Graphical and Kinematical Approach to Cosmological Horizons

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Abstract

We study the apparition of event horizons in accelerated expanding cosmologies. We give a graphical and analytical representation of the horizons using proper distances to coordinate the events. Our analysis is mainly kinematical. We show that, independently of the dynamical equations, all the event horizons tend in the future infinity to a given expression depending on the scale factor that we call asymptotic horizon. We also encounter a subclass of accelerating models without horizons. When the ingoing null geodesics do not change concavity in its cosmic evolution we recover the de Sitter and quintessence-Friedmann-Robertson-Walker models.

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1 Introduction

The difficulty to make compatible string theory with accelerating cosmologies has been studied in references [1] and [2]. The de Sitter (dS) universe driven by a positive cosmological term, as well as quintessence dominated universes, posses event horizons; as a consequence two physical systems can only be separated a finite distance, the distance of the event horizon; then spatially asymptotic states cannot exist and the S-matrix of interactions is not well defined.

This fact contrasts with our understanding of string theory; such a theory can be formulated only perturbatively where the initial asymptotically free states (strings) interact giving rise to a description of the final asymptotically free states in terms of an S-matrix. This is the case for an asymptotically flat background; if the geometry is asymptotically anti-dS the asymptotic states
are realized on the conformal theory on the boundary, following Maldacena's conjecture \[3\] \footnote{For the dS case an analogous dictionary has been proposed by Strominger \[4\] (see also \[5\]).}. In general, gravitational backgrounds with horizons are not compatible with a Hilbert space of infinite dimension like the Hilbert space of string theory; in fact when gravity is present the number of degrees of freedom that a system can support is proportional to the area of the system according to the holographic principle \[6\] \[7\]; the event horizon, when present, is a natural place to project the relevant degrees of freedom living on the bulk.

This theoretical difficulty contrasts with observations \[8\] that, if confirmed, will prove that the observable universe accelerates. Due to the attractive nature of the gravitational force for ordinary matter these observations are compatible only with a positive cosmological term, or with a sort of matter with negative pressure termed quintessence, that can be associated to the vacuum energy for an unstable scalar field evolving towards the minimum of its potential; this sort of tachyonic instability can generate a vacuum energy with negative pressure, a limit of which being the dS solution.

Assuming an equation of state for the matter of the type \( p = w \rho \), it can be shown that for an accelerating universe the range of \( w \) is \(-1 \leq w < -1/3\). For a Friedman Robertson Walker (FRW) universe dominated by quintessence, \( w \) is bounded by the previous values. \( w = -1 \) corresponds to a positive cosmological constant \((\rho = -p)\) and the subsequent geometry is the dS one. In \[1\] and \[2\] the formation of event horizons for the previous values of \( w \) has been studied showing that it is difficult to avoid them and, although the observational data can be explained, it is not compatible with the string/M theory by the reasons commented above.

In spite that for positive cosmological constant \( \Lambda \), as well as for quintessence matter, the models of the universe both have event horizons, they are different. In the case of quintessence the horizon geometry is a cone with a singularity on the apex signaling the big bang singularity, the topology being \( S^2 \times R^+ (time) \). The dS horizon geometry however is a cylinder, \( S^2 \times R (time) \), without the singularity in \( t = 0 \), and the distance to the horizon is constant and equal to \( \sqrt{\frac{2}{3}} \Lambda \) so that the spatial limitation is constant during the cosmological time. However, the proper distance to the horizon for a quintessence dominated universe opens up with the cosmic time, being zero at the big bang and growing linearly as the time elapses; the angle of the vertex of the
horizon cone depends on the value of $w$.

In the present research we study the presence of event horizons in a generic way for metrics of the Robertson Walker (RW) type (spatially isotropic). In order to locate them we use a graphic representation that does not make use of the Penrose diagrams and where the distances on the graphics are physical distances (proper or coordinates); concretely we represent the successive light cones of a fiducial observer comoving with the cosmic fluid, that we place in the origin, as well as the world lines of galaxies without peculiar velocity, that serve as fixed coordinate distance lines. Another useful geometric element is the locus of comoving matter points that have a given, fixed proper velocity with respect to the fiducial observer as a function of cosmic time (iso-velocity lines). This graphical representation is in some sense complementary to the conformal mappings that give rise to the Penrose diagrams, and has been used previously [9] with the aim of clarifying the concept of particle horizons, putting explicitly the presence of superluminal velocities between different comoving matter points of the fluid [10]. The velocity of any object, locally must be smaller than $c$, the velocity of light in vacuo (locally, by the equivalence principle, the theory of special relativity applies), but for separated points, a non local measure, it is the theory of general relativity that governs the phenomena.

For the explicit and exact location of the event horizon in a homogeneous expanding cosmological model we need the explicit form of the scale factor $R(t)$ generally deduced from the dynamic equations. But only with a basic information about the scale factor -without its explicit form- and by means of the three kinematical tools previously commented -past light cones, world line of galaxies and isovelocity lines-, can we determine the existence or not of event horizons. We find that the decelerating cosmological models do not have event horizon. However we also find a particular family of accelerated cosmological models that neither have event horizon. Also we are able to give an approximate expression that asymptotes the event horizon with increasing cosmological time. The criteria that we will use to define the location of event horizons are associated with the history of a photon emitted by the galaxy towards the fiducial observer placed at the origin: when by its travel the photon goes over cosmic regions with increasing proper velocities with

\footnote{The term galaxy is used freely illustrating an emitter of light; we really refer to a comoving dust element, or a geometric point.}
respect to the observer, then the emitted photon would never arrive to the origin; in this case the galaxy is placed beyond the event horizon. On the contrary, if in its path the photon enters regions where the fluid velocity decreases (always with respect to the origin) the photon sooner or later hits the origin. In the border we encounter the horizon (with respect to the observer) that is placed where a emitted photon has a trajectory on constant cosmic fluid proper velocities (with respect to the observer). We emphasize that no reference is made to the velocity of the source, that is arbitrary; also we do not prejudge the accelerated or decelerated character of the universe.

