Invariant connections with torsion on group manifolds and their application in Kaluza-Klein theories

Kubyshin Yu.A.

Departament d’Estructura i Constituents de la Matèria
Universitat de Barcelona
Av. Diagonal 647, 08028 Barcelona, Spain

Malyshenko V.O. and Marín Ricoy D.
Nuclear Physics Institute, Moscow State University,
Moscow 119899, Russia

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Abstract

Invariant connections with torsion on simple group manifolds $S$ are studied and an explicit formula describing them is presented. This result is used for the dimensional reduction in a theory of multidimensional gravity with curvature squared terms on $M^4 \times S$. We calculate the potential of scalar fields, emerging from extra components of the metric and torsion, and analyze the role of the torsion for the stability of spontaneous compactification.
1 Introduction

The ideas of Kaluza and Klein about possible multidimensional nature of our space-time were formulated in the twenties [1] and nowadays are regarded as an important ingredient of many supergravity and superstrings theories [2] and other schemes of unification of interactions.

In the framework of this approach it is common to assume that the multidimensional space-time is of the form $M^4 \times I$, where $M^4$ is the macroscopic four-dimensional part of the space-time and $I$ is a compact space of extra dimensions often called internal space. The compactification of extra dimensions should occur spontaneously, as a solution of the equations of motion, which also determine the size $L$ of $I$ after compactification. In most of the schemes $L \sim L_{Pl} \sim 10^{-33}$ cm.

In the standard Kaluza-Klein approach the bosonic sector of the multidimensional theory includes only pure Einstein gravity, and the electromagnetic and/or non-abelian gauge fields and scalar fields emerge from the extra components of the multidimensional metric after reduction of the theory to the four-dimensional space-time [3]. However, there are some difficulties in this approach. One of them is due to the fact that there are no vacuum solutions with the structure of the space-time $M^4 \times K/H$, where $M^4$ is the Minkowski space-time and $K/H$ is a homogeneous space with non-abelian isometry group $K$, which is necessary for obtaining non-abelian gauge fields after the dimensional reduction. Another difficulty is related to impossibility to obtain chiral fermions in the dimensionally reduced theory. It was found [4] that in order to solve these problems one can generalize the standard Kaluza-Klein approach by adding gauge fields to the multidimensional Lagrangian (see [5], [6] and [7] for reviews). Later other types of generalizations were proposed. One of them is to remain within the pure gravity but to add terms quadratic in the curvature tensor components.

An interesting possibility in the framework of the latter generalization is to consider a model with non-zero torsion. This option was investigated in a series of papers [8] - [11]. There is a big literature on the theory of gravity in the Riemann-Cartan space-time with torsion (see [11] for a review and an extensive list of references on the subject). It is well known that torsion is a non-dynamical variable in the pure Einstein - Cartan theory [12], but it becomes dynamical, if one adds quadratic in curvature terms to the standard Einstein Lagrangian. Such terms are also motivated by the quantum field theory limit of strings [2], [13] - [15]. Problem of spontaneous compactification in multidimensional theories with torsion was investigated in [10], [16] - [18].

The aim of the present paper is to give a description of the class of invariant connections with torsion compatible with the invariant metric on group manifolds $S$, where $S$ is a simple Lie group, and then apply this result to investigation of spontaneous compactification in multidimensional gravity with $R^2$-terms on the space-time $M^4 \times I$ with $I = S$. We should notice that description of invariant connections with torsion, in the case when the internal space $I$ is a homogeneous space $K/H$ from a certain class, was obtained in [19] and we will borrow some of the methods from this paper for our analysis here. But due to some specific features of group manifolds the case of $I = S$ is not covered by the results of [19] and needs a special treatment.

The paper is organized as follows. In section 2 we give a brief description of $K$-invariant metric compatible linear connections on homogeneous spaces $K/H$. In section 3 we consider...
the case of group manifolds $S$. We prove a theorem about decomposition of antisymmetric square of the adjoint representation of a simple Lie algebra, which will enable us to construct the invariant connections on $S$ explicitly and to calculate the components of the multidimensional curvature tensor. In section 4 we consider multidimensional gravity with $R^2$-terms and derive the potential of scalar fields of the dimensionally reduced theory. We analyze this potential and make conclusions about stability of spontaneous compactification in the theory.

2 General properties of invariant connections on homogeneous spaces

In the present section we present the main results from the theory of invariant connections on homogeneous spaces $I = K/H$. They will be used in the next section for explicit construction of such connections in the case when $I$ has the structure of a simple Lie group. Our considerations are based on [20].

