12,13]. Nevertheless, the analysis on Bianchi models, has been restricted almost entirely to the case of diagonal models [9-11], i.e., models in which the metric is purely diagonal. Even if this reduction is completely consistent [11], it would be desirable to carry out the whole quantization program for the full non-diagonal Bianchi models. In this way, one could also study the role played by the extra non-diagonal degrees of freedom in the quantum version of these systems, and discuss the implications of the diagonal reduction from the quantum point of view.

In this paper, we will analyze in detail the full non-diagonal type I and II Bianchi models. These systems are completely soluble, that is, one can obtain the explicit expressions of the general classical solution for both of these models. We will prove that the parameters that appear in the general solution define a good coordinatization of the phase space in these two models. The use of the non-diagonal degrees of freedom turns out to be decisive in determining the ranges of the introduced phase space coordinates. In fact, the reduction to the corresponding diagonal models would lead us to different conclusions about the range of the coordinates that describe the diagonal degrees of freedom. This is essentially due to the fact that, in the diagonal case, one can consistently consider the spatial directions in the homogeneous foliation as fixed once and forever. In the non-diagonal case, however, the requirement of analyticity in the introduced coordinatization of the phase space obliges us to deal exclusively with different homogeneous three-geometries, which are invariant under any interchange of the spatial directions that are not preferred by the symmetries of the model.

On the other hand, using the explicit expressions of the classical solutions, it is
possible to endow the reduced phase space of each of these models with an analytic symplectic structure. The symplectic forms obtained in this way can be interpreted as those associated to real cotangent bundles over some specific reduced configuration spaces. Following the canonical quantization program, we construct, on each of these reduced phase spaces, an over-complete set of classical variables that commute weakly with the constraints of the system and form a closed Lie algebra with respect to the Poisson-brackets structure. This Lie algebra can be identified as the algebra of the Dirac observables of the theory. We will then choose a vector space and find on it an unitary irreducible representation of the algebra of observables, so that the real Dirac observables are represented by self-adjoint operators [14]. The Hilbert spaces determined by this procedure provide us with the spaces of quantum states of the studied models.

The outline of this paper is as follows. In Sec. II we introduce the class of mini-superspace models on which we will concentrate in this work, and the main formulas needed to carry out our analysis in the new variables formalism. The general classical solutions for type I and II Bianchi models are obtained in Sec.III. In Sec. IV we study the symplectic structure of the space of physical solutions for Bianchi type II. The symplectic form in this space is written in terms of the different parameters contained in the general solution. We have to prove then that the chosen set of parameters defines a good coordinatization of the phase space of the model. This is the subject of Sec. V. Our analysis for Bianchi II is generalized to the non-diagonal type I Bianchi model in Sec. VI. Sec. VII deals with the non-perturbative canonical quantization of these two models. In Sec. VIII we present a different approach to the quantization of Bianchi type I, using the symmetries that are present in this model at the classical level to find a complete set of generalized “plane waves” which span the space of quantum states. We also show that the two quantum theories constructed for type I are unitarily equivalent. Finally, we summarize the results in Sec. IX, where we also include some further discussions.

II. Bianchi Models

Bianchi models are spatially homogeneous spacetimes (i.e., they can be foliated by three dimensional Riemannian manifolds) which admit a three dimensional isometry
Lie group $G$ that acts simply transitively on each leaf $\Sigma$ of the homogeneous foliation [12,13]. As a consequence, there exists for each of these models a set of three left-invariant vector fields $L_I$ on $\Sigma$ which form the Lie algebra of the group $G$:

$$[L_I, L_J] = C_{IJK}^L L_K \ ,$$

(2.1)

where $C_{IJK}^L$ are the structure constants of the Lie group. Dual to the vector fields $L_I$, one can introduce a set of three left-invariant one-forms $\chi^I$ which satisfy the Maurer-Cartan equations

$$d\chi^I + \frac{1}{2} C_{IJK}^L \chi^J \wedge \chi^K = 0 \ .$$

(2.2)

If the trace $C_{IJK}$ of the structure constants is equal to zero, the Bianchi model is said to belong to Bianchi class $\mathbf{A}$. For this class of models, the spacetime admits foliations by compact slices [15]. We will restrict ourselves to this case hereafter.

The structure constants for the class $\mathbf{A}$ Bianchi models can always be written in the form [12,16]

$$C_{IJK}^L = \epsilon_{JKL} S^{LI} \ ,$$

(2.3)

with $\epsilon_{JKL}$ the anti-symmetric symbol, and $S^{LI}$ a symmetric tensor. Further classification of the class $\mathbf{A}$ Bianchi models is defined with respect to the signature of the symmetric tensor $S^{IJ}$ [12,16]. The type I and II Bianchi models, the only ones we will consider in this work, are characterized by the signatures $(0,0,0)$ and $(0,0,+)$, respectively. Thus, the structure constants for Bianchi I are given by $C_{IJK}^L = 0$, while for Bianchi II $C_{IJK}^L = \delta_3^I \epsilon_{3JK}$.

In the new variables formalism, one starts by introducing the triads $e^i_a(x)$ on a three-manifold $\Sigma$, where $i = 1, 2, 3$ is a spatial index and $a = (1), (2), (3)$ is an $SO(3)$ vector index which is raised and lowered with the metric $\eta_{ab} = (1, 1, 1)$. The inverse metric on $\Sigma$ can be written in terms of the triads as $g^{ij} = e^i_a e^j_a$. The Ashtekar variables $(\mathcal{E}^i_a, A^a_j)$ are defined as the densitized triad and the spin connection [3]:

$$\mathcal{E}^i_a = (\det g)^{\frac{1}{2}} e^i_a \ , \quad A^a_i = \Gamma^a_i(e) - i K^a_i \ ,$$

(2.4)

with $\Gamma^a_i$ the $SO(3)$ connection compatible with $e^i_a$, and $K^a_i$ the triadic form of the extrinsic curvature.
For the class A Bianchi models and Σ a compact manifold, one can always perform the following transformation of variables [17]:

\[(A_i^a(x,t), E^i_a(x,t)) \rightarrow (A_I^a(t), \tilde{A}_i^a(x,t), E_I^I_a(t), \tilde{E}_a^i(x,t))\]  

(2.5)

where \(t\) is the time introduced by the homogeneous foliation, \(x\) is a set of coordinates on the leaf \(Σ\), and

\[A_I^a = \frac{1}{Ω} \int_Σ d^3x |χ| A_i^a L_i^I, \quad \tilde{A}_i^a = A_i^a - A_I^a χ_i^I, \quad E_I^I_a = \frac{1}{Ω} \int_Σ d^3x E^i_a χ_i^I, \quad \tilde{E}_a^i = E^i_a - E_I^I_a L_i^I |χ|.\]  

(2.6, 2.7)

In Eqs. (2.6, 7), \(χ_i^I = χ_i^I dx^i\), \(L_i^I = L_i^I \partial_i\), \(|χ|\) is the determinant of \(χ_i^I\) and \(Ω = \int_Σ d^3x |χ|\). The reduction of the dynamical degrees of freedom in the class A Bianchi models is accomplished by imposing \(\tilde{A}_i^a = \tilde{E}_a^i = 0\). In this way, one is left only with a finite number of degrees of freedom, given by \((A_I^a(t), E_I^I_a(t))\). The variables \((A_I^a(t), E_I^I_a(t))\) form a canonical set [11,17]:

\[\{A_I^a(t), E_J^b(t)\} = i δ^I_J δ_a^b.\]  

(2.8)

In terms of them, the constraints for the class A Bianchi models adopt the following expressions [17]:

\[G_a = \epsilon_{ab}^c A_I^b E_I^c, \quad V_I = C^K_{IJ} A_K^a E_J^I, \quad S = \epsilon_a^{bc} (-C^K_{IJ} A_K^a + \epsilon^a_{de} A_I^d A_J^e) E_I^I_b E_J^J_c,\]  

(2.9, 2.10, 2.11)

where \(G_a\), \(V_I\), and \(S\) denote, respectively, the Gauss law, the vector constraints and the scalar constraint.

On the other hand, using the left-invariant one-forms \(χ_i^I\), the spatial three-metric on \(Σ\) can be written as \(g = g_{IJ} χ_i^I χ_j^J\), with \(E_I^I_a\) related to the inverse of \(g_{IJ}\) by means of

\[g^{IJ}(\det E) = E_I^I_a E^J_a,\]  

(2.12)

and \(\det E\) denoting the determinant of \(E_I^I_a\).
Finally, the connection $A_I^a$ admits the decomposition

$$A_I^a = \Gamma_I^a(E) - i K_I^a ,$$

where the $SO(3)$ connection $\Gamma_I^a$ and the triadic extrinsic curvature $K_I^a$ can be determined through the formulas

$$\Gamma_I^a = \varepsilon^a_b c \left( \frac{1}{2} C^M J I (E^{-1})^b M E^J c - \frac{1}{4} C^M J I (E^{-1})^d M (E^{-1})^d I E^J b E^L c \right) ,$$

$$K_I^a = K_{IJ} E^J_a (\det E)^{-\frac{1}{2}} ,$$

with $(E^{-1})^a_I$ the inverse of $E^I_a$ and $K_{IJ}$ the extrinsic curvature associated to the metric $g_{IJ}$.

### III. Bianchi Types I and II: Classical Solutions

In the rest of this work we will restrict our attention to the type I and II Bianchi models [12,18]. It is well known that, for these models, one can always reduce the geometrodynamic initial value problem to the diagonal case [19,20]. We will now briefly review the argument that leads to this conclusion.

Suppose that, for either of these two models, we begin by considering a certain set of left-invariant one-forms $\chi^I$, for which the three-metric is given by $g_{IJ}$. Any other set of left-invariant forms $\tilde{\chi}^I$ will be related to $\chi^I$ by a transformation $\tilde{\chi}^I = M^I_J \chi^J$ that maintains the symmetries of the model. The structure constants must, therefore, remain unchanged under the transformation defined by $M^I_J$, \begin{equation}
C^I_{JK} = (M^{-1})^I_L C^L_{PQ} M^P_J M^Q_K .
\end{equation}
The inverse $(M^{-1})$ must always exist, since $\tilde{\chi}^I$ is a set of three linearly independent one-forms.

For Bianchi type I, $C^I_{JK} = 0$, and condition (3.1) is empty. In this case, any invertible matrix $M \in GL(3, \mathbb{R})$ defines a permissible transformation. Thus, for any geometrodynamic initial value data $(g^0_{IJ}, K^0_{IJ})$ in the set $\chi^I$ (with $g^0_{IJ}$ a positive definite metric), we can perform a transformation with a matrix $M \in \frac{GL(3, \mathbb{R})}{SO(3)}$ such that, in the new set of one-forms, the initial metric takes the value $\tilde{g}^0_{IJ} = \delta_{IJ}$. Then, using a transformation under $SO(3)$, we can bring the initial extrinsic curvature $K^0_{IJ}$
to the diagonal form, without altering the identity value for the initial metric [19,20]. Since the diagonal ansatz is compatible with the dynamics of the type I Bianchi model, we conclude as a corollary that any geometrodynamic classical solution for type I can be expressed as

\[ g_{IJ} = (M^t)_I^D g_D M^D_J , \]

(3.2)

where the metric \( g_D \) is a classical solution for the diagonal case, \( M^t \) is a constant invertible matrix, and \((M^t)\) denotes the transpose of \(M\).