The article is organized in five parts. We begin with the present introduction. In the second part we discuss the graphical representation method and we apply it to the quintessence and the dS cases. In the third part we work in a generic cosmological setup and then we obtain general results about event horizons. Finally we present our conclusions. In a mathematical appendix we develop the argument of the main result.

2 Graphical Emergence of Horizons

In this section we begin by introducing the expressions that would be used along the article as well as the notation (that otherwise will be standard); then we use them to study the event horizons for dS and quintessence universes.

The RW geometry is characterized by the line element

\[
 ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2d\Omega_2^2\right) \tag{1}
\]

where \(d\Omega_2^2\) is the line element for the two dimensional unit sphere \(S^2\) and \(k = \pm 1, 0\) is the spatial curvature. The coordinates are comoving with the cosmic fluid; in particular \(t\) is the cosmic time. The velocity of light is \(c = 1\). Due to the isotropy of the geometry (in accordance with the cosmological principle) the origin is a generic point and we can fix one direction (fixing the polar angles) without loss of generality; the metric takes then the form

\[
 ds^2 = -dt^2 + R^2(t)\frac{dr^2}{1 - kr^2}, \tag{2}
\]
where each point of the plane $t - r$ must be considered as a sphere of radius $R(t)r$. Instead of using the radial coordinate $r$, the events can be coordinated using the proper distance to the origin $r = 0$ for a given time $t$; that is taking $dt = 0$ in (2) $D(t, r) = R(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}}$; differentiating we have

$$\frac{dr}{\sqrt{1 - kr^2}} = \frac{dD}{R} - \frac{D}{R^2} \dot{R} \, dt;$$

(3)

substituting in (2) gives

$$ds^2 = -dt^2 + \left( dD - D \frac{\dot{R}}{R} \, dt \right)^2.$$  

(4)

Each point in the plane $t - D$ continues to be a sphere of radius $D$.

Let us remember the three ingredients we will use to fix the event horizons:

1. The locus of the ingoing null geodesics (NGs) expressed in proper distance. Concretely the past light-cone of a fiducial observer that we will place at the origin $r = 0$.

2. The world line of test galaxies without peculiar velocity and its proper distance to the observer (its coordinate distance $r$ will be constant).

3. The proper distance to the origin, as a function of cosmic time, for the galaxies that move with a fixed proper velocity with respect to the observer: isovelocity lines. This last concept will be developed in the next section.

In order to determine the NGs and consequently the causal structure of the geometry we put $ds^2 = 0$ in (4), yielding

$$\frac{dD}{dt} = D \frac{\dot{R}}{R} \pm c;$$

(5)

here we put explicitly $c$, the light velocity. $D$ measures the proper distance to the origin for a photon emitted by a galaxy with a given initial coordinates. There are two contributions to the velocity of the photon with respect to the observer in the origin: $\pm c$ is the local velocity of light with respect to the geometrical background; locally, as a consequence of the equivalence principle, the geometry is minkowskian and are valid the postulates of the theory.
of special relativity; the photon moves with constant velocity $c$ respect to the surrounding matter, $c$ being an upper bound; the two signs of the second term of (5) corresponds to the two possible directions away (+) or towards (−) the origin. The term $H(t)D$ ($H = \frac{\dot{R}}{R}$ is the Hubble constant) represents the photon dragging as a consequence of the variation of the geometry with time; the space-time geometry is not flat and special relativity does not apply, but the general one. (5) shows that the photon velocity with respect to the origin can take arbitrary values $^{3}$; the arbitrariness is encoded in the functional form of the scale factor $R(t)$ that depends on the way the geometry is governed by its coupling with matter (ordinary or vacuum energy).

The relation (5) is purely geometric. To obtain the causal structure of the space-time explicitly we need to know the nature of the matter that couples to the geometry. We suppose that the substratum is an ideal fluid characterized by the equation of state $p = w\rho$. We restrict ourselves to the spatially flat case $k = 0$ that seems to fit well with the observations [8]. Then the Einstein equations in the FRW form are

$$H^2 = \frac{8\pi G \rho}{3}, \quad (6)$$
$$\dot{\rho} + 3H(\rho + p) = 0. \quad (7)$$

The equation of continuity (7) is independent of the constant curvature of the spatial sections. Using the equation of state we can integrate it and obtain the dependence of the density with the scale factor; for $\omega$ constant we obtain

$$\rho \propto \frac{1}{R^{3(1+\omega)}}, \quad (8)$$

The two FRW equations can be recast, integrating (6) with the aid of (8) and differentiating (6) with respect to $t$, with the following result

$$R(t) = R(t_0)\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+\omega)}}, \quad (9)$$
$$\frac{\dot{R}}{R} = -\frac{4\pi G}{3} \frac{1}{1 + 3\omega}(1 + 3\omega)\rho. \quad (10)$$

$^{3}$In (5) we recover the newtonian sum of velocities; this is a signal that the motion of comoving (without proper velocities) galaxies is due to the expansion of the space, not a motion on the space. See [10].
The Hubble constant is
\[ H(t) = \frac{2}{3(1 + \omega)} t. \]  

(11)

We observe that for \( \omega = -1 \) the power law dependence of the scale factor (9) breaks down; in fact for \( p = -\rho \) the fluid is equivalent to the presence of a positive cosmological constant and the appropriate model is the dS universe with an exponential expansion. The equation (10) proves that for \( w < -\frac{1}{3} \) the expansion (\( \dot{R} > 0 \)) is accelerating (\( \ddot{R} > 0 \)). The range \(-1 < \omega < -\frac{1}{3}\) characterizes the quintessence.