Let us consider the principle fibre bundle $O(M)$ with the structure group $SO(d)$, where $d = \dim M$, of orthonormal frames over $M = K/H$ with reductive decomposition of the Lie algebra $K = \text{Lie}(K)$ of the group $K$: $K = \mathcal{H} \oplus \mathcal{M}, \text{ad}H(\mathcal{M}) \subset \mathcal{M}$, where $\mathcal{H} = \text{Lie}(H)$. The group $K$ acts transitively on the base $M$ and induces a natural automorphism on the bundle $L(M)$. We will be interested in metrics $g$ on $M$ and metric compatible connections $\omega$ on the bundle $L(M)$, which are invariant under the action of the group $K$. The construction of $K$-invariant connections on homogeneous spaces is based on the Wang theorem [20]. It states that there is 1-1 correspondence between $K$-invariant connections on the bundle $L(M)$ and linear mappings $\Lambda : \mathcal{M} \to so(d) = \text{Lie}(SO(d))$, which satisfy the following condition

$$\Lambda(\text{Ad}h(X)) = \text{Ad}(\lambda(h))(\Lambda(X)), \quad X \in \mathcal{M}, h \in H,$$

where $\text{Ad}$ denotes the adjoint representation of $H$ and $\lambda$ is the homomorphism $\lambda : H \to SO(d)$ induced by the action of $H$ on the tangent space $T_o(K/H)$ (the corresponding homomorphism of algebras is also denoted by $\lambda$). In terms of the mapping $\Lambda$ the formulas for the invariant torsion and curvature tensors at the point $o = [H]$ acquire simple form, namely

$$T_o(X,Y) = \Lambda(X)Y - \Lambda(Y)X - [X,Y]_{\mathcal{M}}, \quad R_o(X,Y) = [\Lambda(X),\Lambda(Y)] - \Lambda([X,Y]_{\mathcal{M}}) - \lambda([X,Y]_{\mathcal{H}}), \quad X,Y \in \mathcal{M}.$$ 

Here we identified the tangent space $T_o(K/H)$, $\mathcal{M}$ and $R^d$. Notice also that $so(d) \cong R^d \wedge R^d \cong \mathcal{M} \wedge \mathcal{M}$. Let us introduce the mapping $\beta : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ by the formula $\beta(X,Y) = \Lambda(X)Y$. The connection form $\omega$ can be decomposed into the sum of the Levi-Civita connection form $^0\omega$ with zero torsion and the contorsion form $^0\bar{\omega}$. This leads to the corresponding decomposition for $\Lambda = ^0\Lambda + ^0\bar{\Lambda}$ and $\beta = ^0\beta + ^0\bar{\beta}$. For reductive homogeneous spaces the expression for $^0\bar{\Lambda}$ was obtained by Nomizu (see [20]). It is given by the formula

$$^0\beta(X,Y)\equiv ^0\Lambda(X)Y = \frac{1}{2}[X,Y]_{\mathcal{M}} + ^0\bar{U}(X,Y), \quad X,Y \in \mathcal{M},$$
where 0 is a symmetric bilinear mapping, 0: M ⊗ M → M. We will return to it shortly.

K - invariant metrics g on K/H are known to be in 1 - 1 correspondence with adH - invariant bilinear forms B on M, namely \( g_0(X, Y) = B(X, Y), \quad X, Y \in M \cong T_0(K/H). \) The invariance of B with respect to adH means

\[
B([A, X], Y) + B(X, [A, Y]) = 0, \quad X, Y \in M, A \in H.
\]

It is easy to verify that the condition of compatibility of the invariant connection form \( \omega \) with the invariant metric g can be written as:

\[
B(\beta(X, Y), Z) + B(Y, \beta(Z, X)) = 0, \quad X, Y, Z \in M.
\]

We wish to construct the form \( \bar{\beta} \), which describes nonzero torsion. We can represent it as the sum of the symmetric and antisymmetric parts: \( \bar{\beta} = \bar{\beta}_{s} + \bar{\beta}_{as} \) with \( \bar{\beta}_{s}(X, Y) = \bar{\beta}_{s}(Y, X) \) and \( \bar{\beta}_{as}(X, Y) = -\bar{\beta}_{as}(Y, X) \). Combining the condition of metric compatibility (6) with two other formulas obtained from (6) by the cyclic permutation of \( X, Y \) and \( Z \) it is easy to derive the following relation between the symmetric part \( \beta_{s}(X, Y) = 0 \) and the antisymmetric part \( \beta_{as}(X, Y) \):

\[
B(\beta_{s}(X, Y), Z) = B(X, \beta_{as}(Y, Z)) + B(Y, \beta_{as}(Z, X)), \quad X, Y, Z \in M.
\]

Thus, if we construct all mappings \( \bar{\beta}_{as} \), we will be able to find all invariant connections on K/H using eq. (7). Notice that \( \bar{\beta}_{as} \) can be considered as a mapping from \( M \land M \) into \( M \):

\[
\bar{\beta}_{as} : M \land M \rightarrow M.
\]

The condition (8) can be rewritten for \( \bar{\beta}_{as} \) in the infinitesimal form as follows:

\[
\bar{\beta}_{as}(adA \land 1 + 1 \land adA)(\xi) = adA(\bar{\beta}_{as}(\xi)), \quad \xi \in M \land M, A \in H.
\]

