Squeezed States and
Uncertainty Relation at Finite Temperature

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Abstract

We use the quantum Brownian model to derive the uncertainty relation for a quantum
open system in an arbitrarily-squeezed initial state interacting with an environment at
finite temperature. We examine the relative importance of the quantum and thermal
fluctuations in the evolution of the system towards equilibrium with the aim of clarifying
the meaning of quantum, classical and thermal. We show that upon contact with the
bath the system evolves from a quantum-dominated state to a thermal-dominated state
in a time which is the same as the decoherence time calculated before in the context of
quantum to classical transitions. We also use these results to deduce the conditions when
the two basic postulates of quantum statistical mechanics become valid.

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It is a well-known fact in quantum mechanics that a lower bound exists in the product of the variances of pairs of noncommutative observables. Taking the coordinate $x$ and momentum $p$ as examples, the Heisenberg uncertainty principle states that with $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$, and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$, the uncertainty function

$$U_0^{QM} = (\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4} \quad (T = 0, \text{ quantum mechanics})$$

(1)

The existence of quantum fluctuations is a verified basic physical phenomenon. The origin of the uncertainty relation can be attributed as a mathematical property of Fourier analysis [1] which describes quantum mechanics as a wave theory. Recent years have seen effort in establishing a stronger relation based on information-theoretical considerations [2][3].

In realistic conditions quantum systems are often prepared and studied at finite temperatures where thermal fluctuations permeate. At high temperatures the equipartition theorem of classical statistical mechanics imparts for each degree of freedom an uncertainty of $kT/2$. Thus the uncertainty function for a one-dimensional harmonic oscillator approaches the limit

$$U_T^{MB} \approx \left( \frac{kT}{\Omega_0} \right)^2 \quad (\text{high } T, \text{ classical statistical mechanics})$$

(2)

where $\hbar\Omega_0$ is the energy of a normal mode with natural frequency $\Omega_0$. This result, obtained by assuming that the system obeys the Maxwell-Boltzmann distribution, is usually regarded as the classical limit. For a system of bosons in equilibrium at temperature $T$, the application of canonical ensemble gives the result in quantum statistical mechanics as

$$U_T^{BE} = \frac{\hbar^2}{4}[\coth(\frac{\hbar\Omega_0}{2kT})]^2 \quad (\text{all } T, \text{ quantum statistical mechanics})$$

(3)

which interpolates between the two results (1) and (2) at $T = 0$ and $T >> \hbar\Omega_0/k$. This result applies to a system already in equilibrium at temperature $T$.

Our purpose here is to study the corresponding non-equilibrium problem. At time $t_0$ we put the system in contact with a heat bath at temperature $T$ and follow its time evolution. We want to see how the uncertainty function $U_T(t)$ changes from the initial quantum
fluctuation-dominated condition to a later thermal fluctuation-dominated condition. By comparing this result with the decoherence studies recently carried out [4,5,6], where two characteristic times—the decoherence time $t_{\text{dec}}$ and the relaxation time $t_{\text{rel}}$—are defined, one can apply the physics of these two processes involved to examine the following three interrelated issues:

1) the realization of the basic tenets of quantum statistical mechanics from quantum dynamics; 2) the relation between quantum and thermal fluctuations and 3) their role in the quantum to classical transition.

Quantum statistical mechanics of a macroscopic system is derived from the quantum dynamics of its microscopic constituents under two basic postulates [7]: i) random phase, and ii) equal a priori probability. The first condition enables one to assign probability distributions to a system occupying certain quantum states. It requires sufficient diminishing of interference terms in the wave function or that the density matrix of the system be approximately diagonal. The second condition when applied to a closed system forms the basis of the microcanonical ensemble. When applied to an open system in contact with a heat bath described by the canonical ensemble, it ensures that the system equilibrates with its environment. We want to examine the processes by which these two conditions are attained from a more primitive level, starting with the microdynamics of a system of quantum particles. Specifically, we want to see if there is a characteristic time when the phase information is lost (Postulate i) and another time when the system attains equilibrium with its surrounding so that all accessible states in the closed system are equally probable (Postulate ii).