To obtain the possible cosmological horizons the usual way to proceed is the following. Given the matter that fills the universe, the \( \omega \) that can depend on the cosmic time, we solve the equations (6) and (10), and obtain the explicit solution for the scale factor \( R(t) \). Using the NGs of the metric (2) we determine the space-time regions that will never be observed and the proper distance to the event horizon as a function of \( t \):

\[
D_{\text{Hor}}(t) = R(t)r_{\text{Hor}}(t) = R(t) \int_t^\infty \frac{dt'}{R(t')}. 
\]

(12)

In order to determine the cosmological event horizons we use a slightly different method. First, we obtain from the metric (given the explicit function \( R(t) \)) the family of ingoing NGs that reach \( r = 0 \) when \( t = t_0 \), i.e. the family of past light cones for an observer at \( r = 0 \). We represent graphically this family of ingoing NGs using proper distances to coordinate them; then we observe an accumulation of past light cones towards a limiting one that will trace the event horizon. Beyond this limit the ingoing NG will never reach \( r = 0 \) and then those space-time region would never be observed. In the following subsections we apply this procedure to the accelerating models, FRW dominated for quintessence and dS, for the flat spatial case.

2.1 The case of quintessence

We integrate the NG differential equation (5) for the quintessence dominated universe. Using the value of the Hubble constant (11) we can write

\[
D(t) = \pm \frac{3(1 + \omega)}{1 + 3\omega} t + Kt^\frac{2}{(1 + \omega)}, 
\]

(13)
Figure 1: Graphical elements for the location of the event horizons. Ingoing NGs (past light cones), world line of a galaxy (gray line) and isovelocity lines (dotted lines) including the Hubble distance (heavy dotted line). We observe that, for the dS and quintessence cases, the NGs don’t explore all the space-time events. They accumulate on a NG limit (thick line) that corresponds to the event horizon. We observe that the galaxy crosses the event horizon at a finite time but the observation time $t_0$ diverges, as for the black hole case.

where $K$ is an integration constant; forcing the light trajectory to be at a proper distance $D_0$ from the origin when the cosmic time is $t_0$ results in

$$D(t) = \pm \frac{3(1 + \omega)}{1 + 3\omega} t + \left[D_0 \mp \frac{3(1 + \omega)}{1 + 3\omega} t_0\right] \left(\frac{t}{t_0}\right)^{\frac{2}{3(1 + \omega)}}. \quad (14)$$

In this way we construct the two families of NGs, one with growing $D$ and the other with decreasing $D$. $D(t)$ according to (14) is the light cone for each point of the manifold $(D_0, t_0)$. The families (14) determine the causal structure of the universe.

The past light cone for the origin $r = 0 (D = 0)$ at time $t_0$ is given by

$$D(t) = \frac{3(1 + \omega)}{1 + 3\omega} t_0 \left[\left(\frac{t}{t_0}\right)^{\frac{2}{3(1 + \omega)}} - \frac{t}{t_0}\right]. \quad (15)$$

Taking the derivative of the previous expression with respect to the cosmic
time gives us the slope of the past light cone
\[ \dot{D}(t) = \frac{3(1 + \omega)}{1 + 3\omega} \left[ \frac{2}{3(1 + \omega)} \left( \frac{t}{t_0} \right)^{-\frac{1+3\omega}{3(1+\omega)}} - 1 \right]; \] (16)
we see that \( \dot{D}(t) \) is a decreasing function of \( t \), its maximum value occurs for \( t = 0 \); because \( -\frac{3(1+\omega)}{1+3\omega} \) is positive for \(-1 < \omega < -\frac{1}{3}\) the slope of the light cone at the initial time is finite and is
\[ \dot{D}(t = 0, \forall t_0) = \frac{-3(1 + \omega)}{1 + 3\omega} > 0 . \] (17)

This means that the events that can influence the observer placed at the origin are limited to the cone with apex in the big bang and with an angle \( \tan^{-1}\left[-\frac{3(1+\omega)}{1+3\omega}\right] \).

To determine the event horizon we must compare the past light cone of the origin with the world lines of the galaxies. The recession of a galaxy that in \( t_0 \) is placed at \( D_0 \) is described by
\[ D(t) = D_0 \left( \frac{t}{t_0} \right)^{-\frac{2}{1+3\omega}} . \] (18)
\( \omega = -\frac{1}{3} \) separates two regimes with different concavity for \( D(t) \) reflecting the accelerated/decelerated behavior of matter. The derivative of (18) with \( t \) is
\[ \dot{D}(t) \propto t^{-\frac{1+3\omega}{1+3\omega}}; \] this derivative at the origin is zero for the quintessence but diverges for \( \omega > -\frac{1}{3} \); the big bang for the quintessence is not big nor bang, however the quintessence density diverges for \( t \to 0 \) as \( t^{-2} \). Now, it is clear that every pair of galaxies has been causally connected in the beginning of the expansion; also it is clear that the same pair of galaxies would be causally disconnected when one of them crosses the event horizon of the other. The horizon grows linearly with the cosmic time according to
\[ D_{Hor}(t) = -\frac{3(1 + \omega)}{1 + 3\omega} t . \] (19)

For ordinary matter \(-1/3 \leq \omega \leq 1\), \( D_{Hor} = \infty \). We can use (15) to study the future light cone; taking \( t_0 = 0 \) and changing sign we obtain the spatial extension of the future light cone from the big bang i.e. the particle horizon, so that for ordinary matter,
\[ D_{PH}(t) = \frac{3(1 + \omega)}{1 + 3\omega} t , \] (20)
and for quintessence \((-1 < \omega < -\frac{1}{3})\), \(D_{PH} = \infty\); particle and event horizons play a dual role (see Table 1).