This enables us to consider \( \bar{\beta}_{as} \) as an intertwining operator, which intertwines equivalent representations of the algebra \( H \) in the linear spaces \( M \land M \) and \( M \). Thus, the general scheme of construction of the operator \( \bar{\beta}_{as} \) is the following (9). We decompose linear spaces \( M \land M \) and \( M \) into subspaces carrying irreducible representations (irreps) of the algebra \( H \)

\[
M \land M = \sum U_k, \quad M = \sum V_n.
\]

According to Schur’s lemma, the operator \( \bar{\beta}_{as} \) is equal to \( \bar{\beta}_{as} = \sum f_{kn} \beta_{kn} \), where \( \beta_{kn} \) is the unit operator establishing the isomorphism between the subspaces \( U_k \) and \( V_n \) if they carry equivalent irreps and \( \beta_{kn} = 0 \) otherwise. Similar intertwining operators appear in the coset space dimensional reduction of multidimensional Yang-Mills theories. See [6], [7] for the discussion of the problem of the construction of such operators in gauge theories.

In order to illustrate the general scheme of calculation of \( \bar{\beta}_{as} \) let us consider two examples.

1. \( K/H = G_2/SU(3) \). From the results in [21], [22] we have after complexification

\[
\text{ad}H(M) = 3 \oplus 3^*, \quad \text{ad}H(M \land M) = 8 \oplus 3 \oplus 3^* \oplus 1,
\]

where 8 is the adjoint representation of \( H \). We see that there are only two irreps, 3 and 3*, which enter both decompositions. Therefore, the intertwining operator is of the form:
\[ \beta_{as} = f_{33} \beta_{33} + f_{3*3*} \beta_{3*3*}, \] where \( f_{33} \) and \( f_{3*3*} \) are arbitrary complex parameters. The reality condition for \( \beta_{as} \) implies \( (f_{33})^* = f_{3*3*} \).

2. \( K/H = (SU(3) \times SU(3))/\text{diag}(SU(3) \times SU(3)) \cong SU(3) \). In this example

\[
\begin{align*}
\text{ad}H(M \wedge M) &= \text{ad}H \oplus 10 \oplus 10^*, \quad \text{ad}H = 8, \\
\text{ad}H(M) &= \text{ad}H.
\end{align*}
\]

There is only one irrep which enters both decompositions. Therefore, the intertwining operator has the form \( \bar{\beta}_{as} = f \beta_{88} \), i.e. only one real contorsion field exists. This example illustrates the case we are interested in, namely the case of group manifolds represented as a homogeneous space.

More examples of construction of the contorsion form as the intertwining operator are given in [19].

To conclude the section we note that the structure of the algebra \( K \) of reductive spaces admits two natural intertwining operators: \( \phi : M \wedge M \to M \) and \( \psi : M \wedge M \to H \). They are given by

\[
\begin{align*}
\phi(X \wedge Y) &= [X, Y]_M, \\
\psi(X \wedge Y) &= [X, Y]_H, \quad X, Y \in M.
\end{align*}
\]

These operators will be used in the next section.

### 3 Construction of invariant connections on group manifolds

One of the aims of the present paper is to investigate in detail the case of the group manifolds \( S \), represented as a homogeneous space \( S = K/H \) with \( K = S \times S \) and \( H = \text{diag}(S \times S) \).

There are three natural reductive decompositions for \( K \) [20], namely \( K = H \oplus M \) with \( M = M_0, M_+, M_- \),

\[
\begin{align*}
M_0 &= \{(X/2, -X/2), \ X \in S\}, \\
M_+ &= \{(0, -X), \ X \in S\}, \\
M_- &= \{(X, 0), \ X \in S\},
\end{align*}
\]

with \( H = \text{Lie}(H), S = \text{Lie}(S) \). Obviously, \( M \cong H \cong S \). The \( (0) \) - decomposition (the first decomposition in [12]) corresponds to the case when \( S \) is represented as a symmetric space. The subspaces \( M \) and \( H \) carry the adjoint representation of \( S \) only, therefore, to construct the mapping \( \bar{\beta}_{as} \) in this case we must study the decomposition of \( \text{ad}S \wedge \text{ad}S \) into irreps. The fact, that the \( \text{ad}S \) is contained in \( \text{ad}S \wedge \text{ad}S \) at least once is guaranteed by the existence of the non-trivial intertwining operator \( \phi \) (see (10)) in the case of \( (\pm) \) - decomposition and the operator \( j \circ \psi \) (see (11)), where \( j \) is the isomorphism \( H \to M \), for \( (0) \) - decomposition.