Let us consider now the type II Bianchi model. For this model, \( C_{IJK} = \delta^J_3 \epsilon_{3JK} \), and condition (3.1) implies that \( M^t \) must be of the form

\[ M = \begin{pmatrix} \mathcal{M} & 0 \\ M^3_1 & M^3_2 & \text{det} \mathcal{M} \end{pmatrix} , \]

(3.3)

with \( \mathcal{M} \in GL(2, \mathbb{R}) \). Given any initial value data \((g^0_{IJ}, K^0_{IJ})\), defined in the set of one-forms \( \chi^I \), we can always carry out a transformation under a matrix of the type (3.3), with \( \mathcal{M} \in \frac{GL(2, \mathbb{R})}{SO(2)} \), such that, in the new set \( \tilde{\chi}^I = M^t J \chi^J \), the initial metric is equal to \( \tilde{g}^0_{IJ} = \delta_{IJ} + (\tilde{g}^0_3 - 1) \delta_{I3} \delta_{J3} \), and the extrinsic curvature is \( \tilde{K}^0_{IJ} = (M^t)_J^L \tilde{K}^0_{LN} M^N_J \).

Using expressions (2.12-15), the Gauss law constraints (2.9) and the two non-empty vector constraints in (2.10) for Bianchi type II, it is possible to see that, for positive definite metrics \( \tilde{g}^0_{IJ} \) of the form that we have obtained, \( \tilde{K}_{31}^0 \) and \( \tilde{K}_{32}^0 \) must vanish if \((\tilde{g}^0_{IJ}, \tilde{K}^0_{IJ})\) is an admissible set of initial value data. With these conditions, it is clear that \( \tilde{K}_{IJ}^0 \) can be brought to diagonal form by a transformation of the type (3.3) with \( \mathcal{M} \in SO(2) \) and \( M^3_1 = M^3_2 = 0 \) [19]. Under such a transformation, the initial metric \( \tilde{g}^0_{IJ} \) remains unchanged. Since the diagonal case is consistent with the dynamics of the model, we conclude, as for Bianchi I, that any Bianchi type II classical solution can be written in the form (3.2), with \( M \) an invertible constant matrix of the type (3.3).

Therefore, to get the general solution in geometrodynamics for the Bianchi types I and II, it suffices to find the classical solutions for the corresponding diagonal cases. These solutions can in fact be obtained from the analysis of the diagonal Bianchi
models made by Ashtekar, Tate and Uggla in Ref. [9]. Parallelling their notation, we introduce the following parametrization for the diagonal metric \( g_D \), \( D = 1, 2, 3 \):

\[
g_1 = e^{2\sqrt{3}(\beta_0 - \beta_+ + \beta_-)} , 
\quad g_2 = e^{2\sqrt{3}(\beta_0 - \beta_-)} , 
\quad g_3 = e^{2\sqrt{3}\beta_+} . \tag{3.4}
\]

In the Misner’s gauge [21], defined by the lapse function \( N = 12 (\det g_D)^{\frac{1}{2}} = 12 e^{\sqrt{3}(2\beta_0 - \beta_+)} \), the dynamical equations for Bianchi type I adopt the simple expression

\[
\dot{\beta}_0 = -p_0 , 
\dot{\beta}_+ = p_+ , 
\dot{\beta}_- = p_- , \tag{3.5}
\]

where the dot denotes time derivative and \((p_0, p_+, p_-)\) are three real constants which are related through the scalar constraint [9]

\[
S \propto -p_0^2 + p_+^2 + p_-^2 = 0 . \tag{3.6}
\]

Integrating the equations of motion, we arrive at a diagonal metric and a lapse function \(^1\)

\[
g_1 = e^{-2\sqrt{3}(p_0 + p_+ - p_-)t} , 
\quad g_2 = e^{-2\sqrt{3}(p_0 + p_+ + p_-)t} , 
\quad g_3 = e^{2\sqrt{3}p_+ t} , \quad N = 12 e^{-\sqrt{3}(2p_0 + p_+)t} . \tag{3.7}
\]

Finally, from (3.6), we can restrict our analysis to non-negative \( p_0 \) given by

\[
p_0 = \sqrt{p_+^2 + p_-^2} . \tag{3.9}
\]

The solutions corresponding to negative \( p_0 \) can be obtained from those with \( p_0 > 0 \) by changing the sign of the time parameter \( t \) that defines the evolution and flipping the signs of \( p_+ \) and \( p_- \). All the different physical solutions (i.e., solutions with different spacetime geometries) are contained in the sector \( p_0 \geq 0, p_+, p_- , t \in \mathbb{R} \) [9]. We will thus restrict ourselves to this range of the parameters appearing in (3.7-9).

To find the classical solution for Bianchi type II, one needs to perform a canonical transformation that mixes \( g_3 \) with its canonical momentum [9]:

\[
e^{2\sqrt{3}\beta_+} = \frac{\bar{p}_+}{2\sqrt{3}\cosh(2\sqrt{3}\beta_+)} , 
\quad p_+ = -\bar{p}_+ \tanh(2\sqrt{3}\beta_+) , \tag{3.10}
\]

\(^1\) The integration constants for \( \beta_0, \beta_+ \) and \( \beta_- \) can be absorbed by a translation of the origin of time and a redefinition of the matrix \( M \) that appears in (3.2).
with \( \bar{p}_+ \) defined as a strictly positive variable. In Misner’s gauge, the dynamical equations (3.5) and the scalar constraint (3.6) are still valid for the type II diagonal model, with the substitution \((\beta_+, p_+) \rightarrow (\bar{\beta}_+, \bar{p}_+)\), and \((p_0, \bar{p}_+, p_-)\) three real constants. It is then straightforward to derive the general expression for the diagonal metric in the classical solutions

\[
\begin{align*}
g_1 &= e^{-2\sqrt{3}(p_0-p_-)t} \frac{2\sqrt{3} \cosh(2\sqrt{3}\bar{p}_+ t)}{\bar{p}_+}, \\
g_2 &= e^{-2\sqrt{3}(p_0+p_-)t} \frac{2\sqrt{3} \cosh(2\sqrt{3}\bar{p}_+ t)}{\bar{p}_+}, \\
g_3 &= \frac{\bar{p}_+}{2\sqrt{3} \cosh(2\sqrt{3}\bar{p}_+ t)} ,
\end{align*}
\]

where \( t \) is the time coordinate defined by the lapse function

\[
N = 12e^{-2\sqrt{3}p_0t} \left( \frac{2\sqrt{3} \cosh(2\sqrt{3}\bar{p}_+ t)}{\bar{p}_+} \right)^{\frac{1}{2}},
\]

and

\[
\bar{p}_+ > 0, \quad p_- \in \mathbb{R}, \quad p_0 = \sqrt{p_+^2 + p_-^2} > 0.
\]

Using the explicit expressions (3.7,8) and (3.11,12) for the metric and the lapse function in the diagonal type I and II Bianchi models, the relations (2.12-15) and (3.2), and the formula

\[
K_{IJ} = \frac{1}{2N} \hat{g}_{IJ},
\]

valid in the gauge in which the shift functions vanish [13], one can easily compute the general form of the Bianchi I and II triads and spin connections in the physical solutions, restricted to the sector that corresponds to positive definite metrics. The result can be written in the compact notation

\[
\begin{align*}
E^I_a &= \det M (M^{-1})^I_D E_D R^{aD}, \\
A_I^a &= (M^T)^I_D \frac{\omega_D}{E_D} R^{aD},
\end{align*}
\]

where \( R^{aD} \) is a general complex orthogonal matrix, and the sum over \( D = 1, 2, 3 \) is implicitly assumed.
For Bianchi type I, the matrix $M$ belongs to $GL(3, \mathbb{R})$ and

$$
E_1 = e^{-\sqrt{3}(p_0 + p_-)} t, \quad E_2 = e^{-\sqrt{3}(p_0 - p_-)} t, \quad E_3 = e^{-2\sqrt{3}(p_0 + p_+)} t,
$$

$$
\omega_1 = \frac{i}{4\sqrt{3}}(p_0 + p_+ - p_-), \quad \omega_1 = \frac{i}{4\sqrt{3}}(p_0 + p_+ + p_-), \quad \omega_1 = \frac{-i}{4\sqrt{3}}p_+,
$$

(3.17)

with $p_0$ given by (3.9).

In the case of Bianchi II, the matrix $M$ appearing in (3.15,16) must be of the form (3.3), and $(E_D, \omega_D)$ can be expressed as

$$
E_1 = e^{-\sqrt{3}(p_0 + p_-)} t, \quad E_2 = e^{-\sqrt{3}(p_0 - p_-)} t, \quad E_3 = \frac{2\sqrt{3} \cosh(2\sqrt{3}\bar{p}_+ t)}{\bar{p}_+} e^{-2\sqrt{3}p_0 t},
$$

(3.19)

$$
\omega_1 = \frac{1}{4\sqrt{3}}(F + i(p_0 - p_+ - G)),
$$

(3.20a)

$$
\omega_2 = \frac{1}{4\sqrt{3}}(F + i(p_0 + p_- - G)), \quad \omega_3 = \frac{-1}{4\sqrt{3}}(F - iG),
$$

(3.20b)

$$
F = \frac{\bar{p}_+}{\cosh(2\sqrt{3}\bar{p}_+ t)}, \quad G = \bar{p}_+ \tanh(2\sqrt{3}\bar{p}_+ t),
$$

(3.20c)

where $(p_0, \bar{p}_+, p_-)$ must satisfy the restrictions (3.13).

Even if the classical solutions that we have found correspond to positive definite metrics, it is clear from (3.15) that one can always reach degenerate metrics in the limits in which either $E_1, E_2, E_3$ or det$M$ vanish. It is only in this sense that the degenerate solutions are included in our analysis.

**IV. Bianchi Type II: Symplectic Structure of the Space of Solutions**

In this section, we will concentrate our attention on the type II Bianchi model, studying the structure of the space of physical solutions.

Let $\chi^I$ be the set of left-invariant one-forms for which the metric $g_{IJ}$ is diagonal. For Bianchi type II there always exists a preferred one-form $\chi^3$ selected by the symmetries of the model, since the structure constants are given by $C^I_{JK} = S^{IL} \epsilon_{LJK}$, with $S^{33}$ the only non-vanishing component of $S^{IL}$. However, the spacetime geometries remain obviously unaltered under the interchange of $\chi^1$ and $\chi^2$. Two classical solutions which are related by the interchange of indices $I = 1$ and $I = 2$ should then be identified as the same physical solution.\footnote{Nevertheless, one can neglect this identification by considering $\chi^1$ and $\chi^2$ as two preferred one-forms. The space of solutions will then correspond to a different theory, in which the discussed symmetry is not present [9].} From the expressions (3.19,20), this
interchange of indices can be realized as a flip of sign in \( p_- \). Therefore, we can restrict ourselves only to non-negative parameters \( p_- \geq 0 \), so that each physical solution is considered only once.

There is still some redundancy left in the classical solutions of our model. This redundancy comes from the fact that, if \( A \) is a matrix that satisfies

\[
A^t g_D A = g_D
\]

for all diagonal metrics \( g_D \), and such that \( AM \) is of the form (3.3) for every matrix \( M \) of that form, the classical metrics (3.2) associated to the collection of matrices \( AM \) turn out to be identical [22]. The conditions imposed on \( A \) define a discrete group of four elements

\[
\{A_1 \equiv (1,1,1), A_2 \equiv (1,-1,-1), A_3 \equiv (-1,-1,1), A_4 \equiv (-1,1,-1)\},
\]

where \((a,b,c)\) denotes the ordered set of diagonal elements of \( A \), and all the non-diagonal elements are equal to zero. Using the invariance of the physical solutions under multiplication of \( M \) by \( A_2 \), we can choose a positive determinant for the matrix \( M \) appearing in (3.3). We will thus restrict in the following to the case \( \text{det} M > 0 \).

Note that, however, we have still to identify the classical solutions corresponding to \( M \) and \( A_3 M \), since multiplication by \( A_3 \) conserves the sign of \( \text{det} M \). We will return to this point later in this section.