In Fig. 1 we show the world line for an arbitrary galaxy together with the past light cones of an observer placed at the origin; in this way we visualize the event horizon which is also shown. Beyond this limit all the ingoing NGs never reach \(D = 0\) and so we will never observe at \(D = 0\) the events covered by this NG. We have also depicted the same functions for ordinary matter \((\omega > -\frac{1}{3})\) in Fig. 2. Here the expansion is decelerated and as a consequence the slope of the past light cone at the initial time diverges; we need an infinite amount of kinetic energy to overcome the (also infinite) amount of gravitational attraction; the world line of a generic galaxy, having now contrary concavity to the quintessence, will intercept unavoidably the past light cone of the observer.

![Figure 2](image.png)

Figure 2: Causal structure for a decelerated model with \(\omega = 0\). We observe that the photons emitted by a galaxy (grey line) cross the isovelocity lines (dashed lines) always on the decreasing recession velocity sense. There are not real accumulation of past light cones indicating the absence of event horizons for this case; for every space-time event \((D, t)\) there exists an ingoing NG reaching \(D = 0\).

### 2.2 The case of \(\Lambda > 0\).

The same reasoning can be done for the case of a positive cosmological constant \((\Lambda > 0)\) dominated universe. The dS model corresponds to an equation
of state \( p = -\rho \) (\( \omega = -1 \)). For the FRW equations (6) and (7) it follows that the scale factor goes exponentially with time
\[
R(t) = R_0 e^{\pm \sqrt{\frac{\Lambda}{3}} t},
\]
and the continuity equation shows manifestly the constancy of the vacuum energy \( \rho_\Lambda = \frac{\Lambda}{8\pi G} \). \( R_0 \) is an arbitrary scale factor for \( t = 0 \) because the model, possessing an exponential expansion, is self similar. The Hubble constant is in this case truly constant \( H = \pm \sqrt{\frac{\Lambda}{3}} \) and the model is in agreement with the perfect cosmological principle.

To obtain the causal structure we need to know the past light cone of a generic observer as well as the world line of the galaxies \(^4\). Again we make use of the proper metric (4) and integrating the NGs we arrive to
\[
D(t) = e^{-Ht_0} \left( D_0 \pm \frac{1}{H} \right) e^{Ht} \pm \frac{1}{H}.
\]  
(21)
The previous expression is the light cone for an event placed at the point \((t_0, D_0)\); taking the origin \( D_0 = 0 \) \((r_0 = 0)\) at \( t = t_0 \) as our fiducial observer we see that such an event can be influenced by the events inside the region
\[
D(t) = \sqrt{\frac{3}{\Lambda}} \left( 1 - e^{\sqrt{\frac{3}{\Lambda}} (t-t_0)} \right),
\]  
(22)
where obviously \( t < t_0 \). One observes that the past light cone never extends to distances greater than \( \sqrt{\frac{3}{\Lambda}} \). The world lines of galaxies have the form
\[
D(t) = D_0 e^{\sqrt{\frac{3}{\Lambda}} (t-t_0)},
\]  
(23)
and in a finite cosmic time overtake proper distances with respect to the origin higher than \( \sqrt{\frac{3}{\Lambda}} \). The proper distance \( \sqrt{\frac{3}{\Lambda}} \) is the event horizon for the observer placed at the origin and consequently for any observer. The geometry of the event horizon for the dS universe is cylindrical in front to the conical geometry for the quintessence dominated universe. In Fig. 1 we show the dS model with its causal structure; we plot the successive past light cones from \( D = 0 \). A typical galaxy world line crossing the isovelocity lines has also been included.

\(^4\)The galaxies are test particles or geometric tracers; we suppose that all the energetic contribution is due to the cosmological term.
Table 1: Event Horizon, Particle Horizon, Apparent Horizon and the Hubble distance for FRW and dS flat cosmological models. We show also the behavior of the slope of the galaxy world line and past light cone near the Big Bang.

3 General results. The Asymptotic Horizon

In this part we give a general criterion to place an event (the emission of a photon by a galaxy) beyond or below the possible event horizon of a generic observer. We analyze the recession velocity of the regions traversed by the photon emitted by the galaxy towards the observer at the origin. If this cosmic velocity increases we affirm that the photon never reaches the origin and the photon emitter galaxy is hidden by the horizon; because the local velocity with respect to the cosmic fluid is bounded by 1, the same result happens for any other carrier of information leaving the galaxy. On the contrary, if the photon’s path goes over cosmic regions with decreasing proper velocities with respect to the observer, the light sooner or later will overtake the origin; the emitter is causally connected with the origin. The event horizon is the border between the two previous situations.
3.1 Generic Cosmological Models

We are dealing with cosmologies described by a Robertson-Walker geometry characterized by the scale factor $R(t)$ and the curvature that we take null ($k = 0$). We can obtain valuable information by kinematic as well as geometric arguments deduced for the metric independently of the dynamics, that is, the particular way the geometry couples to matter via the Einstein-Friedman equations. To this end we remember some well known expressions that will fix notation.

**Galaxies-Particles-Tracers** The elements of the cosmic fluid are comoving and so the spatial coordinates $r$ are constant. All the spatial points being equivalent, we take the origin as the reference point. The proper distance of a particle with coordinate $r$, that at time $t_0$ is at proper distance $D_0$, is $D_v(t) = rR(t) = D_0 R(t)/R(t_0)$. From this we obtain the Hubble law which can be rewritten and reinterpreted as an isovelocity law

$$D_v(t) = v \frac{R(t)}{R(t)} ,$$

which is the proper distance to the origin of fluid points, with constant proper velocity $v$, as a function of time.

Using (24) we can identify the space-time points where the recession velocity of the galaxies with respect to the origin is 1; locally the special relativity governs the phenomena and the light velocity with respect to the galaxies is also 1; consequently, the photons emitted by such galaxies towards us have zero velocity with respect to the origin; this fact translates in the development of a maximum for the NGs in such points. The locus where this maxima appears for $k = 0$ defines the apparent horizon and matches the isovelocity line for $v = 1$; this defines the Hubble distance $D_{\text{Hub}} = \frac{R}{R}$. The apparent horizon has been proposed as a holographic screen to realize the holographic principle in cosmology [12] [13] and shares many properties with the black hole horizon, as the locus of trapped surfaces.

