QUANTUM MECHANICAL COMPOSITION LAWS IN REPARAMETERIZATION INVARIANT SYSTEMS*

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ABSTRACT

This paper gives a brief description of the derivation of a composition law for the propagator of a relativistic particle, in a sum over histories quantization. The extension of this derivation to the problem of finding a composition law for quantum cosmology is also discussed.

It has been argued that the sum over histories should be regarded as a more generally applicable method of quantization than the canonical framework, in the sense that the former method may be applied to a broader variety of theories.¹ A case in point is general relativity, which admits, at least formally, a sum over histories quantization, but may not admit a canonical one. The existence of a composition law is closely related to the derivation of a canonical formalism from the sum over histories. This observation provides the motivation for our discussion of the derivation of a composition law for the relativistic particle propagator. Although we know that in this particular case, composition laws, of the form,

\[ \mathcal{G}(x'') | x' \rangle = - \int_{\Sigma} d\sigma^\mu \, \mathcal{G}(x'' | x) \, \bar{\mathcal{G}}_{\mu} \, \mathcal{G}(x | x') \]

(1)

and indeed a canonical formulation, exist, it is nevertheless of interest to examine how these can be derived. This will allow us to examine how the derivation might or might not generalize to sums over histories in other reparameterization invariant theories, such as quantum cosmology. A more detailed treatment of the discussion given in this paper may be found in Ref. 2, where proofs are provided for all of the statements below. We shall not include further citations of Ref. 2.

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In non-relativistic quantum mechanics, the propagator \( g(x'', t''|x', t') \) is represented by a sum over histories

\[
g(x'', t''|x', t') = \sum_{p(x', t'\rightarrow x'', t'')} \exp[iS(p)]
\]  

(2)

in which the paths \( p \) move forwards in time \( t \). The composition law

\[
g(x'', t''|x', t') = \int d^3x \ g(x'', t''|x, t) \ g(x, t|x', t')
\]  

(3)

then follows from the fact that each path intersects an intermediate surface of constant time once only, and a partition of paths according to their crossing position may be defined. Thus using (2), (3) may be written as

\[
\sum_{x_i} \sum_{p(x', t'\rightarrow x_i, t)} \sum_{p(x_i, t\rightarrow x'', t'')} \exp \left[ iS(x', t' \rightarrow x_i, t) + iS(x_i, t \rightarrow x'', t'') \right] \\
= \sum_{x_i} g(x'', t''|x_i, t) g(x_i, t|x', t').
\]  

(4)

In relativistic quantum mechanics, by contrast, the propagators may be represented by sums over histories in which the paths go forwards or backwards in time. This arises because the kinetic part of the action for a relativistic particle

\[
S = -mc^2 \int_{\tau'}^{\tau''} d\tau \left[ \frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x^{\nu}}{\partial \tau} \eta_{\mu\nu} \right]^{1/2},
\]  

(5)

involves derivatives with respect to the parameter time \( \tau \) rather than the spacetime time, \( t \). The reparameterization invariance of the action is manifested by the fact that the propagator is independent of the initial and final values of \( \tau \). In fact, it takes the simple form

\[
\mathcal{G}(x''|x') = \int dT \ g(x'', T|x', 0)
\]  

(6)

where \( T \) is proportional to \( \tau'' - \tau' \), and \( g(x'', T|x', 0) \) is the quantum mechanical propagator with Hamiltonian \( H = p^2 - m^2 \), and with \( x'' \) and \( x' \) co-ordinates on \( \mathbb{R}^4 \). The integration range of \( T \) determines the boundary conditions of the propagator \( \mathcal{G} \). For example, integration from 0 to \( \infty \) yields the \( i \) times the Feynman propagator \( G_F \). This expression for the propagator is known as the proper time representation, and similar expressions exist for propagators in other reparameterization invariant systems.

Since the paths summed over in \( g(x'', T|x', 0) \) are free to move forwards or backwards in any co-ordinate time \( x'' \), a general path intersects a surface of constant time arbitrarily many times. As a consequence, a relativistic version of the composition law (3) is not readily recovered from this representation of the relativistic propagator. Below we shall describe how a composition law may be derived, despite this apparent difficulty.
Making use of the proper time representation, (6), where the path integral has the form of a non-relativistic sum over histories, integrated over time, the question of a composition law may be related to the question of factoring the propagators of non-relativistic quantum mechanics across an arbitrary surface in configuration space (in this case \( R^4 \)). This may be achieved using a known result called the Path Decomposition Expansion (PDX).\(^3\)