On the second issue, the demarkation of classical, quantum and thermal regimes are not always clearly noted, their usual definitions or usage oftentimes are imprecise. With the aid of the results obtained here we will point out some existing confusions and make their meanings more precise. The relation between quantum and thermal fluctuations has been studied previously via thermo field dynamics [8] under equilibrium and stationary
conditions. Here we treat the fully nonequilibrium problem. Using the microdynamics of a quantum system as starting point we view thermal fluctuations as statistical variations of the coarse-grained environmental variables with which the quantum system interacts, the exact microdynamics of the system and the environment obeying the laws of quantum mechanics. The problem under study can thus be stated equivalently as finding the uncertainty relation for an open quantum system [9].

On the third issue, in loose terms, one often identifies the high temperature regime of a system as the classical domain. On the one hand, one regards the regime when thermal fluctuations begin to surpass quantum fluctuations as the transition point from quantum to classical. On the other hand, from the wave picture (or consistent history formulation) of quantum mechanics we know that a necessary condition for a system to behave classically is that the interference terms in its wave function (or interference in the histories) have to diminish sufficiently and remain insignificant so that probability can be assigned to classical events [4,5] or that classical decoherent histories can be well-defined [6]. This is known as the decoherence process. Is there any relation between these two criteria of classicality? We show in this problem under the conditions studied that they are indeed equivalent: The time the quantum system decoheres is also the time when thermal fluctuation overtakes quantum fluctuations [see Eq. (22, 25)].

However we issue a warning here that this regime should not be called classical, as is customary in many quantum to classical transition studies. In fact, after the decoherence time only the first postulate of quantum statistical mechanics (QSM) is satisfied, the system can be described by non-equilibrium QSM. Only after the relaxation time, when the second postulate (more precisely, the implied version for canonical ensemble) is satisfied for the closed system, can one use equilibrium QSM. Classical has still a long way to go. It is well-known that quantum statistical effects can be important at very high temperatures (e.g., Fermi temperature for metals). This is due to exchange interactions of identical particles, a distinctly quantum effect. Only when the statistical properties of fermions and bosons
can be approximated by distinguishable particles, usually at high temperatures when the Fermi-Dirac or Bose-Einstein statistics approaches the Maxwell-Boltzmann statistics, can the system be rightfully called \textit{classical}. In this regard \textit{quantum} carries two meanings, one refers to the interference effect and the other refers to spin-statistics effect.

To examine these issues, we use the simple Brownian model \cite{4-6, 10-13} of a quantum open system and the influence functional method \cite{11} to incorporate the statistical effect of the environment on the system. As the microdynamics is explicitly treated in this approach, one can study how the result depends on the properties of the bath and the system-bath interaction in detail. We consider an initial Gaussian wave packet (12) with nonzero averaged initial position and momentum and calculate the time dependence of the spread $<\Delta x>^2, <\Delta p>^2$ and the uncertainty function $U_T$ in the presence of both quantum and thermal fluctuations. The main result is given formally by Eq. (17), which, for an ohmic environment under the conditions of weak damping and high temperature, simplifies to Eqs. (19) and Eq. (24) respectively. As the initial state is arbitrarily-squeezed, we can show clearly how the squeeze parameter enters in these uncertainty relations \cite{14}.

