Classical Models of Subatomic Particles

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N2L 3G1

July 7, 1993  
WATPHYS TH-93/02
Abstract

We look at the program of modelling a subatomic particle—one having mass, charge, and angular momentum—as an interior solution joined to a classical general-relativistic Kerr-Newman exterior spacetime. We find that the assumption of stationarity upon which the validity of the Kerr-Newman exterior solution depends is in fact violated quantum mechanically for all known subatomic particles. We conclude that the appropriate stationary spacetime matched to any known subatomic particle is flat space.
The most basic properties of a subatomic particle are its mass \( M \), charge \( Q \) and spin \( J \) (and lifetime \( \tau \) if it is unstable). As such it would seem natural from a general-relativistic viewpoint to describe the spacetime metric associated with a given subatomic particle by a Kerr-Newman metric [1]

\[
ds^2 = -dt^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + (r^2 + a^2) \sin^2(\theta) d\phi^2 + \frac{rM}{\Sigma} \left( dt - a \sin^2(\theta) d\phi \right)^2
\]

at least at distances large compared to the characteristic size \( R \) of the particle. In (1)

\[
\mathcal{M} \equiv 2M - Q^2/r \quad \Sigma = r^2 + a^2 \cos^2(\theta) \quad \Delta \equiv r^2 + a^2 - \mathcal{M} r
\]

where \( a \equiv J/M \) and units are such that \( G = c = 1 \).

For subatomic particles let \( J = N_s/2 \) and \( Q = N_e e \) where \( e \) is the charge of an electron. Assuming that (1) provides a valid description for \( r > R \) of the spacetime due a given subatomic particle, (so that the effects of strong and weak interactions are neglected) one finds \( a/M >> Q/M >> 1 \) for all known quarks, leptons, baryons, mesons and nuclei (except for spin-zero particles, in which case only the latter inequality is satisfied). The first inequality is violated only when \( N_s < 2N_e \sqrt{\alpha M_{Pl}} \), (where \( \alpha \) is the fine-structure constant) whereas the second is violated when \( N_e < \sqrt{\alpha M_{Pl}} \). These situations only hold macroscopically, when the mass of the body is appreciably larger than the Planck mass \( M_{Pl} \), although the first inequality could be violated for a body with a small mass but a large charge. Hence for all subatomic particles the metric (1), if assumed valid for all values of \( r \), describes the field of a naked singularity.

Unless one is willing to live with such an unattractive scenario (along with whatever empirical difficulties it may cause), it is clear that the Kerr-Newman metric cannot be valid for all values of \( r \) for a subatomic particle; rather it must be matched on to some interior solution to the Einstein equations. Attempts to find such interior solutions have been carried out ever since the Kerr metric was discovered [2]. Such models have been plagued by a variety of unphysical features, including superluminal velocities and negative mass distributions [3, 4, 5]. Indeed some authors [6] have advocated that current data limiting the size of the electron to be smaller than \( 10^{-16} \text{cm} \) (and the
assumption that general relativity is valid at these distance scales) imply that the electron must have a negative rest mass density.

While we regard the search for an interior solution to match onto the Kerr-Newman metric as being of interest in its own right, we argue here that assertions concerning the negativity of the rest mass density of the electron (and all other known elementary particles) are unwarranted. The unphysical features described above arise because the matching is carried out either at $r = 0$ or at half the classical electromagnetic radius $r_Q = Q^2/(2M)$ [7]; this effectively results in ascribing the electromagnetic rest energy of the particle to be too large relative to the laboratory value of its rest mass, and so a negative mass density must be introduced to compensate. However, the matching conditions must not be applied at a distance scale below which the Kerr-Newman metric can no longer be trusted. This distance scale can be set by other physics in one of two ways: either new structure occurs due to the physics of the particle considered, or the assumption of stationarity breaks down.

The former case occurs for (stable) baryons and nuclei: for all nuclei (including Hydrogen), one can easily check that the nuclear radius $r_N \equiv 1.07A^{1/3}$ fm [8] is larger than either the electromagnetic charge radius or the Kerr parameter $a$. Matching the exterior Kerr-Newman metric for baryons and nuclei to an interior solution (whose stress-energy tensor must be determined by nuclear effects) must therefore take place at a distance $r_M \geq r_N >>> a, r_Q$.