Once we have determined the physically different classical solutions, we proceed to show that the space of solutions is endowed with a symplectic structure. We begin with the symplectic structure in the Ashtekar formalism:

\[
i\Omega = dA_I^a \wedge dE^I_a.
\]

Substituting Eqs. (3.15,16) in this formula, we arrive at the expression

\[
i\Omega = d(\omega_I \text{det} M) \wedge \left( \frac{dE_I}{E_I} + d(\text{det} M) - (dMM^{-1})^I_J \right) \\
- \omega_I \text{det} M (dMM^{-1})^I_J \wedge (dMM^{-1})^J_I,
\]

with \((dMM^{-1})^I_J = dM^I_Q (M^{-1})^Q_J \) and \( d(\text{det} M) = \text{det} M \text{Tr}(dMM^{-1}) \). The complex orthogonal matrix \( R^{aD} \) that is present in Eqs. (3.15,16) disappears completely from
the symplectic form (4.4), as it represents only the gauge degrees of freedom associated to the Gauss law constraints (2.9). On the other hand, taking into account that the matrix $M$ is of the form (3.3), with $\det M > 0$, we can always decompose $M$ in the following product of matrices

$$M \equiv M_D M_3 M_T R$$

with $a, b > 0$, $\tilde{M}_1^3, \tilde{M}_2^3, z \in \mathbb{R}$, and $\theta \in S^1$. The diagonal matrix $M_D$ can be absorbed into the diagonal part of the classical solutions, $(E_D, \omega_D)$, by means of the redefinitions

$$\tilde{E}_1 = ab^2 E_1, \quad \tilde{E}_2 = a^2 b E_1, \quad \tilde{E}_3 = ab E_3, \quad \tilde{\omega}_D = (ab)^2 \omega_D.$$ (4.6)

Introducing then the notation $\tilde{M} = M_3 M_T R$, Eq. (4.4) can be rewritten as

$$i\Omega = d(\tilde{\omega}_I) \wedge \left( d(\ln \tilde{E}_I) - (d\tilde{M} \tilde{M}^{-1})^I_J \right) - \tilde{\omega}_I (d\tilde{M} \tilde{M}^{-1})^I_J \wedge (d\tilde{M} \tilde{M}^{-1})^J_I, \quad (4.7)$$

where we have employed that $(dM_D M^{-1}_D)^I_J = \delta^I_J ((dM_D M^{-1}_D)^I_I)$. Let us define $S = M_T R$, so that $\tilde{M} = M_3 S$. It is straightforward to compute that

$$(d\tilde{M} \tilde{M}^{-1})^I_J = d(\tilde{M}_1^3) \delta^I_3 \delta^J_3 + d(\tilde{M}_2^3) \delta^I_3 \delta^J_2 + (M_3)^I_P (dSS^{-1})^Q_P (M_3^{-1})^P_J. \quad (4.8)$$

Then, using that $(dSS^{-1})^I_3 = 0$ and

$$(M_3^{-1})^P_I D_I (M_3)^P_J = D_I + D_I \delta^P_3 (\tilde{M}_1^3 \delta^I_Q + \tilde{M}_2^3 \delta^I_Q) \quad (4.9)$$

for $D$ any diagonal matrix, we conclude that Eq. (4.7) is still valid with the substitution of $S$ for $\tilde{M}$. In this way, the matrix $(M_3)$ drops from the symplectic form $\Omega$, implying that $\tilde{M}_1^3$ and $\tilde{M}_2^3$ do not correspond to real degrees of freedom.

In the parametrization chosen in (4.5), $(dSS^{-1})$ takes the explicit expression

$$(dSS^{-1}) = \begin{pmatrix}
-zd\theta & d\theta & 0 \\
dz - (z^2 + 1)d\theta & zd\theta & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (4.10)$$

13
from which it follows that
\[ i\Omega = d(\omega_D) \wedge d(\ln \tilde{E}_D) + d((\tilde{\alpha}_1 - \tilde{\alpha}_2)z) \wedge d\theta. \]  
(4.11)

Substituting Eqs. (4.6) in this formula, with \((E_D, \omega_D)\) defined by means of (3.19,20) and (3.13), we arrive after some calculations at a symplectic structure of the form
\[ \Omega = d\Pi_+ \wedge dX + d\Pi_- \wedge dY + d\theta \wedge dZ, \]  
(4.12)

where
\[ \Pi_+ = (ab)^2\bar{p}_+, \quad \Pi_- = (ab)^2p_-, \]  
(4.13)

\[ X = \frac{3}{4\sqrt{3}}\frac{\Pi_+}{\Pi_0} \ln(ab), \quad Y = \frac{1}{4\sqrt{3}}\left(3\frac{\Pi_-}{\Pi_0} \ln(ab) + \ln \frac{a}{b}\right), \quad Z = \frac{1}{2\sqrt{3}}\Pi_-z. \]  
(4.14)

Expression (4.12) provides us with the symplectic form for the space of solutions to all physical constraints, i.e., the reduced phase space. Nevertheless, in obtaining this symplectic structure, we have implicitly assumed that the parameters \((p_+, p_-, p_0, t, a, b, z, \theta, \tilde{M}_1^3, \tilde{M}_2^3)\) are good coordinates in the space of physical solutions to all but the scalar constraint; that is, that the transformation from the triad and the spin connection to the given set of parameters is analytic in the sector of the phase space covered by these solutions. We will prove that this is indeed the case in Sec. V.

On the other hand, introducing the notation \(\Pi_0 = (ab)^2p_0\), the scalar constraint (3.6) implies that \((\Pi_+, \Pi_-)\) are a set of coordinates on the light-cone:

\[ -\Pi_0^2 + \Pi_+^2 + \Pi_-^2 = 0. \]  
(4.15)

We point out that, in our model, \(\bar{p}_+, a,\) and \(b\) are positive quantities, and that we have to restrict ourselves to positive \(p_0\) and non-negative \(p_-\) in order to deal exclusively with different physical solutions. With these restrictions, \((X, Y, Z)\) defined in (4.14) run still over the whole real axis, for \(z \in \mathbb{R}\). We can then interpret the two first terms on the right hand side of (4.12) as the symplectic form of the cotangent bundle over \(L^+_{(+, +)}\), the positive quadrant of the future light-cone.

In the last term of (4.12), the angle \(\theta\) belongs to \(S^1\). We recall, however, that, from our previous discussion, there is still some redundancy to be removed if we want
to consider only physically different classical solutions. This redundancy corresponds to the identification of the matrix $M$ appearing in (4.5) with that obtained by multiplication on the left by $A_3 = (-1, -1, 1)$. As a consequence, the solutions associated with the parameters $(\theta, \tilde{M}_1^3, \tilde{M}_2^3)$ and $(\theta + \pi, -\tilde{M}_1^3, -\tilde{M}_2^3)$ are physically identical, and we may restrict our analysis to the interval $\theta \in [0, \pi)$. Note, nevertheless, that $\tilde{M}_1^3$ and $\tilde{M}_2^3$ are not real degrees of freedom (we can always go to the gauge in which $\tilde{M}_1^3 = \tilde{M}_2^3 = 0$), and that the coordinates $\Pi_+, \Pi_-, X, Y, Z$ remain unaltered under the transformation $(\theta, \tilde{M}_1^3, \tilde{M}_2^3) \rightarrow (\theta + \pi, -\tilde{M}_1^3, -\tilde{M}_2^3)$. Therefore, the identification of physical solutions under that transformation obliges us to identify also the boundaries $\theta = 0$ and $\theta = \pi$ of the reduced phase space. In this way, the angle $\Theta = 2\theta$ turns out belong to $S^1$, and the term $d\theta \wedge dZ = d\Theta \wedge d(\frac{Z}{2})$ in (4.12) can be interpreted as the symplectic form of the cotangent bundle over $S^1$. We thus conclude that the space of physical solutions in Bianchi II presents the symplectic structure of the cotangent bundle over the reduced configuration space $\mathcal{L}_{(+,+)}^+ \times S^1$.

V. Bianchi Type II: Analyticity of the Coordinatization of the Space of Solutions

We want to prove that the parametrization employed to describe the classical solutions for Bianchi type II (with $p_0$ regarded as independent of $p_-$ and $p_+$) defines an analytic coordinatization in the space of physical solutions to all but the scalar constraint, in the sense that the transformation from the triad and the spin connection to the chosen set of parameters is analytic in the whole region of the phase space covered by the different physical solutions that we have considered. We note first that the transformation from the triad and the spin connection to the three-metric and the extrinsic curvature is analytic in the sector of the Ashtekar variables that corresponds to non-degenerate metrics. It will suffice then to show that the matrix $M$, appearing in (3.2), and the parameters $(\bar{p}_+, p_-, p_0, t)$, contained in $g_D$, depend analytically on $(g_{IJ}, K_{IJ})$ in the region defined by $\det M, p_+, p_- > 0$, which contains all the different physical solutions.

Let us introduce a matrix $\bar{M}$ of positive determinant (and thus invertible) such that it satisfies the conditions

$$((\bar{M}^{-1})^t)_I^P g_{PQ} (\bar{M}^{-1})^Q_J = \delta_{IJ} + (\bar{g}_3 - 1) \delta_I^3 \delta_J^3,$$

(5.1)
\[ ((\bar{M}^{-1})^t)^P K_{PQ}(\bar{M}^{-1})^Q = \lambda \delta_{IJ} . \]  

(5.2)

It is clear from our previous analysis of the type II Bianchi model that one solution to Eqs. (5.1,2) is provided by

\[ \bar{M} = \left( (g_1)^{\frac{1}{2}}, (g_2)^{\frac{1}{2}}, (g_1 g_2)^{\frac{1}{2}} \right) M , \]  

(5.3)

where we have used a similar notation to that displayed in (4.2), \( M \) is the matrix appearing in (3.2,3), and \((g_1, g_2)\) are given by (3.11) in our parametrization. From Eqs. (3.11,12) and (3.14), one can compute also the explicit expressions of \( \bar{g}_3 \) and \( \lambda_I \) for the solution (5.3):

\[ \bar{g}_3 = 4\lambda^2 \left( \frac{\bar{p}_+}{\cosh(2\sqrt{3} \bar{p}_+ t)} \right)^2, \quad \lambda_1 = \lambda (-p_0 + p_- + \bar{p}_+ \tanh(2\sqrt{3} \bar{p}_+ t)), \]  

(5.4a)

\[ \lambda_2 = \lambda (-p_0 - p_- + \bar{p}_+ \tanh(2\sqrt{3} \bar{p}_+ t)), \quad \lambda_3 = -\lambda \bar{g}_3 \bar{p}_+ \tanh(2\sqrt{3} \bar{p}_+ t), \]  

(5.4b)

\[ \lambda = e^{2\sqrt{3} \rho_0 t} \left( \frac{\bar{p}_+}{2\sqrt{3} \cosh(2\sqrt{3} \bar{p}_+ t)} \right)^{\frac{1}{2}} . \]  

(5.4c)

Since \( g_1 \) and \( g_2 \), given by (3.11), are strictly positive, Eq. (5.3) defines \( M \) analytically in terms of \((\bar{p}_+, p_-, p_0, t, \bar{M})\). All we have to prove then is that \((\bar{p}_+, p_-, p_0, t, \bar{M})\) can be obtained analytically from \((g_{IJ}, K_{IJ})\) in the region of the phase space that we are considering. On the other hand, it is easy to check that the matrix \( \bar{M} \) in (5.3) is of the form (3.3), provided that \( M \) is of this form. Moreover, adopting a parallel notation to that introduced in Eq. (3.3), one can see that \( \det \bar{M} > 0 \) if \( \det M > 0 \). We will thus concentrate in the rest of this section on matrices \( \bar{M} \) of the type (3.3), for which \( \det \bar{M} > 0 \), and such that they verify Eqs. (5.1,2).