Our system is a Brownian particle with mass $M$ and natural frequency $\Omega_0$. The environment is modeled by a set of $n$ harmonic oscillators with mass $m_n$ and natural frequency $\omega_n$. The particle is coupled linearly to the $n$th oscillator with strength $C_n$. The action of the combined system and environment is

$$S[x,q] = \int_{0}^{t} ds \left[ \frac{1}{2} M \dot{x}^2 - \frac{1}{2} M \Omega_0^2 x^2 \right] + \sum_n \left\{ \frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2 \right\} + \sum_{n} \left\{ -C_n x q_n \right\}$$ \hspace{1cm} (4)

where $x$ and $q_n$ are the coordinates of the particle and the oscillators respectively, and $\Omega_0$ is the (bare) frequency of the particle. We are interested in how the environment affects the system in some averaged way. The quantity containing this information is the reduced density matrix of the system $\rho_r(x,x')$ obtained from the full density operator of the system
and environment $\rho(x, q; x' q')$ by tracing out the environmental degrees of freedom $(q, q')$

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, q; x', q', t) \delta(q - q')$$  (5)

The reduced density matrix evolves under the action of the propagator $J_r(x, x', t | x_i, x_i', 0)$ in the following way:

$$\rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx_i' J_r(x, x', t | x_i, x_i', 0) \rho_r(x_i, x_i', 0)$$  (6)

In general this is a very complicated expression since the evolution operator $J_r$ depends on the initial state. If we assume that at a given time $t = 0$ the system and the environment are uncorrelated, i.e. that $\hat{\rho}(t = 0) = \hat{\rho}_s \times \hat{\rho}_e$, and that the environment is in thermal equilibrium at a temperature $T$, the evolution operator for the reduced density matrix has an analytic form given by [5,11-13]: (We use the notation of [5])

$$J_r(x_f, x_f', t | x_i, x_i', 0) = Z_0(t) \exp \frac{i}{\hbar} \left\{ \left[ \dot{u}_1(0) X_i + \dot{u}_2(0) X_f \right] Y_i - \left[ \dot{u}_1(t) X_i + \dot{u}_2(t) X_f \right] Y_f \right\}$$

$$\times \exp \left\{ -\frac{1}{\hbar} \left\{ a_{11}(t) Y_i^2 + [a_{12}(t) + a_{21}(t)] Y_i Y_f + a_{22}(t) Y_f^2 \right\} \right\}$$  (7)

where $X = (x + x')/2$ and $Y = x' - x$. The elementary functions $u_i(s)$ are obtained as solutions of the following differential equations

$$\frac{d^2 u_i(s)}{ds^2} + 2 \int_0^s ds' \eta(s - s') u_i(s') + \Omega_0^2 u_i(s) = 0$$  (8)

with boundary conditions

$$\begin{cases} u_1(0) = 1, & u_1(t) = 0 \\ u_2(0) = 0, & u_2(t) = 1 \end{cases}$$  (9)

and the $a_{ij}(t)$ are obtained from the integral

$$a_{ij}(t) = \int_0^t ds_1 \int_0^{s_1} ds_2 v_i(s_1) v(s_1 - s_2) v_j(s_2)$$  (10)
where \( v_1(s) = u_2(t-s) \) and \( v_2(s) = u_1(t-s) \). Here \( \eta \) and \( \nu \) are the dissipation and noise kernels in the influence functional [11]. They are given respectively by

\[
\nu(s) = \int_0^{+\infty} d\omega \ I(\omega) \coth\left( \frac{\hbar \omega}{2kT} \right) \cos \omega s \tag{11a}
\]

and

\[
\eta(s) = \frac{d}{ds} \gamma(s); \quad \gamma(s) = \int_0^{+\infty} d\omega \ \frac{I(\omega)}{\omega} \cos \omega s. \tag{11b}
\]

Here \( I(\omega) \) is the spectral density function of the environment, \( I(\omega) = \sum_n \delta(\omega - \omega_n) \frac{C_n^2}{\omega_n \omega_n} \). An environment is called ohmic \( I(\omega) \sim \omega \), supra-ohmic \( I(\omega) \sim \omega^n, n > 1 \) or sub-ohmic \( n < 1 \). The most studied ohmic case corresponds to an environment which induces a dissipative force linear in the velocity of the system.