**Null Geodesics** The evolution of the NGs that we have previously identified from the metric (5) can also be obtained imposing that the velocity

\footnote{For a very interesting discussion about the implications of $k = \pm 1$ and the kinematical/dynamical character of the cosmological lens effect see [11]}
of the NG with respect to the galaxies that it crosses equals \( \pm c \); using the Hubble law (24) we obtain the velocity of the NGs with respect to us:

\[
\dot{D}_n(t) = \dot{D}_r(t) \pm 1 = D_n(t) \frac{\dot{R}(t)}{R(t)} \pm 1 ,
\]

where \( D_n(t) \) is the proper distance to the NG. The ingoing NGs that meet the origin at \( t_0 \) can also be obtained by direct integration of the metric (2),

\[
D_n(r = 0, t_0)(t) = R(t) \int_{t_0}^{t} \frac{dt'}{R(t')} ,
\]

and describes the universe we see at \( t_0 \); however we need the function \( R(t) \) given by the dynamics to obtain its explicit form. The NG that traverses the point \( (D_0, t_0) \) can be obtained, using the homogeneity hypothesis, by summing the distance between the galaxy and the origin with the distance between the photon and the galaxy

\[
D_n(D_0, t_0)(t) = D_0 \frac{R(t)}{R(t_0)} + D_n(0, t_0)(t) .
\]

It is easy to verify that (27) satisfies the geodesic condition (25).

**Event horizon** In the previous section we have obtained the event horizon identifying the space-time regions that never can be reached by an ingoing null geodesic; the term *never* implicitly means to take the limit \( t_0 \to \infty \); then we need to know the behavior of the function \( R(t) \) for all time and consequently the dynamics. The expression for the event horizon in terms of the scale factor is given by (12). This function represents the temporal evolution of the ingoing NG frontier between photons observed (reaching \( D = 0 \)) and the unobserved ones (\( D \to \infty \)).

### 3.2 The Asymptotic Horizon

Without recourse to the dynamics and therefore valid for a variety of cosmological models we can study the points where the NGs change its concavity by differentiating (25), with the result

\[
\ddot{D}_n = \frac{D_n \ddot{R} \mp \dot{R}}{R} = 0 \Rightarrow D_{A\text{sH}} = \pm \frac{\dot{R}}{R} ,
\]
where the positive sign for $D_{AsH}$ corresponds to ingoing geodesics; $D_{AsH}$ refers to the \textit{asymptotic horizon}. We will show that the event horizons of any cosmological model, will converge, in the future infinity, with the asymptotic horizon previously defined. The rigorous demonstration is left to the appendix, and here we merely argue its plausibility.

If the cosmological model is of the FRW (quintessence) or dS type it is easy to prove that the asymptotic horizon and the event horizon coincides for all time. For the FRW models, the scale factor is given by (9); then the value of $D_{AsH}$ as defined in (28) is $D_{AsH} = -\frac{3(1+\omega)}{1+3\omega}t$; this matches the value of the event horizon (19) for all time. For the case of a cosmological constant driven universe the scale factor goes as $R = R_0 e^{t\sqrt{\Lambda/3}}$; the asymptotic horizon is placed at $D_{AsH} = \sqrt{3/\Lambda}$ which is the distance to the event horizon independent of the cosmic time.

In any other expanding accelerating model a part of the NGs crosses the asymptotic horizon at least once; let us follow them for the last time they traverse the asymptotic horizon. This can be done in two different ways, namely:

- \textit{inside-outside}. This is the usual case (see Fig. 3 left). The relevant part of the bundle of NGs comes from the inside region of the asymptotic horizon ($D_n < D_{AsH}, \dot{D}_n < 0$) and part of them goes outside ($D_n > D_{AsH}, \dot{D}_n > 0$). All the NGs outside the asymptotic horizon increases the distance to the origin without limit. The event horizon will be traced by the NG that approaches the asymptotic horizon from inside but without crossing it; this will be the NG always below the asymptotic horizon without maximum. In this case $D_{Hor} < D_{AsH}$ and $\dot{D}_{Hor} < 0$.

- \textit{outside-inside}. The part of the bundle of NGs we are interested (see Fig. 3 right), advances as the cosmic time elapses in the outside part of the asymptotic horizon ($D_n > D_{AsH}, \dot{D}_n > 0$); part of the bundle will traverse it and enters the inside part ($D_n < D_{AsH}, \dot{D}_n < 0$). Then they reach a maximum distance to the origin and after this, approaches $D = 0$ until they meet the origin and are observed. The horizon is defined by the NG that approaches progressively the asymptotic horizon from the outside part but without traversing it. In that case $D_{Hor} > D_{AsH}$ and $\dot{D}_{Hor} > 0$. 

16
Figure 3: The convergence between the event horizon and the asymptotic horizon is shown for the two cases discussed in text. On the left a universe with dust ($\omega = 0$) and a positive cosmological constant is represented. This case corresponds to the inside-outside evolution of the NGs (see the text). The figure on the right, outside-inside evolution, corresponds to a mixture of quintessence ($\omega = -3/4$) and matter ($\omega = 2/3$). We see also the concavity change for the NGs when crossing the asymptotic horizon.

The two situations previously described are the more general ones. The proof is left to the mathematical appendix.

There is a relation between the deceleration parameter $q = -\frac{\ddot{R}R}{R^2}$ and $D_{AsH}$ namely

$$q = -\frac{D_{Hub}}{D_{AsH}};$$

it is easy to show that

$$\dot{D}_{Hub} = 1 + q;$$

now, if $\ddot{R} > 0$ ($D_{AsH} > 0$) then $q < 0$; if $D_{Hub} < D_{AsH}$ then $q \in (-1, 0)$ and using (30) it follows that $\dot{D}_{Hub} > 0$; more concretely $\dot{D}_{Hub} \in (0, 1)$.