For charged leptons, quarks or mesons, the Kerr-Newman solution will not be applicable if the particle is not stationary. By the uncertainty principle, the particle would have to be moving at relativistic speeds on distance scales shorter than the Compton wavelength $\lambda_c$ of the particle, the length scale at which the average quantum zero-point kinetic energy of the particle is comparable to its rest energy. Requiring that $|p| << m$ implies that the matching must take place at $r_M >> \lambda_c$ (e.g. $r_M \sim 100\lambda_c$). This scale is much larger than the parameter $a = N_s\lambda_c/2$ for all known non-baryonic subatomic particles.

If the solution were stationary, one could employ the matching conditions using the charged Lense-Thirring metric

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 + \frac{2M}{r} - \frac{Q^2}{r^2})(dx^2 + dy^2 + dz^2)$$
\[ +2 \frac{Q^2 - 2Mr}{r^4} \left( \vec{x} \times \frac{\vec{J}}{M} \right) \cdot d\vec{x} dt \]  

(3)

which is a post-Newtonian solution to the Einstein equations for the metric exterior to a charged spinning sphere of constant density where \( M/r << 1 \), \( J/r^2 << 1 \) and \( Q^2/r^2 << 1 \). Under these conditions, this metric is equivalent to (1) provided \( J/M = a \). In principle one could determine the values of \( M \), \( J \) and \( Q \) for a body by Gaussian integration of the gravitational field at a large distance \( r_M \).

However in order to perform such integrations it is necessary that the body be confined to a region \( r < r_M \). Quantum mechanically the uncertainty principle requires that such confinement impart a root-mean square momentum \( \Delta p \geq \hbar/r_M \) to the particle, necessitating corrections to the metric (3). For a given imparted momentum \( \vec{P} \) these corrections modify (3) to be

\[
ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\vec{x} \cdot \vec{D}}{r^3}) dt^2 \\
+ (1 + \frac{2M}{r} - \frac{Q^2}{r^2} + \frac{\vec{x} \cdot \vec{D}}{r^3})(dx^2 + dy^2 + dz^2) \\
+ (2 \frac{Q^2 - 2Mr}{r^4} (\vec{x} \times \frac{\vec{J}}{M}) - \frac{\vec{P}}{r}) \cdot d\vec{x} dt
\]

(4)

where \( r = |\vec{x}| \), where \( \vec{D} \) is the gravitational dipole moment resulting from the particle no longer being at the origin.

For macroscopic bodies such corrections are negligible, but for subatomic particles this is not the case. The term \( \frac{\vec{P}}{r} \sim \frac{\vec{x}}{r_M} \) which is the same order of magnitude as the \( \frac{2Mr}{r^4} (\vec{x} \times \frac{\vec{J}}{M}) \) term in (4) since \( J \sim \hbar \). Similarly \( \frac{\vec{x} \cdot \vec{D}}{r^3} \sim M/r_M \) and so it is of the same order of magnitude as the first term in \( g_{00} \) in (4). The charge terms are of order \( Q^2/r_M^2 \sim M r_Q/r_M^2 << M/r_M \), and so, even though there will be corrections to these terms due to the uncertainty principle introducing electric and magnetic dipole moments, the charge terms are already negligible relative to the mass and spin terms we have kept. Since the root-mean-square quantum corrections are always of the same magnitude as the largest terms we have kept in the post-Newtonian expansion, we cannot trust keeping those classical terms.

Thus we suggest that appropriate matching to the Kerr-Newman geometry for the electron is constrained by stationarity to take place at radial
distances from the particle much larger than the Compton wavelength. The interior solution will be modelled by a quantum distribution. But, however large the matching radius \(r_M\) is taken to be, the act of measuring the space-time curvature on a surface at that distance (i.e. measuring the parameters of the Kerr-Newman metric), would, again by the uncertainty principle, kick the momentum of the electron by \(\Delta p \geq m\lambda_c/r_M\), introducing quantum non-stationarity corrections to the metric of order \(\lambda_c/r_M\). These corrections for an electron are the same order as the \(a/r\) term kept even in the Lens-Thirring approximation to the Kerr-Newman geometry. This means the uncertainty principle should make it impossible to measure the Kerr-Newman or even the charged Lens-Thirring parameters, and the appropriate stationary solution matching a quantum electron is flat.

Unfortunately, this conclusion tends to undermine one good motivation for trying to model subatomic particles as Kerr-Newman sources in the first place. This is the fact that the Kerr-Newman metric predicts the Dirac value of the electron’s gyromagnetic ratio. The above argument, though, would lead us to regard this as a coincidence.

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

References