We now consider a Brownian oscillator with an initial wave function

\[
\psi(x,0) = \sqrt{N_0} \ \exp\left\{ -\frac{(x-x_0)^2}{4\sigma^2} + \frac{i}{\hbar} p_0 x \right\} \tag{12}
\]

where \( \sigma \) is the initial spread, and \( x_0, p_0 \) are the averaged initial position and momentum of the Gaussian wave packet. This gives \( \rho_r(x,x',0) = \psi^*(x,0)\psi(x',0) \). One can calculate \( \rho_r(x_f,x'_f,t) \) by performing the Gaussian integrals over \( x_i \) and \( x'_i \) and get

\[
\rho_r(x_f,x'_f,t) = \tilde{Z}_0(t) \ \exp\left\{ -\frac{1}{2} X^T Q X + B^T X \right\} \tag{13}
\]

where the prefactor \( \tilde{Z}_0(t) \) depends only on time. Here \( X = (X_f,Y_f)^T \). The parameters \( x_0, p_0 \) enter in the elements \( B_i \) of \( B \) but do not appear in the uncertainty function. The elements \( Q_{ij}(t), i,j = 1,2 \) of \( Q \) are given below.

To calculate the averages of observables, it is convenient to use the Wigner function defined as

\[
W_r(X,p,t) = \int dY e^{ipY} \rho_r(X-Y/2,Y+Y/2,t), \tag{14}
\]

The quantum statistical average of a system observable, e.g., \( x^n \) or \( p^n \), with respect to the reduced density matrix \( \rho_r(t) \) is given by

\[
<x^n>_T = \int dx x^n \rho_r(x,x,t) = \int dx \int \frac{dp}{2\pi \hbar} x^n W_r(X,p,t) \tag{15}
\]
where the subscript $T$ indicates that the environment is at temperature $T$. Obviously the average contains both quantum and thermal contributions. We get the variances

$$
(\Delta x)^2_T \equiv \langle x^2 \rangle_T - \langle x \rangle_T^2 = \frac{1}{Q_{11}(t)}; \quad (\Delta p)^2_T \equiv \langle p^2 \rangle_T - \langle p \rangle_T^2 = \hbar^2 \frac{\det Q}{Q_{11}(t)}
$$

and arrive at the finite temperature uncertainty function:

$$
U_T(t) = (\Delta x)^2_T (\Delta p)^2_T = \hbar^2 \frac{\det Q(t)}{|Q_{11}(t)|^2} = \hbar^2 \left\{ \frac{Q_{22}(t)}{Q_{11}(t)} - \frac{Q_{12}(t)}{Q_{11}(t)} \frac{Q_{21}(t)}{Q_{11}(t)} \right\}.
$$

The matrix elements in (17) are given by

$$
\frac{Q_{22}(t)}{Q_{11}(t)} = \frac{1}{4} \frac{[\dot{u}_1(t)]^2}{[u_2(0)]^2} + \frac{2\sigma^2}{\hbar |u_2(0)|^2} \left\{ [\dot{u}_1(t)]^2 a_{12}(t) - \dot{u}_1(0) \dot{u}_1(t) [a_{12}(t) + a_{21}(t)] \right\}
$$

+ \frac{\hbar^2}{4\sigma^4} + \left[ \dot{u}_1(0) \right]^2 \left[ a_{22}(t) \right] + \frac{1}{[u_2(0)]^2} \left\{ 4a_{11}(t)a_{22}(t) - [a_{12}(t) + a_{21}(t)]^2 \right\}

and

$$
\frac{Q_{12}(t)}{Q_{11}(t)} = \frac{Q_{21}(t)}{Q_{11}(t)} = \sigma^2 \frac{i \dot{u}_2(t)}{\hbar [u_2(0)]^2} \left\{ \frac{\hbar^2}{4\sigma^4} + \left[ \dot{u}_1(0) \right]^2 - \frac{\dot{u}_2(0)}{u_2(t)} \dot{u}_1(0) \dot{u}_1(t) \right\}
$$

+ \left[ \frac{i}{\dot{u}_2(0)} \left\{ \frac{2 \dot{u}_2(t)}{[u_2(0)]^2} a_{11}(t) + a_{12}(t) + a_{21}(t) \right\} \right]

The complete result can be obtained from solving the equations for the $u_i(t)$ functions and the $a_{ij}(t)$ coefficients. It is also instructive to look at some special cases where one can find simpler analytical expressions.