In fact, we can prove the following

**Theorem.** For simple Lie algebras \( S \) the decomposition of \( \text{ad}S \wedge \text{ad}S \) into irreps of \( S \) has one of the following forms

\[
\begin{align*}
\text{ad}S \wedge \text{ad}S &= \text{ad}S \oplus \gamma \oplus \gamma^*, \quad \text{for } S = A_n, \\
\text{ad}S \wedge \text{ad}S &= \text{ad}S \oplus \gamma, \quad \text{for other simple Lie algebras},
\end{align*}
\]

(13)
where $\gamma$ and $\gamma^*$ are irreps different from $adS$. Namely,
\[
\dim \gamma = n(n - 1)(n + 2)(n + 3)/4 \quad \text{for } S = A_n,
\]
\[
\dim \gamma = n(n - 1)(2n - 1)(2n + 3)/2 \quad \text{for } S = B_n \text{ and } C_n,
\]
\[
\dim \gamma = n(n + 1)(2n - 1)(2n - 3)/2 \quad \text{for } S = D_n,
\]
\[
\dim \gamma = 77 \quad \text{for } G_2, 1274 \quad \text{for } F_4, 2925 \quad \text{for } E_6, 8645 \quad \text{for } E_7 \text{ and } 30380 \quad \text{for } E_8.
\]

Proof. For the proof of the theorem it is convenient to complexify $\mathcal{M}$ and $\mathcal{H}$ and apply results from the theory of Lie algebras [23,24]. Here we will present the proof for the case $S = A_n$, other cases can be proved in a similar way. The main idea is rather simple. As it has been mentioned above the adjoint representation $adS$ is contained in $adS \wedge adS$ at least once. We will find another irrep $\gamma$ contained in $adS \wedge adS$, calculate its dimension $\dim \gamma$ and show that
\[
\dim(adA_n \wedge adA_n) = \dim adA_n + \dim \gamma + \dim \gamma^*.
\] (14)

We will also see that $\dim \gamma \neq \dim A_n$, and this will complete the proof of the theorem for $S = A_n$. The important tool in the proof is Weil’s formula for the dimension of an irrep [23]. It is known that any irrep $\gamma$ of Lie algebra $S$ is characterized by its highest weight $\Omega = (\Omega_1, \ldots, \Omega_n)$, where $\Omega_i$ are the Dynkin coefficients of $\Omega$ with respect to a system of the simple roots $\{\alpha_i, \ i = 1, \ldots, n = \text{rank } S\}$ of $S$, i.e. $\Omega = \sum_{i=1}^n \Omega_i \alpha_i$. Weil’s formula states that the dimension of the irrep $\gamma(\Omega)$ is equal to
\[
\dim \gamma(\Omega) = \frac{\sum_{\alpha > 0} \sum_i K_i(\alpha)(1 + \Omega_i) \langle \alpha_i, \alpha_i \rangle}{\sum_i K_i(\alpha) \langle \alpha_i, \alpha_i \rangle},
\]
where the first sum goes over all positive roots $\alpha_i$ of $S$ and $K_i(\alpha)$ are the coefficients of the root $\alpha$ with respect to $\{\alpha_i\}$, $\alpha = \sum_{i=1}^n K_i(\alpha) \alpha_i$. Here $\langle \cdot, \cdot \rangle$ is the canonical scalar product in the space dual to the Cartan subalgebra of $S$ induced by the non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle$ in $S$ (proportional to the Killing form). Nonzero weights of the adjoint representation $adS$ are roots of the algebra $S$. Our first step is to find any irrep $\gamma$ in the decomposition of $adS \wedge adS$. There is a general procedure of finding the so called highest irrep in the antisymmetric tensor product (see for example ref. [22]). Let us denote by $\Omega_{ad}$ the heighest weight of $adS$ and by $\tilde{\Omega}_{ad}$ one of the next to the heighest weights, i.e. the weight which is obtained from $\Omega_{ad}$ by subtraction of one of the simple roots $\alpha_i$ of $S$. Then the highest weight $\Omega$ of the highest irrep in $adS \wedge adS$ is given by the formula $\Omega = \Omega_{ad} + \tilde{\Omega}_{ad}$.

For $S = A_n$ there are two next to the highest weights. Therefore, two highest irreps $\gamma_1$ and $\gamma_2$ in $adS \wedge adS$ exist. Their highest weights are $\Omega_1 = (0, 1, 0, \ldots, 0, 2)$ and $\Omega_2 = (2, 0, \ldots, 0, 1, 0)$. It is known for $S = A_n$ that if two irreps have the highest weights $\Omega_1 = (a_1, \ldots, a_n)$ and $\Omega_2 = (a_n, \ldots, a_1)$, then they are conjugate to each other. Thus, $\gamma_2 = \gamma_1^*$.