It is convenient to employ the following decomposition for the matrix \( \bar{M} \):

\[ \bar{M} \equiv \bar{R} \bar{M}_D \bar{M}_T \bar{M}_3 , \]  

(5.5)

with \( \bar{R}, \bar{M}_D, \bar{M}_T, \) and \( \bar{M}_3 \) parametrized as their respective counterparts in Eq. (4.5) with the substitutions \((a, b, \bar{M}_1^3, \bar{M}_2^3, z, \theta) \rightarrow (\bar{a}, \bar{b}, \bar{M}_1^3, \bar{M}_2^3, \bar{z}, \bar{\theta})\), so that \( \bar{a}, \bar{b} > 0, \bar{z}, \bar{M}_1^3, \bar{M}_2^3 \in \mathbb{R} \) and \( \bar{\theta} \in S^1 \). Let us show then that all the elements of these matrices present, through Eqs. (5.1,2), an analytic dependence on \((g_{IJ}, K_{IJ})\) in the region of physical interest.
We first determine $\bar{M}_3$ by the conditions

$$\bar{g}_{13} = \bar{g}_{23} = 0, \quad \text{with} \quad \bar{g}_{IJ} = \left((\bar{M}_3^{-1})^t\right)_I^P g_{PQ} (\bar{M}_3^{-1})^Q_J,$$

(5.6)

from which one obtains that $\bar{M}_3^1 = g_{13}/g_{33}$ and $\bar{M}_3^2 = g_{12}/g_{33}$. $\bar{M}_3$ is thus analytic in $g_{IJ}$, since $g_{IJ}$ is a positive definite metric, and therefore $g_{33} > 0$. Let us define then the two-by-two matrix $\tilde{h}$ constructed with the two first rows and columns of $\tilde{g}$, and, similarly, the matrix $\tilde{K}$ obtained from the extrinsic curvature $\tilde{K} = (\bar{M}_3^{-1})^t K (\bar{M}_3^{-1})$. 3

We can use the degrees of freedom $\bar{a}$, $\bar{b}$, and $\bar{z}$ in $\bar{M}$ to diagonalize $\tilde{h}$ to the identity

$$\left((\bar{M}_D)^{-1}\right)^t (\bar{M}_T)^{-1} \tilde{h} (\bar{M}_T^{-1})(\bar{M}_D^{-1}) = I,$$

(5.7)

where $\bar{M}_D$ and $\bar{M}_T$ are the two-by-two matrices constructed from $\bar{M}_D$ and $\bar{M}_T$ by the procedure explained above. Eq. (5.7) fixes $\bar{a}$, $\bar{b}$, and $\bar{z}$ uniquely, for $\bar{a}$ and $\bar{b}$ strictly positive,

$$\bar{a} = \left(\frac{\text{det}\tilde{h}}{\tilde{h}_{22}}\right)^\frac{1}{4}, \quad \bar{b} = (\tilde{h}_{22})^\frac{1}{4}, \quad \bar{z} = \frac{\tilde{h}_{12}}{\tilde{h}_{22}}.$$

(5.8)

From the expression (5.8) we conclude that $\bar{a}$, $\bar{b}$, and $\bar{z}$ are analytic in $\tilde{h}$ (and then in $g$) for $\tilde{h}$ positive definite.

We can now diagonalize $\tilde{K} = (\bar{M}_D^{-1})^t (\bar{M}_T^{-1})^t \tilde{K} (\bar{M}_T^{-1})(\bar{M}_D^{-1})$ by the SO(2) transformation given by the matrix $\bar{R}$ contained in $\bar{R}$. This transformation leaves invariant the value $\hat{h} = I$ reached in (5.7). We are thus left with the eigenvalue problem

$$F_{IJ} \equiv \bar{R}_I^P \bar{K}_{PQ} (\bar{R}^t)^Q_J - \lambda I \delta_{IJ} = 0.$$

(5.9)

The element $F_{12}$ of the system of equations (5.9) defines $\theta$ as an implicit function of $\tilde{K}$. Taking into account that, from our previous analysis, $\tilde{K}$ is analytic in $(g_{IJ}, K_{IJ})$, all we have to prove is that the implicit dependence of $\theta$ on $\tilde{K}$, imposed by Eq. (5.9), is analytic. Let us suppose that $(\theta^0, \tilde{K}^0_{IJ})$ is a particular solution to the equation $F_{12} = 0$ that determines $\theta$ as a function of $\tilde{K}$. Then, $\theta(\tilde{K}^0) = \theta^0$ defines an analytic germ [23] around $\tilde{K}^0$ which can be continued analytically as long as $\partial_\theta F_{12}(\theta(\tilde{K}), \tilde{K}) \neq 0$

3 From our discussion of the type II Bianchi model in Sec. III, it follows that, given the form of $\tilde{g}$, $K$ must satisfy $\tilde{K}_{13} = \tilde{K}_{23} = 0$ if $(\tilde{g}_{IJ}, \tilde{K}_{IJ})$ corresponds to a classical solution to all but the Hamiltonian constraint.
(in particular, if $\theta^0$ and $\tilde{K}^0_{IJ}$ are real and $\theta(\tilde{K})$ can be continued analytically around $\tilde{K}^0_{IJ}$, it is possible to show that $\theta(\tilde{K})$ remains real for $\tilde{K} \in \mathbb{R}$). After a simple computation, one arrives at the identity
\[
\partial_\theta F_{12} = \lambda_2 - \lambda_1 ,
\] (5.10)
so that the obtained solution $\theta(\tilde{K})$ depends analytically on $\tilde{K}$ as far as $\lambda_1 \neq \lambda_2$.  
Note that, in our parametrization, $\lambda_1 = \lambda_2$ only if $p_- = 0$ (see Eqs. (5.4)). Since we are restricting our attention to the sector of positive definite metrics and extrinsic curvatures for which $p_- \geq 0$, we conclude that $\theta(\tilde{K})$ is analytic in that region, except at the boundary $p_0 = 0$. With this caveat, the matrix $\bar{M}$ turns out then to be analytic in the sector of metrics and extrinsic curvatures associated to the physical solutions analyzed in Sec. IV.

Employing this result, Eqs. (5.1,2) provide us with $\bar{g}_3$ and $\lambda_I$ ($I = 1, 2, 3$) as analytic functions of $(g_{IJ}, K_{IJ})$, for $(\bar{M}^{-1})$ is always well-defined, and is analytic if $\bar{M}$ is analytic. We can use now the explicit expressions (5.4) to determine $(\bar{p}_+, p_-, p_0, t)$ as analytic functions of $\bar{g}_3$ and $\lambda_I$, and, therefore, of $(g_{IJ}, K_{IJ})$. The relations obtained are
\[
p_- = \bar{p}_+ F_1 / F_3 , \quad p_0 = \bar{p}_+ F_2 / F_3 ,
\] (5.11a)
\[
t = \frac{1}{2 \sqrt{3 \bar{p}_+}} \cosh^{-1} \left( \frac{F_3}{F_4} \right) ,
\] (5.11b)
\[
\bar{p}_+ = 2 \sqrt{3} F_3 \left( \frac{4}{F_4} e^{-2 \frac{F_2}{F_3} \cosh^{-1} \left( \frac{F_3}{F_4} \right)} \right)^{\frac{1}{3}} ,
\] (5.11c)
where
\[
F_1 = \frac{1}{2} (\lambda_1 - \lambda_2) , \quad F_2 = -\frac{\lambda_3}{\bar{g}_3} - \frac{1}{2} (\lambda_1 + \lambda_2) ,
\] (5.12a)
\[
F_3 = \left( \frac{\bar{g}_3}{4} + \left( \frac{\lambda_3}{\bar{g}_3} \right)^2 \right)^{\frac{1}{2}} , \quad F_4 = \frac{(\bar{g}_3)^{\frac{3}{2}}}{2} .
\] (5.12b)

\footnote{Eq. (5.9) can be interpreted as an eigenvalue problem. Each row of the matrix $\bar{R}$ can be identified as an unit eigenvector of the matrix $\tilde{K}$. When $\lambda_1 = \lambda_2$, the eigenvalue problem is degenerate: any unit vector is an eigenvector of $\tilde{K}$. As a consequence, $\bar{R}$ is ill-defined in that case.}

\footnote{It is clear then that, without a restriction of this type, $p_-$ would have not been a good coordinate in the phase space of the full type II Bianchi model.}
For real positive definite metrics (so that $\bar{g}_3 > 0$) and real extrinsic curvatures, all the functions $F_n$ (with $n = 1, \ldots, 4$) are analytic in $(\bar{g}_3, \lambda_I)$, and $F_3$ and $F_4$ turn out to be positive. Then, $(\bar{p}_+, p_-, p_0, t)$, given by (5.11), result to be analytic in $(g_{IJ}, K_{IJ})$.

From our previous discussion, it follows that the matrix $M$, defined by means of (5.3), is analytic in $(g_{IJ}, K_{IJ})$ in the sector of solutions to all but the Hamiltonian constraint that we are studying. The only point that remains to be proved is that the specific parametrization (4.5), employed for $M$ in Sec. IV, is analytic with respect to its dependence on the elements of $M$. Using this parametrization, and the notation in (3.3), we have that

$$M = \begin{pmatrix} M^1_1 & M^1_2 \\ M^2_1 & M^2_2 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b_z & b \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

(5.13)

$$M^3_1 = ab\left(\tilde{M}^3_1 \cos \theta + \tilde{M}^3_2 (z \cos \theta - \sin \theta)\right),$$

(5.14a)

$$M^3_2 = ab\left(\tilde{M}^3_1 \sin \theta + \tilde{M}^3_2 (z \sin \theta + \cos \theta)\right),$$

(5.14b)

where $a$ and $b$ are strictly positive, and $\det M > 0$ by construction. From Eq. (5.13), one can easily arrive at the explicit expressions of $a, b,$ and $z$ in terms of $M, M'$. These expressions are analytic in the elements of $M$ for $\det M > 0$, and allow us to define $a$ and $b$ as strictly positive functions: $a(M), b(M) > 0$. Eq. (5.13) provides us also with the relations $\cos \theta = M^1_1/a(M)$ and $\sin \theta = M^2_2/a(M)$, from which we conclude that $\cos \theta$ and $\sin \theta$ are analytic in $M$, as $a(M) > 0$. 6 Substituting then $a, b, z, \cos \theta$ and $\sin \theta$ as functions of $M$, the equations in (5.14) determine $\tilde{M}^3_1$ and $\tilde{M}^3_2$ as two implicit functions of the elements of the matrix $M : (\tilde{M}^3_1(M), \tilde{M}^3_2(M))$. The Jacobian of these equations with respect to $\tilde{M}^3_1$ and $\tilde{M}^3_2$ can be computed straightforwardly to be equal to $a^2(M)b^2(M)$. Therefore, given a particular analytic germ for $(\tilde{M}^3_1, \tilde{M}^3_2)$, we can always continue it analytically to the whole range of matrices $M$ with $\det M > 0$ (so that $a(M) \text{ and } b(M) \text{ are strictly positive}$) [23,24].

In conclusion, we have proved that the parameters $(\bar{p}_+, p_-, p_0, t, a, b, z, \theta, \tilde{M}^3_1, \tilde{M}^3_2)$ are good coordinates in the space of non-degenerate real solutions that satisfy all but the Hamiltonian constraint, when restricted to the region

$$\bar{p}_+, p_-, a, b > 0, \quad \theta \in S^1, \quad p_0, t, z, \tilde{M}^3_1, \tilde{M}^3_2 \in \mathbb{R}.$$

(5.15)

---

6 Indeed, these relations define the angle $\theta$ analytically, because the Jacobian matrix of $(\cos \theta, \sin \theta)$ with respect to $\theta$ has always a rank equal to the unity.
If we want to go to the reduced phase space, we have to impose in addition the constraint (3.6), and restrict the range of $p_0$ to the positive real axis. As a consequence, the symplectic structure obtained in Sec. IV results to be analytic everywhere in the cotangent bundle over the reduced configuration space $L^{(+,+)}_0 \times S^1$, with its boundary excluded.