For an ohmic environment $\gamma(t) = 2\gamma_0 \delta(t)$ and $a_{ij}(t)$, $u_i(t)$ become harmonic and exponential functions. Let us introduce the following physical parameters: the effective frequency $\Omega \equiv (\Omega_0^2 - \gamma_0^2/4)^{1/2}$, the energy parameter $\epsilon \equiv \frac{\Omega}{2\kappa T} \equiv 1/\tau$, the damping parameter $\alpha \equiv \frac{\gamma_0}{2\Omega}$ and the squeeze parameter $\delta \equiv \frac{2\sqrt{\epsilon^2 + \sigma^2}}{\hbar}$ which measures the spread in the initial Gaussian wavepacket. For ohmic weak damping, $\alpha << 1$, $\Omega \simeq \Omega_0$ the uncertainty function is given for all temperatures to first order in $\alpha$ by

$$
\frac{1}{\hbar^2} U_T(t) \sim \frac{1}{4} \left\{ \frac{e^{-\gamma_0 t}}{\coth \epsilon \left[ 1 - e^{-\gamma_0 t} \right]} \right\}^2
$$

+ $\coth \epsilon \left\{ \frac{(1 - \delta)^2}{4\delta} \left[ 1 - e^{-\gamma_0 t} \right] - \frac{1 - \delta^2}{4\delta} \alpha \sin 2\Omega t \right\} e^{-\gamma_0 t}$

+ $\left\{ \frac{1 - \delta^2}{4\delta} \sin 2\Omega t + \alpha \left[ \coth \epsilon - \frac{1 + \delta^2}{2\delta} \right] \sin^2 \Omega t \right\}^2 e^{-2\gamma_0 t}$

$$
\frac{1}{\hbar^2} U_T(t) \sim $$
We see that there are two factors at play here: time and temperature. Time is measured in units of the relaxation time proportional to $t_{rel} = \gamma_0^{-1}$, and temperature is measured with reference to the ground state energy $\hbar \Omega_0 / 2$ of the system. There are also two parameters involved, $\alpha$ and $\delta$. One can deduce various limits from this expression. For example, it is obvious that at $t = 0$, when the initial uncorrelated conditions is assumed valid, $U_T(0) = \hbar^2 / 4$, which is the Heisenberg relation (1). At very long time ($t >> \gamma_0^{-1}$), $U_T(t)$ is insensitive to $\delta$ and approaches $U_T^{BE}$ as in (3) at finite temperature (as $\Omega \simeq \Omega_0$). That means the Brownian particle approaches an equilibrium quantum statistical system. (For supraohmic bath this may not always be true). We can also see that for a $T = 0$ bath (coth $\epsilon = 1$), $U_T(t)$ has a leading term given by $\hbar^2 / 4$ (the Heisenberg relation) followed, for squeezed states $\delta \neq 1$, by terms of order $\alpha^0$ and $\alpha$ depicting both decay and oscillatory behavior. We understand that this is due to the action of quantum fluctuations alone. For a minimum-uncertainty initial state ($\delta = 1$), we get for all finite temperatures

$$U_T(t) = \frac{\hbar^2}{4} \left\{ e^{-\gamma_0 t} + \coth \epsilon \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 + O(\alpha^2)$$

(20)

Notice that there is no linear order damping term.