Weil’s formula gives
\[
\dim \gamma_1 = \frac{n(n - 1)(n + 2)(n + 3)}{4}
\]

Now we calculate $\dim S = \dim (adS)$ and $\dim (adS \wedge adS)$:
\[
\dim (adS) = n(n + 2), \quad \dim (adS \wedge adS) = \frac{n(n + 2)(n(n + 2) - 1)}{2}.
\]

We see immediately that $\dim \gamma_1 \neq \dim (adS)$, therefore $\gamma_1$ and $\gamma_1^*$ are not equal to the adjoint representation, and can check easily that the formula (14) is true. This finishes the proof for the case $S = A_n$.$\square$. 


This theorem is the development of the known result stating that \( adS \) is always contained in the tensor product \( adS \otimes adS \) \cite{22,24,25}. Using the theorem proved above we can construct the mapping \( \tilde{\beta}_{as} \), satisfying the intertwining condition \((8)\), explicitly. Indeed, Schur’s lemma implies that \( \tilde{\beta}_{as} \) must be proportional to the intertwining operator \( \phi \) or \( j \circ \psi \) (see \cite{11}, \cite{11}) and the result \( (13) \) guarantees that there are no other intertwining operators mapping from \( \mathcal{M} \wedge \mathcal{M} \) into \( \mathcal{M} \). Thus, we have

\[
\tilde{\beta}_{as}(\tilde{X} \wedge \tilde{Y}) = \frac{f}{2}[\tilde{X},\tilde{Y}]_{\mathcal{M}} \quad \text{for } (\pm) \text{- decomposition},
\]

\[
\tilde{\beta}_{as}(\tilde{X} \wedge \tilde{Y}) = 2f \circ j([\tilde{X},\tilde{Y}]_{\mathcal{H}}) \quad \text{for } (0) \text{- decomposition},
\]

where \( \tilde{X}, \tilde{Y} \in \mathcal{M} \) and \( f \) is an arbitrary real parameter.

As for the symmetric part \( \beta_s \), we can show using eq.\((7)\) that \( \beta_s \) is identically zero. To do this we take an \( adK \) - invariant bilinear form \( B(\tilde{X},\tilde{Y}) = \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle \) on the Lie algebra \( K \) of \( K \). Here \( \tilde{X} = (X_1, X_2) \), \( \tilde{Y} = (Y_1, Y_2) \), \( \tilde{X}, \tilde{Y} \in \mathcal{M}, \ X_i, Y_i \in \mathcal{S} \), and as in the proof of the theorem \( \langle \cdot , \cdot \rangle \) is an \( adS \) - invariant bilinear symmetric form on \( \mathcal{S} = \text{LieS} \), which in our case is proportional to the bi - invariant metric \( g \) on \( S \) and to the Killing form. We see now that for \( \beta_{as} \), given by eq. \((13)\), the r.h.s. of \((7)\) vanishes, thus \( \beta_s = 0 \).

Finally, the mappings \( \Lambda \), corresponding to the invariant connection with torsion on the group manifold \( S \), form a 1 - parameter family given by

\[
\Lambda(\tilde{X})\tilde{Y} = \frac{1+f}{2} [\tilde{X},\tilde{Y}]_{\mathcal{M}}, \quad \text{for } (\pm) \text{- decomposition},
\]

\[
\Lambda(\tilde{X})\tilde{Y} = 2f j([\tilde{X},\tilde{Y}]_{\mathcal{H}}) \quad \text{for } (0) \text{- decomposition}.
\]

The mapping \( \Lambda \) with \( f = 0 \) corresponds to the Levi-Civita connection with zero torsion on \( S \) (see eq.\((13)\)). When \( f = -1 \) for \((\pm) \) - decomposition and \( f = 0 \) for \((0) \) - decomposition \( \Lambda \) describes the canonical connection. Notice that for \((0) \) - decomposition canonical connection coincides with the Levi-Civita connection. We would like to underline here that we constructed non-trivial invariant connection for the case of the \((0) \) - decomposition in \((12)\) when the group manifold \( S \) is represented as a symmetric homogeneous space \( K/H \). This differs from the case of simply connected compact irreducible symmetric spaces \( K/H \) which are not group manifolds. In the latter case, as it was shown in \cite{13} ( Proposition 3.1), the Levi-Civita connection is the only \( K \) - symmetric metric compatible connection on \( K/H \).

Introducing the isomorphism \( i : S \rightarrow \mathcal{M} \) and using that \( \tilde{R}(\tilde{X}_k,\tilde{X}_p)\tilde{X}_j \) on \( K/H \) equals \( i(R(X_k, X_p)X_j) \) on \( S \) where \( i(X_k) = \tilde{X}_k, \ i(X_p) = \tilde{X}_p, \) etc., one gets from \((3)\)

\[
R_0(X_k, X_p)X_j = F(f)[[X_k, X_p], X_j], \quad F(f) = \frac{f^2 - 1}{2}, \quad X_k, X_p, X_j \in \mathcal{S},
\]

which yields for the curvature tensor components

\[
R_{ijkp} = F(f)C^{a}_{kp}C^{b}_{aj}g(X_b, X_i),
\]

where \( g(\cdot , \cdot) \) is the bi - invariant metric on \( S \) and \( C^{a}_{ij} \) are the structure constants of the algebra \( S \).