Let us discuss when $D_{Hub} < D_{AsH}$ and the previous arguments applies. In general the ingoing geodesics have maxima that force them to converge until they are observed. The maxima are localized on $D_{Hub}$ and being maxima $\ddot{D}_n < 0$. In the accelerating models, because at $D_{AsH}$ there is a change
of concavity for the NGs, $D_{\text{Hub}}$ must be below the asymptotic horizon. Nevertheless, due to changes of the sign of the acceleration it is possible that $D_{\text{Hub}} > D_{\text{AsH}}$; this would causes the existence of minima in the NGs. It is easy to prove that the presence of minima in the NGs implies the violation of the dominant energy condition [14]. For an equation of state of the type $p = \omega \rho$, we have with the use of the Friedman equations (6) and (10),

$$q = -\frac{\ddot{R}}{R H^2} = \frac{1 + 3\omega}{2}.$$  \hspace{1cm} (31)

If the deceleration parameter is smaller than $-1$ (which is the condition for the presence of minima in the NGs) then $\omega < -1$, violating the dominant energy condition. So, for an expanding model, we will consider the realistic assumption $\dot{D}_{\text{Hub}} > 0$.

### 3.3 No-horizon Universes

We can now prove using kinematical arguments the following statement: *Universes with a final expansive decelerating epoch do not have event horizons.* If $\ddot{R} < 0$ then $\dot{R}$ is a decreasing function, so $\lim_{t \to \infty} \dot{R} = a < \infty$; then for later times $t > t_c$ there exist $\tilde{a} > a$ so that the scale factor has an upper bound $\tilde{a} t > R(t)$; the distance to the horizon diverges

$$D_{\text{Hor}}(t) = R(t) \int_{t_c}^{t} \frac{dt'}{R(t')} > R(t) \int_{t_c}^{\infty} \frac{dt'}{\tilde{a} t'} \to \infty.$$ \hspace{1cm} (32)

We have shown that the necessary condition for the emergence of cosmological horizons in expansive universes is that they have to be accelerated at their final epoch. Then $D_{\text{AsH}} > 0$ and so the emergence of event horizons imply the concavity change of some of the NGs (the NGs that intersect $D_{\text{AsH}}$). But not all the accelerating universes have event horizon; if $\ddot{R}(t) \geq 0$ and $\lim_{t \to \infty} \ddot{R} = 0^+$, although $\dot{R}(t) > 0$ for all $t$, all the NGs intersect $D_{\text{AsH}}$ from outside to inside and finally reach $r = 0$. The demonstration of this last statement is identical to the previous one. Again, we can bound from above the scale factor with a linear function of $t$ and proceed as in (32). It is easy to prove also that this family of universes verify $\lim_{t \to \infty} D_{\text{Hub}} = 1^-$

\[ \text{Unless the asymptotic horizon coincides with the event horizon for all times, which is the case of quintessence and dS models.} \]
and that \( \lim_{t \to \infty} \dot{D}_{A\&H} = \infty \) (see (41) in the Appendix). There is an infinity of functions obeying this condition and it is possible to study (with the dynamical equations) the relation between the equation of state, that is the value of \( \omega(t) \), and the functional form of the scale factor (work in progress).

Figure 4: For an accelerating model with acceleration tending to zero there is not accumulation of past light cones and therefore all the space-time events are traversed by them signaling the absence of event horizons. We also represent the Asymptotic Horizon (thick line); we can check its divergent slope and the concavity change of the ingoing NGs there.

### 3.4 Kinematical deduction of the event horizons for FRW and dS universes

Let us follow the path for a photon emitted by a galaxy in order to see if it can reach the origin and can be observed or not. If the velocity of the galaxy is bigger than 1 the photon will travel to us with negative velocity and will go away. If the universe is accelerating, the photon in its travel traverses regions with higher fluid velocity and its relative proper velocity with respect to the origin grows. The first statement is true; the second false, because the photon evolves also in the temporal direction, and in this direction the velocity of
the galaxies always decreases; this can be seen differentiating the Hubble law
\[ v(t) = \frac{D}{D_{Hub}} \] for fixed \( D \)

\[ \frac{\partial v}{\partial t} = -D \frac{\dot{D}_{Hub}}{D_{Hub}^2} < 0, \] (33)

because \( \dot{D}_{Hub} > 0 \); in fact \( \dot{D}_{Hub} \in (0, 1) \) in the accelerating epochs 7.

We see that the final history of the photon depends on the way the galaxy recession velocity field is distributed. This is described by the isovelocity bundle (24); therefore, if the evolution of the photon traverses the isovelocity lines towards growing values then this photon goes away from the origin with increasing relative velocity being \textit{probable} it will never return to \( r = 0 \); and vice versa. In the first case \( \ddot{D}_n > 0 \), in the other case \( \ddot{D}_n < 0 \).

The \textit{probable} horizon will be placed where \( \ddot{D}_n = 0 \), and we have seen that such points are localized by the function \( D_{AsH}(t) \) (28). What happens in \( D_{AsH} \) is that a NG ceases to traverse the isovelocity lines in the increasing (decreasing) sense, during some instants is parallel to one of them and finishes traversing the bundle in the decreasing (increasing) direction. The positive sign for \( D_{AsH} \) and consequently the existence of this \textit{probable} horizon implies an accelerating universe 8. This horizon is \textit{probable} and not \textit{sure}; it can happens that a NG exploring regions of growing recession velocities, and that we suspect to be unobservable, in a given time crosses the line \( D_{AsH} \) and by changing its concavity begins to travel regions characterized by decreasing recession velocities respect to the origin arriving at the end to \( r = 0 \). To avoid this situation and in this way to confirm that the \textit{probable} horizon is the \textit{sure} horizon, we must be certain than each of the NGs cannot change concavity. This does not means that all of them have the same concavity because in this case \( D_{AsH} < 0 \). The situation we consider corresponds to two families of NGs with different concavities separated by a lineal NG that forms the border between them and that matches \( D_{AsH} \).