At short times ($t << \gamma_0^{-1}$),

$$U_T(t) \simeq \frac{\hbar^2}{4} \left[ 1 + 2(\delta \coth \frac{\hbar \Omega}{2kT} - 1)\gamma_0 t + O(t^2) \right]$$

(21)

This simple expression is revealing in several aspects: The first term is the ubiquitous quantum fluctuation, the second term is the thermal contribution, which depends on the initial spread and increases with increasing dissipation and temperature. The time when thermal fluctuations overtake quantum fluctuations is (assuming the temperature is higher than the ground state energy):

$$t_1 = \frac{1}{2\gamma_0 (\delta \coth \frac{\hbar \Omega}{2kT} - 1)}$$

(22)

We expect this to be equal to the decoherence time scale $t_{dec}$ calculated for weak coupling for all temperatures in the study of quantum to classical transitions [15].
At zero coupling, i.e., for an isolated harmonic oscillator,

\[ U_T(t) = \frac{1}{4} \hbar^2 \left\{ 1 + \frac{(1 - \delta^2)^2}{4 \delta^2} \sin^2 2\Omega_0 t \right\} \geq \frac{1}{4} \hbar^2 \]  

(23)

This is the quantum uncertainty relation for squeezed states. The time-dependent term is the result of quantum dispersion.

At high temperatures, \( \tau_0 = \frac{kT}{\pi\Omega_0} \gg 1 \), both the noise kernel and the dissipation kernel for ohmic dissipation become local, we have for the full range of damping strength,

\[
\frac{1}{\hbar^2} U_T(t) = \frac{1}{4} \left\{ e^{-\gamma_0 t} + \tau_0 \left[ 1 - e^{-\gamma_0 t} \right] \right\}^2 \\
+ \tau_0 \left\{ \frac{(1 - \delta^2)^2}{4 \delta^2} \left[ 1 - e^{-\gamma_0 t} \right] - \alpha \left[ \frac{1 - \delta^2}{4 \delta} \right] \sin 2\Omega t - \alpha^2 \left[ \tau_0 - \frac{1 + \delta^2}{2 \delta} \right] \sin^2 \Omega t \right\} e^{-\gamma_0 t} \\
+ \frac{\Omega_0^2}{\Omega^2} \left\{ \frac{1 - \delta^2}{4 \delta} \sin 2\Omega t + \alpha \left[ \tau_0 - \frac{1 + \delta^2}{2 \delta} \right] \sin^2 \Omega t \right\}^2 e^{-2\gamma_0 t}
\]  

(24)

At short times \( (t \ll \gamma_0^{-1}) \),

\[ U_T(t) \simeq \frac{\hbar^2}{4} \left[ 1 + 2(\tau_0 \delta - 1)\gamma_0 t + O(t^2) \right] \]  

(25)

The time when thermal fluctuations overtake quantum fluctuations is given by

\[ t'_1 = \frac{\hbar \Omega_0}{4kT\gamma_0 \delta} \]  

(26)

This is equal to the decoherence time scale \( t_{dec} \) calculated before for high temperature for the same model [4,5].

To complete the description of the evolution of the system, the second time scale of importance is the relaxation time scale, \( t_{rel} = \gamma_0^{-1} \) we referred to earlier, when the particle reaches equilibrium with the environment. It is at this time that equilibrium QSM can be applied to the description of the system. After this, for ohmic and subohmic environments the uncertainty relation takes on the Bose-Einstein form (3). At high temperatures the system reaches the Maxwell-Boltzmann limit and the uncertainty relation takes on the classical form (2). For supraohmic environments at low temperature, the highly nonlocal
frequency response may make it difficult for the system to settle down. The decoherence
time scale is longer, and the relaxation can even be incomplete. This is the regime where
one expects to find more intricate and interesting behavior in the interplay of quantum
and thermal effects. The nonohmic results [16] and details of the present study [9] are to
be presented elsewhere.

After this paper was submitted for publication we learned of the work of Anderson and
Halliwell [3] on an information-theoretical definition of uncertainty which, among reporting
other interesting results, also confirmed our main findings here.

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References

librium Statistical Physics, ed. G. T. Moore and M. O. Scully (Plenum, N. Y., 1986); E.


[15] Although this decoherence time was not given explicitly, it could be derived from the results of [5] and Paz et al in [4].