Analogously, eq.\((2)\) and \((16)\) imply that the torsion tensor on \( S \) equals

\[
T_0(X, Y) = f[X, Y].
\]
4 Dimensional reduction of multidimensional gravity with torsion

We investigate the theory with the action

\[ S = \int d^4x \sqrt{-\hat{g}} \{ \hat{\lambda}_0 + \hat{\lambda}_1 R + \hat{\lambda}_2 R^2 \}, \]  

(19)

where \( R^2 = \kappa R_{ABCD}R^{CDAB} - 4R_{AB}R^{BA} + R^2 \), on the space-time \( E = M^4 \times S \), \( \kappa = 0, 1, 2 \), \( A, B, C, D = 0, 1, 2, \ldots, d + 3 \). As it has been pointed out in the introduction such action arises in the field theory limit of strings with \( \kappa = 0 \) for the superstring [13], \( \kappa = 1 \) for the heterotic string [15] and \( \kappa = 2 \) for the bosonic strings [14]. We choose the metric tensor in the block diagonal form

\[ \hat{g} = \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & L^2 \theta_a^m(\xi) \theta_b^n(\xi) \delta_{mn} \end{pmatrix}, \]  

(20)

where \( \alpha, \beta = 0, 1, 2, 3 \), \( a, b = 4, \ldots, d + 3 \), \( L \) is a constant of the dimension of length characterizing the size of the space \( S \), \( x \in M^4 \), \( \xi \in S \), \( \theta_a^m(\xi) \) are the vielbeins. Substituting (20) in (19) and taking the invariance of the metric and connections into account, we get

\[ S = v_d \int d^4x L^d \sqrt{-\hat{g}} \{ \hat{\lambda}_0 + \hat{\lambda}_1 (\bar{R}^{(4)} + R^{(d)}) + \hat{\lambda}_2 R^2 \}, \]  

(21)

where \( L^d v_d \) is the volume of the internal space, \( \bar{R}^{(4)} \) and \( R^{(d)} \) are the scalar curvatures of the spaces \( M^4 \) and \( S \) respectively, and \( g = \det g_{\alpha\beta} \). To separate the term corresponding to the pure four-dimensional Einstein gravity we introduce the true physical metric \( \eta(x) \) on \( M^4 \) related to \( g(x) \) in the following way:

\[ g_{\alpha\beta}(x) = (\frac{L}{L_0})^{-d} \eta_{\alpha\beta}(x), \]

where \( L_0 \) is the constant of the dimension of length to be fixed later on. Then the action (21) takes the form

\[ S = \int d^4x \sqrt{-\eta} \{ \check{\lambda}_1 \bar{R}^{(4)} - W(L, f, d, \kappa, L_0, \check{\lambda}_0, \check{\lambda}_2) \}, \]  

(22)

\[ W(L, f, d, \kappa, L_0, \check{\lambda}_0, \check{\lambda}_2) = -\{ \check{\lambda}_2 (\frac{L}{L_0})^d (\bar{R}^2)^{(4)} + 2 \check{\lambda}_2 \bar{R}^{(4)} R^{(d)} \} \]

\[ + \check{\lambda}_0 (\frac{L}{L_0})^{-d} + \check{\lambda}_1 (\frac{L}{L_0})^{-d} R^d + \check{\lambda}_2 (\frac{L}{L_0})^{-d} (\bar{R}^2)^{(d)} \}, \]  

(23)

where \( \check{\lambda}_i = \check{\lambda}_i L_0^d v_d \). If we had considered the contorsion form (16) and the metric (20) with the parameter \( f(x) \) and the size \( L(x) \) depending on the coordinates of \( M^4 \), then after the dimensional reduction we would have obtained the Einstein gravity on \( M^4 \) with the metric tensor \( \eta_{\alpha\beta}(x) \) coupled to the scalar fields \( \psi(x) = \ln \{ L(x)/L_0 \} \) and \( f(x) \) with kinetic terms, higher derivatives and the potential arising from (23). If \( L \) and \( f \) are constant, as in the case considered in the present paper, we are left with (22). Thus, \( W \) is the effective potential.
of scalar fields $\psi$ and $f$ of the four dimensional reduced theory. It determines vacua, i.e. constant with respect to four dimensional coordinates solutions of the equations of motion. We are going to analyze the form and properties of the potential and find its minima.