What cosmological model verifies than \( D_{AsH} \) is the locus of an ingoing NG? If this condition is satisfied we can affirm that \( D_{AsH} \) is the event horizon of such models of the universe. If \( D_{AsH} \) is an ingoing NG by no means will be traversed by any other one (the NGs form a bundle that only at the singular points like the Big Bang can meet). As a consequence the rest of

7 This arguments can be used to clarify the issues of the \textit{chained galaxy} paradox [15].
8 We do not consider the decelerated contracting universes where also \( D_{AsH} > 0 \).
the NGs have a definite and invariable concavity; \( D_{AsH} \) will be then linear 
\( D_{AsH} = at + b \). We appreciate two possibilities:

- Case \( a \neq 0 \). Introducing \( D_{AsH} \) in the geodesic equation (25) we obtain

\[
\frac{a + 1}{at + b} = \frac{\dot{R}(t)}{R(t)} \Rightarrow R(t) = R_0 t^{1 + \frac{1}{a}}, \tag{34}
\]

where we impose \( R(0) = 0 \) (\( b = 0 \)). We have obtained a FRW solution
with the temporal power given in terms of the slope \( a \) of the event horizon instead of the parameter of the equation of state \( \omega = p/\rho \). The slope of the event horizon is then

\[
a = -\frac{3(1 + \omega)}{1 + 3\omega}. \tag{35}\]

- Case \( a = 0 \Rightarrow b \neq 0 \). Substituting the value of \( D_{AsH} \) into (25) it follows

\[
\frac{1}{b} = \frac{\dot{R}(t)}{R(t)} \Rightarrow R(t) = R_0 e^{t/b}. \tag{36}
\]

In this case we obtain a dS universe with \( b \), the distance to the horizon, related with the cosmological constant \( \Lambda \) according to

\[
b = \sqrt{\frac{3}{\Lambda}}. \tag{37}\]

4 Discussion and conclusions

The coordinates used in this work in order to describe the causal structure
of RW cosmological scenarios are not the usual ones; we have used proper
distances to comoving observers as coordinates to represent the light ray
paths and we have been able to extract the event horizons of the models
studied in this way. The standard method used to obtain the causal structure
is to bring the infinities at a finite coordinate value making use of a conformal
transformation of the metric; the light cones in the new metric are the same
as in the original one, but we pay the price that the proper distances are
highly distorted.
The physical position of the event horizon is important due to the holographic principle. The area of the horizon at a given time bounds the entropy of the universe at this time by the projection of the inside degrees of freedom onto the horizon by an holographic prescription; the way in which this projection is made will give important insights on the nature of the way gravity must be consistent with the quantum world; we hope our physical coordinates complement the conformal ones to improve the understanding of the causal structure of the cosmological models.

We emphasize that our analysis has been made, in the main part of this work, using only the kinematics inferred for the line element of the RW model and as a consequence our results are valid for different dynamical scenarios; we require only the agreement with the cosmological principle for the energy momentum tensor that couples to the geometry. In this way we have shown that expanding universes with a final decelerating epoch do not have event horizons. We have introduced the concept of asymptotic horizon as the locus where the NGs change concavity and we have proved that for an expanding cosmological model, if it is accelerating, its future horizon converges with the asymptotic horizon. For universes with constant concavity light rays the asymptotic horizon matches the event horizon for all times; this is the case for the FRW quintessence dominated universe and for the dS model.

The fact that we have an expression for the asymptotic distance of the event horizon can be useful in view of the holographic conjecture; the event horizon is the natural place where the degrees of freedom in the bulk are projected, and its area is a measure of the entropy of the system; asymptotically the area of the horizon is then an absolute upper bound on the number of degrees of freedom of the universe and is given by the area of the asymptotic horizon.

We would like to comment also the teleological nature of the horizon. We cannot be sure at a given time if a model has event horizons or not; this question depends on the scale factor for all the times and specifically on the value the scale factor takes asymptotically. What we can only do is to extrapolate our local knowledge to all future times and extract the consequences. The horizon is a fragile concept as that of the black hole; in this last case the horizon disappears as time evolves but due to Hawking evaporation; in the cosmological case the probable horizon can be diluted but now due to kinematic arguments. Can this two processes be related?

To finish, let us remember that the origin of this research was motivated
by the conflict among the accelerating cosmologies and string theory; the ob-
observation that the actual universe is accelerated and probably will continue
this regime forever, forbids the existence of physical states separated an in-
finiteness distance in an operational sense; that is we cannot make measures on
two states which distance is greater than the horizon; however an S-matrix
description requires asymptotic accessible states. On the other hand we have
been able to identify an accelerating model without event horizon. We hope
this work can help to resolve this challenge.

Appendix

In this Appendix we demonstrate the following statement:

The event horizon of any RW cosmological model, regardless of
the dynamical theory in which is based, converges to the asymp-
totic horizon coinciding both at the future infinity.

We will show that asymptotically the intersection angle between the NGs
that traverse the asymptotic horizon and the proper asymptotic horizon tend
to zero. In this way we can affirm that the asymptotic horizon tends to
coincide with a NG; this limiting NG will be the event horizon. First we
calculate the slope of the NG when it intersects the asymptotic horizon;
introducing $D_{AsH}$ in (25) and using (29) and (30):

$$
\dot{D}_n\big|_{AsH} = \frac{D_{Hub}}{D_{AsH}} - 1 = \frac{-1 - q}{q} = \frac{\dot{D}_{Hub}}{1 - \dot{D}_{Hub}}.
$$

(38)

Then we show that the previous expression for large times is equal to the slope
of the asymptotic horizon ($\dot{D}_n\big|_{AsH} = \dot{D}_{AsH}$). We do this for the different
values that takes the slope of the Hubble distance at infinite time; we define
$l \equiv \lim_{t \to \infty} \dot{D}_{Hub}$. Because the universe is accelerating $0 \leq l \leq 1$. We study
the following cases.