VI. Type I Bianchi Model

Let us proceed now to generalize the study of Secs. IV and V to the case of the type I Bianchi model. We begin by analyzing the sector of classical solutions that corresponds to different spacetime geometries.

In the set of left-invariant one-forms $\chi^I$ in which the metric is diagonal, the spacetime geometries are invariant under permutations of all $\chi^I$, because the structure constants vanish identically in this model, so that there is no preferred one-form. Therefore, the classical diagonal solutions that are related under any interchange of the indices $I = 1, 2, 3$ must be identified. From expressions (3.17,18), each of the planes $\Pi_1 \equiv p_- = 0, \Pi_2 \equiv p_0 + 2p_+ - p_- = 0$ and $\Pi_3 \equiv p_0 + 2p_+ + p_- = 0$ divide the space $(p_0, p_+, p_-) \in \mathbb{R}^3$ into two regions which can be interchanged under the respective permutations of indices: $1 \leftrightarrow 2, 1 \leftrightarrow 3$ and $2 \leftrightarrow 3$. If we take into account also the constraint (3.9), which implies that $(p_0, p_+, p_-)$ lie in the future light-cone, the requirement of considering only different classical solutions may be implemented by the following restrictions in the ranges of our parameters:

$$p_+ \geq 0, \quad p_- \geq 0 \quad p_0 = \sqrt{p_+^2 + p_-^2} \geq 0,$$

i.e., $(p_0, p_+, p_-) \in L^{(+,+)}_0$.

Parallel to the situation in Bianchi type II, we have to identify also the solution (3.15,16) for a given matrix $M \in GL(3, \mathbb{R})$ with all other solutions obtained from matrices of the form $AM$, where $A$ is any orthogonal matrix that commutes with the diagonal subgroup of $GL(3, \mathbb{R})$ (see Eq. (4.1)). We can then fix the determinant of $M$ to be strictly positive, because either $M$ or $-M$ has a positive determinant, and $A = (-1, -1, -1)$ is orthogonal and commutes with the diagonal subgroup. There is, however, some redundancy left, because there exist matrices $A$ which conserve the sign of the determinant of $M$. These matrices are given by the discrete group of
four elements defined in (4.2) [22]. We will discuss the corresponding identification of classical solutions later in this section. Finally, we point out that, given the explicit expression (3.7) for the diagonal metric in Bianchi type I, it is always possible to absorb the determinant of \( M \) by a redefinition of the origin of the time coordinate \( t \). We will thus restrict hereafter to matrices \( M \in SL(3, \mathbb{R}) \).

Let us introduce the following parametrization for the matrices \( M \in SL(3, \mathbb{R}) \):

\[
M \equiv M_D M_T R = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & \frac{1}{ab}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
y & z & 1
\end{pmatrix} \begin{pmatrix}
R
\end{pmatrix}, \tag{6.2}
\]

where \( a \) and \( b \) are strictly positive, \( x, y \) and \( z \) are real and \( R \in SO(3) \). Similar to the notation for Bianchi type II, we will call

\[
\hat{E}_1 = \frac{E_1}{a}, \quad \hat{E}_2 = \frac{E_2}{b}, \quad \hat{E}_3 = abE_3. \tag{6.3}
\]

Then, it is straightforward to generalize the discussion presented in Sec. IV to arrive at a symplectic structure of the form (4.7) for the space of solutions in Bianchi type I, with \( \hat{M} = M_T R \) and \( \hat{\omega}_I = \omega_I \) (note that now \( \det M = 1 \)).

The symplectic form (4.7) can be simplified as follows. Decompose first the matrix \( \hat{M} \) as \( \hat{M} = M_T R \), and define \( T^{IJ} = dR^{IP} R^{JP} \). From the expression of \( M_T \) in (6.2), one can explicitly check that

\[
\sum_P d(M_T)^I_P (M_T^{-1})^P_I = 0 \tag{6.4}
\]

for every \( I = 1, 2, 3 \). On the other hand, the matrix of one-forms \( T^{IJ} \) turns out to be antisymmetric, since \( R^{IP} R^{JP} = \delta^{IJ} \), and so \( dR^{IP} R^{JP} = -dR^{JP} R^{IP} \). Using this fact, it is possible to prove that

\[
dT^{IJ} = T^{IP} \wedge T^{PJ} \tag{6.5}
\]

Conveniently rearranging the different terms in (4.7), we arrive at the conclusion

\[
i\Omega = d\hat{\omega}_I \wedge d\ln(\hat{E}_I) - d(L_{IJ} T^{IJ}), \tag{6.6}
\]

\[
L_{IJ} = (M_T^{-1})^{|I|}_P \omega_P (M_T)^{|J|}_P, \tag{6.7}
\]
where the indices $I, J$ are raised and lowered with the metric $\eta_{IJ} = (1, 1, 1)$ and $[ \ ]$ denotes antisymmetrization. The first term on the right hand side of Eq. (6.6) can be calculated from Eqs. (6.3), (3.17,18) and (3.9). The result is

$$d\omega_I \wedge d\ln(\hat{E}_I) = \frac{i}{4\sqrt{3}}(dp_+ \wedge dX + dp_- \wedge dY) ,$$

(6.8)

with

$$X = -\left(\frac{p_+ + 2}{p_0}\right) \ln(ab) , \quad Y = \ln\left(\frac{a}{b}\right) - \frac{p_-}{p_0} \ln(ab) .$$

(6.9)

The one-form $\text{Tr}(-L^T)$ that appears in (6.6) can be interpreted as the pre-symplectic structure in $SO(3)$. From the parametrizations for $\omega_I$ and $M_T$ given by (3.18) and (6.2), one can explicitly compute the matrix $L_{IJ}$. Introducing the notation

$$L_{IJ} = \epsilon_{IJK}l^K,$$

we find that

$$l_1 = (p_0 + 2p_+ + p_-)z ,$$

(6.10a)

$$l_2 = -\left(2p_-(xz-y) + (p_0 + 2p_+ + p_-)y\right) , \quad l_3 = -2p_-x .$$

(6.10b)

Let us parametrize now the matrices $R \in SO(3)$ in terms of the Euler angles $(\alpha, \beta, \theta)$

$$R(\alpha, \beta, \theta) = R(1)(\alpha)R(3)(\beta)R(1)(\theta) ,$$

(6.11)

where $R(I)(\alpha)$ is a rotation of an angle $\alpha$ around the axis defined by the direction $I$, and $\alpha, \theta \in S^1, \beta \in [0, \pi]$. This parametrization is unique and well-defined for all matrices $R \in SO(3)$, except at $\beta = 0$ and $\beta = \pi$. After a short calculation using Eqs. (6.10,11), we conclude that the pre-symplectic structure $\text{Tr}(-L^T)$ can be written as

$$L_{IJ}T^{IJ} = \frac{i}{4\sqrt{3}}(F_1d\alpha + F_2d\beta + F_3d\theta) ,$$

(6.12)

with

$$F_1 = l_1 , \quad F_2 = l_2 \sin \alpha + l_3 \cos \alpha ,$$

(6.13a)

$$F_3 = -l_3 \sin \alpha \sin \beta + l_2 \cos \alpha \sin \beta + l_1 \cos \beta .$$

(6.13b)

Therefore, the symplectic form (6.7) has the expression

$$\Omega = \frac{1}{4\sqrt{3}}(dp_+ \wedge dX + dp_- \wedge dY + d\alpha \wedge dF_1 + d\beta \wedge dF_2 + d\theta \wedge dF_3) .$$

(6.14)
For $a$ and $b$ strictly positive and $x, y, z$ real, the variables $X, Y, F_1, F_2$ and $F_3$ run over the whole real axis $\mathbb{R}$. As it stands, the symplectic structure (6.14) might be interpreted as that corresponding to the cotangent bundle over $\mathcal{L}^+_{(+, +)} \times SO(3)$. Nevertheless, we have still to identify the classical solutions obtained from all matrices of the form $AM$, with $M \in SL(3, \mathbb{R})$ and $A$ any matrix in the discrete group (4.2).

In the parametrizations (6.2) and (6.11), the change of $M$ to $A_2M$ can be realized as the transformation $(\alpha, x, y) \rightarrow (\alpha + \pi, -x, -y)$, which leaves invariant all the variables appearing in (6.14) except $\alpha$ (see Eqs. (6.10) and (6.13)). In order to consider only different physical solutions, we can restrict ourselves to the range $\alpha \in [0, \pi)$ [22]. The boundaries $\alpha = 0$ and $\alpha = \pi$ must be identified in the reduced phase space, since, from our previous discussion, the classical solutions for $\alpha = 0$ and $\alpha = \pi$ are physically identical. In this way, $\tilde{\alpha} = 2\alpha$ belongs to $S^1$, and we can interpret the symplectic form $d\alpha \wedge dF_1 + d\beta \wedge dF_2$ as that of the cotangent bundle over the two-sphere, $S^2$, parametrized by the angles $\beta$ and $\tilde{\alpha}$. The singularities $\beta = 0$ and $\beta = \pi$ of the parametrization (6.11) then correspond to the poles of this two-sphere.

Let us now study the identification of the matrices $M$ and $A_3M$. The interchange of these two matrices is performed by the transformation $(\alpha, \beta, \theta, x, z) \rightarrow (\pi - \alpha, \pi - \beta, \theta + \pi, -x, -z)$, where, we recall, $\alpha \in [0, \pi)$. We could then impose the restriction $\theta \in [0, \pi)$, so that each physical solution is considered only once [22]. Note that, under the above transformation, $p_+, p_-, X, Y, F_3$ remain invariant, while $F_1$ and $F_2$ flip their sign. We should thus identify the points $(\alpha, \beta, \theta = 0, F_1, F_2, F_3)$ and $(\pi - \alpha, \pi - \beta, \theta = \pi, -F_1, -F_2, F_3)$ at the boundaries $\theta = 0$ and $\theta = \pi$ of the reduced phase space. Nevertheless, we will adopt a different approach for the quantization of the model, leaving $\theta$ to run over the whole of $S^1$ and imposing restrictions on the space of physical states associated to the identification of points

$$(\alpha, \beta, \theta, F_1, F_2, F_3) \quad \text{and} \quad (\pi - \alpha, \pi - \beta, \theta + \pi, -F_1, -F_2, F_3). \quad (6.15)$$

We will return to this issue in the next section.

Finally, the matrices $M$ and $A_4M$ are interchanged through the transformation $(\sin \alpha, \beta, \theta, x, z) \rightarrow (-\sin \alpha, \pi - \beta, \theta + \pi, -x, -z)$, which leaves invariant $p_+, p_-, X, Y,$ and $F_3$, and reverses the signs of $F_1$ and $F_2$. For $\alpha \in [0, \pi), \sin \alpha$ is non-negative, and the only possible redundancy left is at $\alpha = 0$. However, we notice that the
corresponding identification of points \((\alpha = 0, \beta, \theta, F_1, F_2, F_3)\) and \((\alpha = 0, \pi - \beta, 
theta + \pi, -F_1, -F_2, F_3)\) has already been taken into account by identifying the points in (6.15), and \(\alpha\) with \(\pi + \alpha\).

In conclusion, the space of physical solutions in the type I Bianchi model has the symplectic structure of the cotangent bundle over the reduced configuration space \(L^{+}_{(+, +)} \times S^2 \times S^1\). To consider only different physical solutions we still must impose the identification of points described in (6.15). We will implement this condition in the quantum version of the model as a restriction on the physical states.

We have been assuming so far that the parametrization used for the classical solutions corresponds to a good set of coordinates in the phase space. In fact, employing parallel arguments to those presented in Sec. V for the type II Bianchi model it is possible to show that the transformation from the triad and the spin connection to the chosen set of parameters is analytic in the considered physical solutions to all but the scalar constraint (except at \(\beta = 0, \pi\)). We will now briefly discuss the main lines of this proof.