Assuming that $M^4$ is the Minkowski space-time and $\eta_{\alpha\beta}$ is the Minkowski metric, the potential $W(L,f,\ldots)$ takes the form

$$W(L,f,\ldots) = -(\frac{L}{L_0})^d\{\tilde{\lambda}_0 + \tilde{\lambda}_1 R^{(d)} + \tilde{\lambda}_2 (R^2)^{(d)}\}$$

(24)

Hereafter we will drop the symbol ”(d)” for the components corresponding to the space $S$. The components of the curvature and Ricci tensors can be expressed in terms of the eigenvalue of the Casimir operator $C_2$. Using the fact that in our case $C_2 = 1$ for the adjoint representation [26] we find for $R$ and $R^2$

$$R = -\frac{1}{L^2} F(f) d,$$

(26)

Substituting (25), (26) into (24) and introducing the field $\psi(x) = \ln\{L(x)/L_0\}$ we obtain the following expression for the potential of the scalar fields $\psi$ and $f$ of the dimensionally reduced theory

$$W(\psi, f) = e^{-\psi d}\{\lambda_0 + \lambda_1 Fe^{-2\psi} + \lambda_2 F^2 e^{-4\psi}\},$$

(27)

where

$$F(f) = \frac{f^2 - 1}{4}, \quad \lambda_0 = -\tilde{\lambda}_0, \quad \lambda_1 = \tilde{\lambda}_1 \frac{d}{L_0^2} > 0, \quad \lambda_2 = -\frac{\tilde{\lambda}_2 d(d + \kappa - 4)}{L_0^4}.$$  

Our next step is to investigate possible cases corresponding to the values of the parameters $\lambda_0, \lambda_1, \lambda_2, \kappa$ and $d$ for which $W(\psi, f)$ has a minimum. We would like to underline here that the point $(\psi_{\text{min}}, f_{\text{min}})$, where the potential has its minimum, is a constant solution of the equations of motion of the theory corresponding to spontaneous compactification of the extra dimensions to the compact space $S$, so that the space-time has the form $M^4 \times S$ (see [27]). We will use the notation $\Delta \equiv \lambda_0^2/4\lambda_0\lambda_2$.

4.1 Case 1: $\lambda_0, \lambda_2 > 0, \Delta = 1$

It can be checked that in this case the potential has degenerate minima at the points $(\psi_{\text{min}}, f_{\text{min}})$ located on the curve $\psi_{\text{min}}(f) = \frac{1}{2} \ln\{\lambda_2 (1 - f^2)/2\lambda_1\}, \quad |f| < 1$, and its values at these points are equal to zero, i.e. $W(\psi_{\text{min}}, f_{\text{min}}) = 0$ (see Fig. 1). The four-dimensional cosmological constant $\Lambda^4$, which is determined by the value of $W$ at the point corresponding to the vacuum solution, vanishes. We may fix the parameter $L_0$ by the requirement $\psi_{\text{min}}(0) = 0$. This gives

$$L_0 = \sqrt{\frac{-\tilde{\lambda}_2(d + \kappa - 4)}{2\lambda_1}},$$

which is thus the size of the internal space in the vacuum corresponding to spontaneous compactification with zero torsion.
4.2 Case 2: $\lambda_0, \lambda_2 > 0$, $\frac{d(d+4)}{(d+2)^2} < \Delta < 1$

In this case the potential is positive for all $(\psi, f)$ and vanishes when $\psi \to +\infty$. It can be verified that the potential has the minimum at the point

$$(\psi_{\text{min}}, f_{\text{min}}) = \left( \frac{1}{2} \ln \frac{\lambda_2(d+4)}{2\lambda_1(d+2)[1 + \sqrt{1 + \frac{d(d+4)}{(d+2)^2} \Delta^{-1}}]}, 0 \right).$$

Since $W(\psi_{\text{min}}, 0) > 0$ the minimum is local and the corresponding vacuum state is metastable. We should note that the potential $W(\psi, f)$ has two gutters in the region $|f| < 1$ and $\psi < \frac{1}{2} \ln 2\lambda_1 / \lambda_2$, which join each other at the point $\left( \frac{1}{2} \ln 2\lambda_1 / \lambda_2, 0 \right)$ and ascend when $\psi \to -\infty$. We fix the parameter $L_0$ by the requirement that $\psi_{\text{min}} = 0$. This gives two values for $L_0^2$, and we choose the one for which $L_0^2 = 0$ when $\lambda_2 = 0$ (this corresponds to the collapse of the internal space and absence of stable spontaneous compactification solution for the pure Einstein gravity, as it has been discussed in the Introduction):

$$L_0^2 = \frac{(d+2)\hat{\lambda}_1}{2\hat{\lambda}_0}[{-1 + \sqrt{1 - \frac{\hat{\lambda}_0 \lambda_2}{\hat{\lambda}_1^2}(d + \kappa - 4)(d - 4)}}].$$

(28)

The experimental bound for the four-dimensional cosmological constant $\Lambda^4$ gives $|16\pi G \Lambda^4| < 10^{-120}$. By making the parameter $\lambda_0$ approaching $\lambda_2^2 / 4\lambda_2$ from above we can obtain arbitrary small values for $\Lambda^4 = W(0, 0)$. Trying to make $\Lambda^4 = 0$ it is easy to see that this is possible if and only if $\Delta = 1$, i.e. in case 1.