The PDX is based on a partition of paths according to their first crossing position and time of a surface in configuration space, and may be stated as follows. Let \( g(x'', T|x', 0) \) be a quantum mechanical propagator from a point \( x' \) in a region \( C_1 \) of configuration space, to a point \( x'' \) in a region \( C_2 \) of configuration space, and let \( \Sigma \) be the boundary between \( C_1 \) and \( C_2 \). We assume that the quantum mechanical system is derived from a Lagrangian \( L = \frac{1}{2} M \dot{x}^2 - V(x) \). Then the PDX states that the propagator may be decomposed into the composition of a restricted propagator in \( C_1 \) from \((x', 0)\) to \((x_\sigma, t)\), composed with a standard unrestricted propagator in \( C_2 \) from \((x_\sigma, t)\) to \((x'', T)\), with summations over both \( x_\sigma \) and \( t\), as

\[
g(x'', T|x', 0) = \int_0^T dt \int_{\Sigma} d\sigma \ g(x'', T|x_\sigma, t) \frac{i}{2M} \ n \cdot \nabla g^{(r)}(x, t|x', 0)|_{x=x_\sigma} . \tag{7}
\]

Here, \( d\sigma \) is an integration over the surface \( \Sigma \), and \( n \) is the unit normal to \( \Sigma \) pointing into \( C_2 \). The quantity \( g^{(r)} \) is the restricted propagator in \( C_1 \). It can be defined as a sum over paths lying only within \( C_1 \), or as the propagator satisfying the same differential equation as \( g \), but with the condition that it vanish on \( \Sigma \).

It is also possible to partition the paths according to their last crossing position and time, and this leads to a slightly different composition law,

\[
g(x'', T|x', 0) = -\int_0^T dt \int_{\Sigma} d\sigma \ \frac{i}{2M} \ n \cdot \nabla g^{(r)}(x'', T|x, t)|_{x=x_\sigma} g(x_\sigma, t|x', 0) \tag{8}
\]

It is important to emphasize that both (7) and (8) can be derived using only sum over paths representations of \( g \) and \( g^{(r)} \).

Before we can make use of the PDX, it is necessary to eliminate the restricted propagator \( g^{(r)} \) from the expressions (7) and (8), since we do not know how to relate it to the relativistic propagators \( G \). This may be achieved by making use of a simple method of images type argument\(^4\) (which again may be derived from the sum over paths), which states that provided that the potential \( V(x) \) satisfies obvious symmetry properties (reflection symmetry about \( \Sigma \),

\[
\left. n \cdot \nabla g^{(r)}(x, t|x', 0) \right|_{x=x_\sigma} = 2 \left. n \cdot \nabla g(x, t|x', 0) \right|_{x=x_\sigma} . \tag{9}
\]

As a consequence, (7) becomes

\[
g(x'', T|x', 0) = \int_0^T dt \int_{\Sigma} d\sigma \ g(x'', T|x_\sigma, t) \frac{i}{M} \ n \cdot \nabla g(x, t|x', 0)|_{x=x_\sigma} . \tag{10}
\]
and (8) takes a similar form.

We are now ready to derive the composition law for the relativistic propagator defined by (6). Let us concentrate on the Feynman Green function where \( T \) runs from 0 to \( \infty \). In this case, a straightforward use of both (10) and its analogue derived from (8), yields the familiar composition law,

\[
G_F(x''|x') = -\int_{\Sigma} d\sigma \ G_F(x''|x) \ \delta_n \ G_F(x|x'),
\]

(11)

where \( \Sigma \) is a spacelike hypersurface in \( R^4 \). The PDX can be used to derive composition laws for all the relativistic propagators with proper time representations.

If we attempt to generalize the construction to a particle propagating in a curved spacetime, we find that all the previous steps go through, with the exception of the use of the method of images. This is only true if the propagator is symmetric about each member of a family of factoring surfaces \( \Sigma \). A sufficient condition is that the metric have a timelike Killing vector \( \partial_t \), and that the chosen surfaces be surfaces of constant \( t \), i.e. the metric is static (we anticipate that this will extend to stationary spacetimes). On an arbitrary spacetime a composition law cannot be derived. This observation should be compared with the result\(^5\) that a canonical formulation for a quantized relativistic particle exists only if the ambient spacetime has a timelike Killing vector.

Turning to quantum cosmology, where one may construct a closely analogous propagator between three metrics, one can ask whether such an object might satisfy a composition law. An important result\(^5\) is that there is no Killing vector on superspace. We therefore conclude that it is not possible to derive a composition law, and hence a canonical formulation of quantum cosmology, using the techniques outlined above. This result supports the notion that the sum over histories may be more general than the canonical approach to quantization. However, it is important to emphasize that without a canonical framework, there remains the important problem of how the sum over histories may be used to construct probabilities, i.e. the question of interpretation.

References


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