Let \((g_{IJ}, K_{IJ})\) be the metric and extrinsic curvature for the type I Bianchi model, which are analytic in the triad and the spin connection for \(g_{IJ}\) a non-degenerate metric. We first introduce the matrix \(\bar{M}^I_J = (g^I_J)^{1/2}M^I_J\), where \(M^I_J \in SL(3, \mathbb{R})\) is the matrix which appears in (3.2) and leads to a diagonal form for the metric \(g_{IJ}\), and \(g_I\) are the components of the corresponding diagonal metric. The matrix \(\bar{M}\) satisfies the relations

\[
\det \bar{M} > 0, \quad ((\bar{M}^{-1})^t)^P g_{PQ} \bar{M}^{-1)^Q}_I = \delta_{IJ}, \tag{6.16}
\]

\[
((\bar{M}^{-1})^t)^P K_{PQ} \bar{M}^{-1)^Q}_I = \lambda_I \delta_{IJ}, \tag{6.17}
\]

with

\[
\lambda_I = \frac{\dot{g}_I}{2Ng_I}, \quad \det \bar{M} = (g_1g_2g_3)^{1/2}. \tag{6.18}
\]

One can use Eqs. (6.16,17) to determine \(\bar{M}\) in terms of \(g_{IJ}\) and \(K_{IJ}\). It is convenient to adopt the decomposition \(\bar{M} = \bar{R}\bar{M}_T\), with \(\bar{R} \in SO(3)\), and \(\bar{M}_T\) a lower-triangular matrix that includes diagonal elements different from the unity. Eq. (6.16) fixes \(\bar{M}_T\) analytically as a function of \(g_{IJ}\). On the other hand, the non-diagonal components of the system of equations (6.17) provide us with the matrix \(\bar{R}\) as an analytic function.
of \((g_{IJ}, K_{IJ})\), as far as \(\lambda_I \neq \lambda_J\) for all different \(I, J = 1, 2, 3\). From expressions (6.18) and Eqs. (3.7,8), one can easily check that this is indeed the case if \(p_+, p_-\), \((p_0 + 2p_+ - p_-) > 0\). Substituting the resulting matrix \(\bar{M}\), the diagonal components of the system (6.17) determine \(\lambda_I\) analytically in terms of \((g_{IJ}, K_{IJ})\). One can then identify \(\lambda_I\) and \(\det \bar{M}\) with their explicit expressions in our parametrization (through Eqs. (6.18)), and subsequently compute the parameters \(p_0, p_+, p_-\) and \(t\) as analytic functions of \((g_{IJ}, K_{IJ})\). Inserting these functions into the expressions (3.7) for \(g_I\), we obtain the matrix \(M^I_J = (g_I)^{-1/2} \bar{M}^I_J\), which turns out to depend analytically on \((g_{IJ}, K_{IJ})\). Finally, the parametrization used for \(M^I_J\) in (6.2) and (6.11) is well-defined and analytic, except at \(\beta = 0\) and \(\beta = \pi\), points that can be interpreted as the poles of the two-sphere coordinatized by \(\tilde{\alpha} = 2\alpha\) and \(\beta\). We consider then the non-analyticity at \(\beta = 0\) and \(\beta = \pi\) as a failure in the coordinatization of \(S^2\). In order to go to the reduced phase space, one has only to impose the scalar constraint (3.6) on \(p_0, p_+\) and \(p_-\), and restrict oneself exclusively to positive \(p_0\) (then, \((p_0 + 2p_+ - p_-) > 0\) is automatically satisfied).

**VII. Bianchi types I and II: Quantization**

Once we have identified the symplectic structures and the reduced configuration spaces for Bianchi I and II, we turn to the task of the canonical quantization of these models. Let us start with Bianchi type II. In this case the reduced phase space is the cotangent bundle over \(\mathcal{L}^+_{(+,+)} \times S^1\). Keeping the notation introduced in Sec. IV, we parametrize \(S^1\) by the angle \(\Theta\), while \(\mathcal{L}^+_{(+,+)}\) is defined by the constraint \(\Pi_0 = \sqrt{\Pi_+^2 + \Pi_-^2}\), with \(\Pi_0, \Pi_+, \Pi_- \in \mathbb{R}^+\). A natural set of elementary variables in this reduced configuration space is then provided by \((\Pi_+, \Pi_-, c \equiv \cos \Theta, s \equiv \sin \Theta)\). As generalized momentum variables we choose [9]

\[
L_+ = \Pi_+ U + \frac{\Pi_+^2}{\Pi_0} T, \quad (7.1)
\]

\[
L_- = \Pi_- V + \frac{\Pi_-^2}{\Pi_0} T, \quad (7.2)
\]

\[
L_\Theta = Z, \quad (7.3)
\]

where the momenta \((T, U, V, Z)\) are canonically conjugate to \((\Pi_0, \Pi_+, \Pi_-, \Theta)\). Our set of reduced phase space variables commute with the constraint (4.15), with respect to
the Poisson-brackets structure. On the other hand, the only non-vanishing Poisson-
brackets among the configuration and momentum variables are:

\[
\{ \Pi_+, L_+ \} = \Pi_+ , \quad \{ \Pi_-, L_- \} = \Pi_-, \quad \{ c, L_\Theta \} = -s, \quad \{ s, L_\Theta \} = c .
\]

So, under the Poisson-brackets, the chosen set of variables forms the Lie algebra
\[
L(T^* GL(1, \mathbb{R}) \times T^* GL(1, \mathbb{R}) \times E_2) \text{ where } T^*GL(1, \mathbb{R}) \equiv \mathbb{R} \oplus \mathbb{R}^+ \text{, } \oplus \text{ is the semi-direct product, and } E_2 = \mathbb{R}^2 \oplus SO(2) \text{ is the Euclidean group in two dimensions [14].}
\]

Our next step consists of finding a unitary irreducible representation of the Lie algebra
(7.4-6). We choose as our representation space the space of distributions \( \psi(\Pi_+, \Pi_-, \Theta) \) over \( L^+_\times S^1 \), and define on it a set of operators \( (\hat{\Pi}_\pm, \hat{c}, \hat{s}, \hat{L}_+, \hat{L}_-, \hat{L}_\Theta) \) such that their only non-vanishing commutators correspond to \( i\hbar \) times the Poisson-brackets (7.4-6). The action of these operators on \( \psi(\Pi_+, \Pi_-, \Theta) \) can be consistently defined in the form

\[
\begin{align*}
(\hat{\Pi}_+ \psi)(\Pi_+, \Pi_-, \Theta) &= \Pi_+ \psi(\Pi_+, \Pi_-, \Theta) , \\
(\hat{c} \psi)(\Pi_+, \Pi_-, \Theta) &= \cos \Theta \psi(\Pi_+, \Pi_-, \Theta) , \\
(\hat{s} \psi)(\Pi_+, \Pi_-, \Theta) &= \sin \Theta \psi(\Pi_+, \Pi_-, \Theta) , \\
(\hat{L}_+ \psi)(\Pi_+, \Pi_-, \Theta) &= -i\hbar \Pi_+ \frac{\partial}{\partial \Pi_+} \psi(\Pi_+, \Pi_-, \Theta) , \\
(\hat{L}_- \psi)(\Pi_+, \Pi_-, \Theta) &= -i\hbar \Pi_- \frac{\partial}{\partial \Pi_-} \psi(\Pi_+, \Pi_-, \Theta) , \\
(\hat{L}_\Theta \psi)(\Pi_+, \Pi_-, \Theta) &= -i\hbar \frac{\partial}{\partial \Theta} \psi(\Pi_+ \Pi_-, \Theta) .
\end{align*}
\]

The inner product in the space of quantum physical states can then be expressed as:

\[
< \phi | \psi > = \int_Q \mu \bar{\phi} \psi ,
\]

where \( Q \equiv \mathcal{L}^+_\times S^1 \), and the measure \( \mu \) is given by

\[
\mu = \frac{d\Pi_+}{\Pi_+} \frac{d\Pi_-}{\Pi_-} d\Theta .
\]

The measure \( \mu \) is simply the product of the measures that correspond to the unitary irreducible representations of the groups \( T^*GL(1, \mathbb{R}) \) (twice) and \( E_2 \) [14]. It is
straightforward to check that our operators are self-adjoint with respect to the scalar product (7.13). Furthermore, there is no nontrivial subspace in our representation space which remains invariant under the action of the operators (7.7-12). This proves that the constructed representation of the algebra (7.4-6) is unitary and irreducible.

Let us consider now the type I Bianchi model. In this case, the reduced configuration space can be identified with \( L^+_{(+,+)} \times S^2 \times S^1 \) (with certain restrictions still to be imposed in the space of physical states that come from the identification of points in (6.15)). The sphere \( S^2 \) is parametrized by the angles \( \tilde{\alpha} \) and \( \beta (\beta \in [0, \pi), \tilde{\alpha} \in [0, 2\pi)) \), the circle \( S^1 \) by the angle \( \theta \), and \( L^+_{(+,+)} \) is defined by the constraint \( p_0 = \sqrt{p_+^2 + p_-^2} \), with \( p_+, p_-, p_0 \in \mathbb{R}^+ \). From now on, we adopt the compact notation \( p = (p_+, p_-) \), \( \gamma = (\tilde{\alpha}, \beta, \theta) \). Given the form of the reduced configuration space, it is natural to choose the following over-complete set of configuration variables:

\[
(p_+, p_-, c, s, k_1, k_0, k_{-1}) ,
\]

where

\[
c \equiv \cos \theta , \quad s \equiv \sin \theta ,
\]

\[
k_1 \equiv Y^1_1(\tilde{\alpha}, \beta) , \quad k_0 \equiv Y^0_1(\tilde{\alpha}, \beta) , \quad k_{-1} \equiv Y^{-1}_1(\tilde{\alpha}, \beta) ,
\]

and \( Y^m_l(\tilde{\alpha}, \beta) \) are the spherical harmonics on the two-sphere [25]. All the functions in (7.15) are real, except \( k_1 \) and \( k_{-1} \), which satisfy

\[
(k_1)^*(\tilde{\alpha}, \beta) = -k_{-1}(\tilde{\alpha}, \beta) .
\]

In this case, our generalized momentum variables are

\[
L_+ = p_+ U + \frac{p_+^2}{p_0} T , \quad L_- = p_- V + \frac{p_-^2}{p_0} T ,
\]

\[
L_{\tilde{\alpha}} = \bar{F}_1 , \quad L_{(\tilde{\alpha}, \beta)} = e^{\pm i \tilde{\alpha}} \left( \pm i F_2 - \cot \beta \bar{F}_1 \right) , \quad L_\theta = F_3 ,
\]

with \( (T, U, V, \bar{F}_1, F_2, F_3) \) the momenta canonically conjugate to \( (p_0, p_+, p_-, \tilde{\alpha}, \beta, \theta) \). Following a similar discussion to that presented for Bianchi II, one can check that our set of variables forms the Lie algebra \( L(T^*GL(1, \mathbb{R}) \times T^*GL(1, \mathbb{R}) \times E_3 \times E_2) \) (with respect to the Poisson-brackets structure), where \( E_3 = \mathbb{R}^3 \otimes SO(3) \) is the Euclidean group in three dimensions [14]. We now proceed to find a unitary irreducible
representation of the corresponding Lie algebra of Dirac observables on the space of distributions \( \psi(p, \gamma) \) over the reduced configuration space \( \mathcal{L}^+_{(+,+)} \times S^2 \times S^1 \). Parallel to the situation in type II, the configuration operators that correspond to the variables (7.15) act as multiplicative operators on the chosen representation space, while the action of the momentum operators (7.18,19) can be defined by

\[
\begin{align*}
\left( \hat{L}_+ \psi \right)(p, \gamma) &= -i\hbar p_+ \frac{\partial}{\partial p_+} \psi(p, \gamma), \\
\left( \hat{L}_- \psi \right)(p, \gamma) &= -i\hbar p_- \frac{\partial}{\partial p_-} \psi(p, \gamma), \\
\left( \hat{L}_\theta \psi \right)(p, \gamma) &= -i\hbar \frac{\partial}{\partial \theta} \psi(p, \gamma), \\
\left( \hat{L}_\alpha \psi \right)(p, \gamma) &= -i\hbar \frac{\partial}{\partial \alpha} \psi(p, \gamma), \\
\left( \hat{L}_{(\alpha, \beta)}^\pm \psi \right)(p, \gamma) &= \hbar e^{\pm i\alpha} \left( \pm \frac{\partial}{\partial \beta} + i \cot \beta \frac{\partial}{\partial \alpha} \right) \psi(p, \gamma).
\end{align*}
\]