4.3 Case 3: $\lambda_0, \lambda_2 > 0$, $\Delta > 1$

Now the potential has no minimum. As in the previous case for $\psi < \frac{1}{2} \ln 2\lambda_1 / \lambda_2$ and $|f| < 1$, the potential has two gutters which join each other at $\left( \frac{1}{2} \ln 2\lambda_1 / \lambda_2, 0 \right)$, but contrary to the case 2, lower to $(-\infty)$ when $\psi \to -\infty$. These features of the potential are of some importance for understanding of the spontaneous compactification issue as will be discussed in the next section.

5 Discussion of the results

Let us start our discussions with case 2. We have found that the minimum of the scalar potential $W(\psi, f)$ of the reduced theory is at $(\psi_{\text{min}}, 0)$ (for our choice of $L_0$, $\psi_{\text{min}} = 0$) that corresponds to spontaneous compactification of extra dimensions to the group manifold $S$ with characteristic size $L_0$ given by (28) and zero torsion. This minimum is stable with respect to fluctuation with non zero torsion and classically stable with respect to fluctuation in $\psi$ - direction. Since $W(0, 0) > 0$ and $W(\infty, 0) = 0$ the corresponding vacuum state is metastable, and the system can pass to the region of large $\psi$ via quantum tunnelling. This corresponds to decompactification of the space of extra dimensions. Analogous phenomena for the multidimensional Einstein - Yang - Mills system without torsion were considered in [28], [29]. But if the parameters of the lagrangian are tuned in such way that the four-dimensional cosmological constant $\Lambda^4 = W(0, 0)$ is small enough to satisfy the experimental
bound $|16\pi G\Lambda^4|<10^{-120}$, the lifetime of the metastable vacuum exceeds the lifetime of the Universe (see estimations in [29]).

Another interesting problem which can be addressed here is the dynamics of compactification of extra dimensions in the framework of the Kaluza-Klein cosmology (the analogous issue for the Einstein-Yang-Mills system without torsion was considered in [28]). The main question is the following: if at the early stage the multidimensional Universe had started its classical evolution with large negative $\psi$ (small size $L$) and large $|\dot{\psi}|$, would it have found itself in the minimum ($\psi = 0, f = 0$) corresponding to spontaneous compactification of extra dimensions? Since the height of the barrier separating the minimum from the region where $\psi \to +\infty$ (decompactification of extra dimensions) is finite, the system can have enough energy to overcome the barrier in spite of the loss of energy due to the friction terms which are present in such sort of theory. In any case this question needs more detailed investigation for which the explicit form of non-static terms in the reduced action must be known. This is beyond the scope of the present paper.

In case 1, the minimum of the potential is degenerate: $W(\psi_{min}(f), f) = 0$ for $0 \leq |f| \leq 1$. The vacuum with $(\psi_{min}(0), 0)$, corresponding to compactification with zero torsion, is not separated by any barrier from another vacua with the same energy but non-zero torsion.

The situation changes even more drastically in case 3. The potential $W(\psi, f)$ does not have any minimum at all. But if we analyzed the same theory without torsion, we would see (as in case 1 also) that the potential $W(\psi, 0)$ has minimum and expect to have spontaneous compactification solution. However, taking torsion as additional degree of freedom in the theory into account changes the situation. The vacuum $(\psi_{min}(0), 0)$ is not stable and small fluctuations of the fields and their time derivatives may initiate transitions of the system to another state with the same (case 1) or less (case 3) energy. Non-zero torsion is developed in such transitions.

We think that this example is rather instructive for deeper understanding of the spontaneous compactification problem. It illustrates some of the hidden difficulties that the Kaluza-Klein approach may face.

In conclusion we would like to underline that the analysis of the Kaluza-Klein $\mathcal{R}^2$-gravity with torsion was carried out for arbitrary $S \times S$-invariant configurations of the metric and connection form. The mathematical results, obtained in Section 3, enabled us to describe all metric compatible invariant connections with non-zero torsion on group manifolds $S$, and thus the solution of the spontaneous compactification problem for this class of metrics and connections is complete.

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References


   Coquereaux R. and Jadczyk A. Riemannian Geometry, Fiber Bundles, Kaluza-Klein
   Theories and All That... Lecture Notes in Physics, vol. 16, (World Scientific, Singapure,
   1988).


   Gauge Theories, Spontaneous Compactification and Model Building, Lecture Notes in


[10] Richter O. and Rudolph G. Spontaneous Compactification with Dynamical Torsion,

    393.


Figure 1 Shape of the potential $W(\psi, f)$ when $\lambda_0, \lambda_0 > 0$ and $\Delta = 1$ (see Sect. 4.1). The minimum of the potential is degenerate and is located on the curve $\psi - \frac{1}{2} \ln \frac{(1-f^2)}{2\lambda_1} = 0$ in the $(\psi, f)$-plane.