A $l = 0$ and $D_{Hub} \to H < \infty$ (Horizontal Asymptote).

Using (38) we have $\dot{D}_n\big|_{AsH} \to 0$; then we have $q = -1$ at late times and

$$
D_{AsH} \to D_{Hub} \Rightarrow D_{AsH} \to H \Rightarrow \dot{D}_{AsH} \to 0.
$$

(39)
Examples of this behavior are finally dS dominated models (final exponential grow). See Fig. 3 left.

**B** \( l = 0 \) and \( D_{\text{Hub}} \to \infty \) (Parabolic grow).

Again \( \dot{D}_n |_{\text{AsH}} \to 0 \). Now, because \( D_{\text{AsH}} > D_{\text{Hub}}, D_{\text{AsH}} \to \infty \); so together with (29) and (30) we use the l’Hôpital theorem. We find a solution identical to the previous one

\[
\frac{D_{\text{Hub}}}{D_{\text{AsH}}} \to 1 \Rightarrow \frac{\dot{D}_{\text{Hub}}}{\dot{D}_{\text{AsH}}} \to 1 \Rightarrow \dot{D}_{\text{AsH}} \to 0.
\] (40)

We don’t know any simple realistic model obeying this behavior.

**C** \( l \in (0, 1) \) (Oblique Asymptote).

Also with (29) and (30) and the l’Hôpital theorem we find

\[
\lim_{t \to \infty} \frac{D_{\text{AsH}}}{D_{\text{Hub}}} = \frac{1}{1 - l} \Rightarrow \lim_{t \to \infty} \dot{D}_{\text{AsH}} = \frac{l}{1 - l},
\] (41)

which exactly corresponds to the limit of (38). In this case \( D_{\text{Hub}} \) and \( D_{\text{AsH}} \) tend to a linear function. Examples of this behavior are finally FRW quintessence dominated models (final potential grow). See Fig. 3 right.

**D** \( l = 1 \) (Limiting Oblique Asymptote).

This case corresponds to an acceleration tending to zero from above and it has been shown that there are not event horizons (see subsection 3.3). With (41) we find that now \( \dot{D}_{\text{AsH}} \) diverges and so \( D_{\text{AsH}} \) has no asymptote. Examples of this behaviour are finally FRW models with \( \omega \to -\frac{1}{3} \) (final linear grow). See Fig. 4.

We have covered all the possible finally accelerated scenarios and therefore we have completed our demonstration.

The Table 2 resume our classification including the decelerated universes (E and F) and an exotic accelerated universe (Sub A).

Finally we want to comment an interesting property. It can be shown that if the event horizon has an asymptote, then it is the unique NG having an asymptote. The rest of the NGs reconverge or recede without asymptote.
<table>
<thead>
<tr>
<th>Type</th>
<th>$\omega \to$</th>
<th>$R(t) \sim$</th>
<th>$D_{Hub} \to$</th>
<th>$\dot{D}_{Hub} \to$</th>
<th>$D_{AsH} \to$</th>
<th>$D_{Hor} \to$</th>
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<td>A</td>
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<td>$e^{at}$</td>
<td>$\frac{1}{a}$</td>
<td>$0^+$</td>
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<td>$D_{AsH}$</td>
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<tr>
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<td>$e^{at^n}$</td>
<td>$n&lt;1$</td>
<td>$\frac{1}{n} t^{1-n}$</td>
<td>$0^+$</td>
<td>$\frac{1}{n} t^{1-n}$</td>
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<tr>
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<td>$b&gt;1$</td>
<td>$\frac{1}{b} t$</td>
<td>$1$</td>
<td>$\pm \infty$</td>
</tr>
<tr>
<td></td>
<td>$-\frac{1}{3}$</td>
<td>$d t$</td>
<td>$t$</td>
<td>$1$</td>
<td>$\pm \infty$</td>
<td>$D_{AsH} \to \infty$</td>
</tr>
<tr>
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<td>$t^b$</td>
<td>$b&lt;1$</td>
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<tr>
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<td>$\log t$</td>
<td>$t \log t$</td>
<td>$\log t$</td>
<td>$-t$</td>
<td>$\mathbb{F}$</td>
</tr>
<tr>
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<td>$e^{at^n}$</td>
<td>$n&gt;1$</td>
<td>$\frac{1}{n} t^{m-1}$</td>
<td>$0^-$</td>
<td>$D_{Hub}$</td>
</tr>
</tbody>
</table>

Table 2: Classification of the RW cosmological models based on the growing rate of $D_{Hub}$ associated with different ranges for $\omega$. The values for $D_{AsH}$ and $D_{Hor}$ are also exposed. Compare each case with its graphic representation. The last two rows corresponds to pathological universes.

Using the l’Hôpital heorem in the NG equation (25) we found that if $\lim \dot{D}_n \neq \infty$ then

$$\lim_{t \to \infty} \dot{D}_n = \lim_{t \to \infty} \frac{\dot{D}_n}{\dot{D}_{Hub}} - 1 \Rightarrow \lim_{t \to \infty} \dot{D}_n = \frac{l}{1 - l} \in (0, \infty) \quad (42)$$

25
which is identical to (41) and can be easily extrapolated to (39). So, for the cases A and C we can prescribe this rule: the event horizon is the unique NG having an asymptote and the asymptotic horizon has exactly the same asymptote. This property could be used for a better delimitation of the event horizons by means of the Legendre’s transformation which relates a mathematical function to its family of tangent straight lines.

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