The only non-vanishing commutators in this algebra are

\[
\begin{align*}
[\hat{\Pi}_+, \hat{L}_+] &= i\hbar \hat{\Pi}_+ , \quad [\hat{\Pi}_-, \hat{L}_-] = i\hbar \hat{\Pi}_- , \\
[\hat{c}, \hat{L}_\theta] &= -i\hbar \hat{s}, \quad [\hat{s}, \hat{L}_\theta] = i\hbar \hat{c}, \\
[\hat{L}_\alpha, \hat{L}_{(\alpha, \beta)}^\pm] &= \pm \hbar \hat{L}_{(\alpha, \beta)}^\pm, \quad [\hat{L}_{(\alpha, \beta)}^+, \hat{L}_{(\alpha, \beta)}^-] = 2\hbar \hat{L}_\alpha , \\
[\hat{L}_\alpha, \hat{k}_{\pm 1}] &= \pm \hbar k_{\pm 1}, \quad [\hat{L}_{(\alpha, \beta)}^\pm, \hat{k}_m] = \hbar \sqrt{2 - m(m \pm 1)} \hat{k}_m ,
\end{align*}
\]

where \( m = 1, 0, -1 \). The inner product in the space of quantum states, without imposing the quantum analogue to the identification of points given by (6.15), takes on the expression

\[
< \phi | \psi > = \int_Q \mu \bar{\phi} \psi ,
\]

where \( Q = \mathcal{L}^+_{(+,+)} \times S^2 \times S^1 \), and the measure \( \mu \) is

\[
\mu = \frac{dp_+ dp_-}{p_+ - p_-} \sin \beta d\beta d\alpha d\theta .
\]

Note that all the Dirac observables are represented by self-adjoint operators, except \( \hat{k}_m \) and \( \hat{L}_{(\alpha, \beta)}^\pm \), which satisfy

\[
\begin{align*}
(\hat{k}_m) &= (-1)^m \hat{k}_{-m} , \\
(\hat{L}_{(\alpha, \beta)}^+) &= \hat{L}_{(\alpha, \beta)}^- .
\end{align*}
\]
Let us impose now the identification of points (6.15) in the reduced phase space as a restriction on the physical states \( \psi \in L^2(L^+_{(+,+)} \times S^2 \times S^1, \mu) \). In the representation that we have chosen, the required restriction can be stated as

\[
\psi(p, \tilde{\alpha}, \beta, \theta) = \psi(p, -\tilde{\alpha}, \pi - \beta, \theta + \pi).
\] (7.33)

The operators that correspond to the canonical variables \( \tilde{F}_1 \) and \( F_2 \) in the reduced phase space are clearly given, in our representation, by \( \hat{L}_{\tilde{\alpha}} \) and \( \hat{L}_{\theta} \), respectively. It is easy to check, using Eq. (7.33), that

\[
\left( \hat{L}_{\tilde{\alpha}} \psi \right)(p, \tilde{\alpha}, \beta, \theta) = -\left( \hat{L}_{\tilde{\alpha}} \psi \right)(p, -\tilde{\alpha}, \pi - \beta, \theta + \pi),
\] (7.34)

and similarly

\[
\left( \hat{L}_{\theta} \psi \right)(p, \tilde{\alpha}, \beta, \theta) = -\left( \hat{L}_{\theta} \psi \right)(p, -\tilde{\alpha}, \pi - \beta, \theta + \pi).
\] (7.35)

On the other hand, and apart from the factor ordering ambiguities that must be irrelevant in the classical limit, the variable \( F_2 \) in the reduced phase space can be represented by the symmetrized operator corresponding to \( L_\beta = -i(e^{-i\tilde{\alpha}}L_+ + \cot \beta L_{\tilde{\alpha}}) \), where \( e^{i\tilde{\alpha}} \) and \( \cot \beta \) must be expressed in terms of \( k_m, m = 1, 0, -1 \). It is then possible to show that

\[
\left( \hat{L}_\beta \psi \right)(p, \tilde{\alpha}, \beta, \theta) = -\left( \hat{L}_\beta \psi \right)(p, -\tilde{\alpha}, \pi - \beta, \theta + \pi) + o(\hbar).
\] (7.36)

Eqs. (7.33-36) guarantee that, in the classical limit, the points (6.15) are identified in the reduced phase space, recalling that \( \tilde{\alpha} = 2\alpha \) and, thus, \( \tilde{F}_1 = \frac{1}{2}F_1 \).

Therefore, the space of quantum physical states for the type I Bianchi model is simply the Hilbert subspace of functions in \( L^2(L^+_{(+,+)} \times S^2 \times S^1, \mu) \) that satisfy relation (7.33). Using Eqs. (7.29,30) and (7.33), the inner product in this space is easily computed to be

\[
<\phi|\psi> = 2 \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{dp_+}{p_+} \int_{\mathbb{R}^+} \frac{dp_-}{p_-} \int_0^{\pi} \sin \beta d\beta \int_0^{2\pi} d\tilde{\alpha} \int_0^\pi d\tilde{\alpha} \phi(p, \gamma)\psi(p, \gamma),
\] (7.37)

so that we can restrict our attention to the sector \( \theta \in [0, \pi) \), and identify the space of physical states with \( L^2(L^+_{(+,+)} \times S^2 \times [0, \pi), \mu) \). The restriction to this Hilbert
space of the set of operators previously defined in $L^2\left(L^+_{(+,+)} \times S^2 \times S^1, \mu\right)$ have then a well-defined action in the quantum physical states for Bianchi type I.

We have thus succeeded in quantizing the full type I and II Bianchi models following the non-perturbative canonical approach. The physical interpretation of the quantum theories so-constructed is completely analogue to that presented in Ref. [9] for the diagonal Bianchi models. The results will be discussed elsewhere [26].

**VIII. Bianchi Type I: An Alternative Quantization**

We want to discuss now a different approach to the quantization of the type I Bianchi model, making use of the symmetries that are present in the scalar constraint at the classical level [9]. We will show that the quantum theory obtained in this way is equivalent to the reduced phase space quantization of this Bianchi model.

We will restrict our attention to the sector of positive definite metrics. These metrics can be represented by real triads in $GL(3, \mathbb{R})$, which can be uniquely written in the form

$$E^I_a = (M_T)^I_J (M_D)^J_K R^{aK} (\gamma_L) ,$$  \hspace{1cm} (8.1)

with $R^{aK} (\gamma_L) \in SO(3)$, $\gamma_L (L = 1, 2, 3)$ the associated Euler angles (see Eq.(6.11)), and $M_T$ and $M_D$, respectively, an upper-triangular and a diagonal matrix. Let us introduce the following basis of generators for the upper-triangular and diagonal groups:

$$(T^1)_I^J = \delta^J_1 \delta^I_2 , \quad (T^2)_I^J = \delta^J_1 \delta^I_3 , \quad (T^3)_I^J = \delta^J_2 \delta^I_3 ,$$

$$(T^4)_I^J = \delta^J_1 - 3 \delta^J_2 \delta^I_3 , \quad (T^5)_I^J = \sqrt{3}(\delta^J_1 \delta^I_2 - \delta^J_2 \delta^I_1) , \quad (T^6)_I^J = 2 \delta^J_1 .$$  \hspace{1cm} (8.2)

Then, the matrices $M_T$ and $M_D$ can be uniquely expressed in terms of the exponentials of these generators as

$$M_T = \Pi_{i=1}^3 e^{x_i T_i} , \quad M_D = \Pi_{i=4}^6 e^{x_i T_i} .$$  \hspace{1cm} (8.4)

We can consider $x_i, i = 1, ..., 6,$ and $\gamma_L, L = 1, 2, 3,$ as a new set of configuration variables. Let us designate their canonically conjugate momenta by $p^i$ and $p^\gamma_L$, and perform a canonical transformation from the Ashtekar variables $(A^a_L, E^I_a)$ to the new
set \((x_i, \gamma_L, p^i, J^\gamma_L)\). Instead of dealing with \(p^\gamma_L\), it is more convenient to use the angular momenta \(J^\gamma_L\) of the Euler angles:

\[
J^\gamma_L = (Q^{-1})_L^K p^\gamma_K .
\]  

(8.5)

The matrix \(Q_L^K\) that appears in this equation is defined by means of the relation [27]

\[
\frac{\partial R^a_M}{\partial \gamma_L} = \epsilon^{P}_{\ M N} R^a_P Q^N_L ,
\]

(8.6)

which implicitly employs the fact that \(dR^a_M R^a_P\) is an antisymmetric one-form.

The spin connection \(A^a_I\) can be obtained by integrating the system of differential equations

\[
\{ A^a_I, E^J_b \} = -\frac{\partial A^a_I}{\partial p^i} \frac{\partial E^J_b}{\partial x_i} - \frac{\partial A^a_I}{\partial J^\gamma_L} (Q^{-1})_L^K \frac{\partial E^J_b}{\partial \gamma_L} = i \delta^J_I \delta^a_b .
\]

(8.7)

The solution to Eq. (8.7) turns out to be a complicated algebraic expression, although the calculations leading to it are relatively simple. We will proceed to discuss the conclusions that can be inferred from the result of these computations without displaying the explicit form of \(A^a_I(x_i, \gamma_L, p^i, J^\gamma_L)\).

We first note that, since the \(SO(3)\) connection \(\Gamma^I_a\) vanishes in Bianchi type I, \(A^a_I\) is purely imaginary. As a consequence, all the momenta \((p^i, J^\gamma_L)\) can be restricted to be real. On the other hand, substituting \(E^I_a\), given by (8.1-4), and \(A^a_I(x_i, \gamma_L, p^i, J^\gamma_L)\) in Eq. (2.9), the Gauss law constraints for Bianchi type I in the introduced set of canonical variables can be rewritten

\[
\mathcal{G}_a = \frac{i}{2} \epsilon_{abc} R^{bl} R^{cJ} J^\gamma_L \epsilon_I J^L \approx 0 ,
\]

(8.8)

from which we conclude that

\[
J^\gamma_L \approx 0 .
\]

(8.9)

The vector constraints (2.10) are empty for the type I Bianchi model. We are thus left only with the scalar constraint corresponding to (2.11):

\[
S = \left( \text{Tr} AE \right)^2 - \text{Tr} \left( AE \right)^2 .
\]

(8.10)
Substituting \((A^a, E^I_a)\) as functions of \(x_i, \gamma_L\) and \(p^i\) in (8.10) (with \(J^\gamma_L\) set equal to zero), we arrive at the expression

\[
S = -\frac{1}{6} (p^6)^2 + \frac{1}{6} (p^4)^2 + \frac{1}{6} (p^5)^2 + 2 \left( e^{6x_4 - 2\sqrt{3}x_5} (p^3)^2 + e^{6x_4 + 2\sqrt{3}x_5} (p^2)^2 + e^{4\sqrt{3}x_5} (p^1 - x_3 p^2)^2 \right).
\]

(8.11)

Eq. (8.11) can be considered as a quadratic constraint on the cotangent bundle over the six dimensional space coordinatized by the variables \(x_i\). We point out that, from Eq. (8.1), this space can be identified with the space of positive definite metrics, since

\[
g^{IJ}(x_i) = (\det E)^{-1} E^I_a E^J_a = e^{-6x_6} (MT)^I_K (MD)^K_L (MD)^P_L (MT)^J_P.
\]

(8.12)

The quadratic form that appears in (8.11) endows this space with the natural metric [22]:

\[
ds^2 = -6(dx_6)^2 + G^{ij} d\tilde{x}_i d\tilde{x}_j,
\]

(8.13)

where \(\tilde{x}_i = x_i, i = 1, \ldots, 5\), and

\[
G^{ij} = 6(dx_4)^2 + 6(dx_5)^2 + \frac{1}{2} \left( e^{-6x_4 + 2\sqrt{3}x_5} (dx_3)^2 + e^{-4\sqrt{3}x_5} (dx_1)^2 + e^{-6x_4 - 2\sqrt{3}x_5} (dx_2 + x_3 dx_1)^2 \right).
\]

(8.14)

The coordinate \(x^6\) plays then the role of a time, and \(\partial_6\) is a global time-like Killing vector of the metric (8.13). From Eqs. (8.12) and (8.2-4), the five dimensional space coordinatized by \(\tilde{x}_i\) is simply the space of positive definite metrics of unit determinant. Following now a completely similar analysis to that carried out by Henneaux, Pilati and Teitelboim in Ref. [22], it is possible to prove that this five dimensional space, provided with the metric (8.14), can be identified with the coset space \(SL(3, \mathbb{R})/SO(3)\) [28]. Furthermore, they showed that, in the metric representation, the quantum states for Bianchi type I can be decomposed in the form

\[
f(g) = \int_{\mathbb{R}} d\lambda_+ \int_{\mathbb{R}} d\lambda_- \mu(\lambda) \int_0^\pi d\alpha \int_0^\pi \sin \beta d\beta \int_0^\pi d\theta \tilde{f}(\lambda, \gamma)e_{\lambda, R(\gamma)}(g),
\]

(8.15)

where \(\lambda = (\lambda_+, \lambda_-)\), \(\mu(\lambda)\) is a specific measure over \(\mathbb{R}^2\), \((\alpha, \beta, \theta) \equiv (\gamma_L)\) are the Euler angles that parametrize the matrices \(R(\gamma_L) \in SO(3)\), and \(e_{\lambda, R(\gamma)}(g)\) are generalized
“plane waves” which satisfy the operator constraint associated to the classical constraint (8.11). For any matrix $R(\gamma_L) \in SO(3)$, $e_{\lambda,R(\gamma_L)}(g)$ is defined as the “plane wave” of the rotated metric $g$: $e_{\lambda,R(\gamma)}(g) = e_{\lambda,I}(RgR^t)$, with $e_{\lambda,I}(g)$ given in the parametrization (8.2-4) by the expression

$$e_{\lambda,I}(g) = e^{-i\lambda_0(\lambda)x_6 + (i\lambda_++3)x_4 + (i\lambda_-+\sqrt{3})x_5}.$$ (8.16)

The “plane waves” (8.16) are eigenfunctions of the momenta operators $\hat{p}_4, \hat{p}_5$ and $\hat{p}_6$, defined as self-adjoint with respect to the metric (8.13,14) [22]. The respective eigenvalues are $\lambda_+, \lambda_-$ and $\lambda_0$. In order to fulfill the operator constraint corresponding to (8.11), $(\lambda_0, \lambda_+, \lambda_-)$ must satisfy the relation

$$-\lambda_0^2 + \lambda_+^2 + \lambda_-^2 = 0 ,$$ (8.17)

which is a direct analogue of the scalar constraint (3.6). In fact, it is possible to check (from Eqs. (8.2-4), (8.12), and (3.4)) that the set of parameters $(\beta_0, \beta_+, \beta_-)$ employed in Sec. III to describe the diagonal metrics for the type I model are related to the coordinates $(x^4, x^5, x^6)$ by means of the transformation

$$\beta_0 = \frac{1}{\sqrt{3}}(2x^6 + x^4) + C_1, \quad \beta_+ = \frac{1}{\sqrt{3}}(x^6 + 2x^4) + C_2, \quad \beta_- = -x^5 + C_3,$$ (8.18)

with $C_1, C_2, C_3$ some unspecified constants. The canonically conjugated variables to the $\beta$’s, $(p_0, p_+, p_-)$, used throughout our analysis of the Bianchi type I, can then be obtained from $(\lambda_0, \lambda_+, \lambda_-)$ by completing the canonical transformation (8.18):

$$p_0 = \frac{1}{\sqrt{3}}(2\lambda_0 + \lambda_+), \quad p_+ = \frac{1}{\sqrt{3}}(-\lambda_0 + 2\lambda_+), \quad p_- = -\lambda_-.$$ (8.19)

If we want to consider exclusively inequivalent spacetime geometries, and not only different positive definite spatial metrics, we have to restrict the range of $p_0, p_+$ and $p_-$ to lie in the positive real axis, as we proved in Sec. VI. Changing coordinates from $(\lambda_0, \lambda_+, \lambda_-)$ to $(p_0, p_+, p_-)$, imposing that $p_0 = \sqrt{p_+^2 + p_-^2}$, and restricting $p_+, p_- \in \mathbb{R}^+$, we arrive, from (8.15), at the following decomposition for the quantum states of the type I spacetime geometries:

$$F(g) = \int_{\mathcal{L}_{(+,+)}^+} dp_+ dp_- \tilde{\mu}(p) \int_0^\pi d\alpha \int_0^\pi \sin \beta d\beta \int_0^\pi d\theta \tilde{F}(p, \gamma)e_{p,R(\gamma)}(g).$$ (8.20)
where \( p \equiv (p_+, p_-), \tilde{\mu}(p) \) is certain measure over \( L_{(+,+)}^+ \) and \( e_{p,R(\gamma)}(g) \) are the generalized “plane waves” expressed in terms of \( p_+, p_- \) and \( \gamma \). The wave function \( \tilde{F}(p, \gamma) \) characterizes uniquely the quantum state \( F(g) \) [22], so that, instead of the metric representation, we can select the representation \((p, \gamma)\) for the quantization of the model. The decomposition (8.20) induces the following inner product:

\[
< \tilde{F}, \tilde{G} > = \int \int \int \int dp^+ dp^- \tilde{\mu}(p) \int_0^\pi d\alpha \int_0^\pi \sin \beta d\beta \int_0^\pi d\theta \tilde{F}^*(p, \gamma) \tilde{G}(p, \gamma). \quad (8.21)
\]

It is obvious that the representation \((p, \gamma)\) coincides with that used in Sec. VII for the quantization of Bianchi I. Furthermore, the inner product (8.21) in the space of quantum states differs only in the choice of the measure \( \tilde{\mu}(p) \) from that determined in (7.37). As a consequence, we conclude that the two quantum theories here discussed for Bianchi type I result to be unitarily equivalent.

**IX. Conclusions**

We have succeeded in completing the canonical quantization of the type I and II Bianchi models while keeping the totality of degrees of freedom of these homogeneous gravitational minisuperspaces. Our analysis generalizes the works existing so far in the literature [9,10], which had concentrated their attention on the diagonal reduction of these systems.

We have first calculated the explicit expressions of the general solution for these two Bianchi types, using the fact that the classical evolution problem can always be brought to the diagonal form by a change in the set of left-invariant one-forms on the leaves of the homogeneous foliation. This is the first time, to our knowledge, that the general solution for Bianchi II has been explicitly displayed.

The classical solutions have been written in both geometrodynamical and Ashtekar variables, restricted to the sector of positive definite metrics. We have determined the sets of solutions that correspond to different spacetime geometries, eliminating the overcounting of physical states. For both types I and II, the parameters used to describe the relevant non-degenerate physical solutions have been shown to define a good (analytic) set of coordinates in the phase spaces of these models. The presence of the non-diagonal degrees of freedom plays an essential role in the proof of this
statement. If we had performed a similar analysis in the reduced diagonal models, the phase space coordinates associated to the diagonal degrees of freedom could have been extended to a wider range of analyticity, because, in this case, one can consistently consider as fixed the set of left-invariant one-forms in the homogeneous foliation. The physical states of the quantum theories constructed thereafter for the diagonal models can be interpreted as dependent on three-geometries with some preferred directions. In our description, however, one is forced to treat the directions in the homogeneous slices as being interchangeable, as far as this is allowed by the symmetries of the Bianchi model. Apart form the maintenance of the extra degrees of freedom, this is the main difference that arises in the study of the full Bianchi types I and II with respect to the analysis of their diagonal counterparts. The constraints of the systems are identical for both kinds of models, since the whole classical evolution can always be put into diagonal form.

Performing a transformation from the Ashtekar variables to the sets of phase space coordinates introduced for Bianchi I and II, we have endowed the reduced phase space of each of these minisuperspaces with an analytic symplectic structure. For the type II Bianchi model, the symplectic form obtained in this way can be identified with that of the real cotangent bundle over the reduced configuration space \( L^{+}(+,+) \times S^{1} \); for type I, the symplectic form corresponds to the real cotangent bundle over \( L^{+}(+,+) \times S^{2} \times S^{1} \), with an additional identification of points in \( S^{2} \times S^{1} \) that we have chosen to impose at the quantum level.

We have proceeded to quantize the models by selecting a complete and closed \(*\)-algebra of Dirac observables in the reduced phase space of each of these systems. We have then constructed an explicit unitary irreducible representation of that algebra on the space of distributions in the reduced configuration space. In this way, the \(*\)-relations in the algebra of observables have been straightforwardly implemented in the adopted representation as adjointness relations among the quantum operators and the inner product in the space of quantum states has been subsequently fixed. The physical interpretation of the mathematical framework obtained here is essentially the same as that presented in Ref. [9] for the diagonal reduction of the Bianchi models.

Finally, we have outlined a different approach to the quantization of Bianchi type I,
using the existence of a conditional symmetry [29] to span the quantum states in terms of generalized “plane waves”. This alternative approach may be useful for minisuperspace models whose algebra of Dirac observables is not sufficiently known, so that the canonical quantization program cannot be yet applied to completion in these cases. For Bianchi I, the two quantum theories analyzed in this work have been shown to be unitarily equivalent.

We have thus provided the full type I and II Bianchi models with the framework needed to address cosmological problems from the quantum point of view. Our study also illuminates the role played by the additional non-diagonal degrees of freedom, which turn out to be frozen in the Hamiltonian description. Let us finally point out that, from our discussion in Sec. VII, the Hilbert subspace of states that are separable as a function in $\mathcal{L}^+_{(+,+)}$ times a function in the “non-diagonal” part of the reduced configuration space can be immediately associated with quantum physical states for the corresponding diagonal case, by simply neglecting the dependence on the non-diagonal configuration variables. One can thus analyze the implications of the quantum reduction of degrees of freedom for these two kinds of minisuperspace models, as toy models in which one can check the possible validity of the minisuperspace approximation in the full quantum theory of gravity [30].

Acknowledgements

The authors are greatly thankful to A. Ashtekar and J. Louko for helpful discussions and useful comments, and to D. Marolf for suggesting some corrections to the original manuscript. N. Manojlović would also like to thank C. Isham, who initially proposed the subject of this work. He gratefully acknowledges the financial support provided by the Relativity Group at Syracuse University, as well as their warm hospitality. G. A. Mena Marugán was supported by the Spanish Ministry of Education and Science Grant No. EX92 06996911. He is grateful for the hospitality and partial support of the Department of Physics at Syracuse University.

References


