Lepton Universality, Rare Decays and Split Fermions

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Abstract

We investigate the constraint on the split fermions in extra dimensions by considering the universality of $W$ leptonic decays $W \rightarrow l_i \nu_i$, the lepton decays $l_i \rightarrow l_j \nu_i \bar{\nu}_j$, the lepton flavor violating process $l_i \rightarrow \bar{l}_j l_k l_h$ and the $l_i \rightarrow l_j \gamma$ transition where $l_i = e, \mu$ or $\tau$. For the Standard Model (SM) background of $W \rightarrow l_i \nu_i$, we extended the one loop quantum correction to include effects of order $m_i^2/M_W^2$ and the Higgs mass dependence. We find that in general the split fermion scenarios the 4D effective Yukawa matrix of the Kaluza-Klein Higgs bosons is misaligned with respect to the fermion mass matrix. This leads to decays of $l_i \rightarrow \bar{l}_j l_k l_h$ at tree level and $l_i \rightarrow l_j \gamma$ transition at one loop level. Interestingly the leptonic universality of $W$ boson decays are not affected.
1 Introduction

Recently new avenues of exploring physics beyond the Standard Model (SM) have opened up by assuming that there exists large extra dimensions beyond the four we are familiar with [1],[2],[3]. The graviton and possibly SM gauge singlet particles are allowed to propagate in the extra dimensions. This picture can also provide a natural geometrical understanding of the hierarchy of fermion masses by postulating that the chiral fermions of the SM are localized at different points in the extra dimensions [4]; i.e. they are split from each other. By the same token different families of fermions also occupy different points in bulk space. The localization of a chiral fermion is represented by a Gaussian wave function in the extra dimension $y$. The mass of a fermion is generated via a five dimensional Yukawa term. In four dimensions, after integrating out $y$, a small Yukawa coupling arises due to the small overlap of the wave functions of the left- and right- handed components of a fermion. In this way a hierarchy in the effective 4D Yukawa couplings is obtained without invoking new symmetries. A detail model for the observed quark and lepton masses in terms of their displacements in $y$ has been given [5].

Besides offering a new vista on the Yukawa coupling hierarchy this scenario also point to a novel way of looking at the question of gauge coupling universality. Historically, the branching ratio (Br) of $\pi \to e\nu/\pi \to \mu\nu$ provided the crucial evidence that the charged weak current couples with the same strength to the first two lepton families. This universality study has since been extended to leptonic $\tau$ decays and also to the leptonic branching ratios of the $W$ boson. These are cornerstones that support the SM and they are very accurately predicted in the SM. An example is the $\text{Br}(W \to l_i\nu/W \to l_i\nu) = 1 + O(\alpha)$ where $l_i = e,\mu, or \tau$. In the SM the deviation from unity is a function of lepton masses and the lepton energy
cut used in a given experiment [6]. The dependence on the unknown Higgs boson mass is very weak. As a by-product of our investigation we will give the complete 1-loop SM result. It is very important to examine how proposed new physics will altered these predictions.

Another generic feature of the split fermions scenario is the existence of effective flavor changing neutral currents which we shall demonstrate are related to the separation between two chiral fermions belonging to different families. In this paper we concentrate on the issue of lepton flavor violation (LFV) interactions partly because they involve less theoretical uncertainties.

To see more quantitatively how these various reactions can be used to probe the split fermion scenario we construct the simplest model with one extra dimension and concentrate on the three lepton families. We will not discuss the issue of neutrino mass which is very interesting and beyond our scope; and thus no $\nu_R$ is introduced.

2 Model setup

The model we employed is the 5D SM similar to that introduced in [7] augmented by the distributions of chiral fermions located at different points in the 5th dimension. In particular the left-handed ($L$) lepton doublet is separated from the right-handed ($R$) lepton. For the minimal matter content of the SM there are twelve relative distances $y_i^a - y_j^b$ where $i$ and $j$ are family indices and $a, b \in \{L, R\}$ stand for chiralities. The 4D effective theory is obtained by compactifying the bulk fields on a $S_1/Z_2$ orbifold where $S_1$ is a circle define by $-\pi R \leq y \leq \pi R$. It is natural to assume that $R \sim \text{TeV}^{-1}$ Then we implement the idea that chiral fermions can be trapped at topological domain wall in such a setting [8] and also at different
locations [4], [9]). The zero mode of a fermion is chiral and is given a narrow Gaussian distribution in $y$. At this point we state explicitly the scales involved. We adopt a universal Gaussian width $\sigma$ for all the fermions. The thickness of the brane in the 5th dimension in which the fermions and gauge bosons are localized is denoted by $T$. While localizing fermions are relatively well understood there are still issues regarding (quasi)-localization of gauge fields and some are discussed in [10]. The third scale is the overall radius of bulk space where gravity and the Higgs boson can propagate. It is natural to as $\sigma < T < R$. Coordinates in Minkowski space is denoted by $x^\mu, \mu = \{0 \cdots 3\}$ and in bulk space by $x^M, M = \{0 \cdots 3, y\}$. Also the fifth Dirac matrix is chosen to be $\gamma^y = i\gamma_5$.

The 5D SM Lagrangian with standard notations is given by

$$L_5 = - \frac{1}{4} F^{MN} F_{MN} - \frac{1}{4} G^{(a),MN} G^{(a)*}_{MN} + \overline{L}(x,y)i\gamma^M D_M L'(x,y)$$

$$+ (D_M \Phi(x,y))^\dagger (D^M \Phi(x,y)) - \kappa R \left( |\Phi(x,y)|^2 - \frac{v^2}{2} \right)^2$$

$$- \sqrt{\pi} \lambda_{ij} \overline{L}_i(x,y) \Phi(x,y) E'_j(x,y) + h.c. + \cdots$$

(1)

where $i, j$ are the family indices and $L'$ and $E'$ are respectively the $SU(2)$ doublet and singlet lepton fields and $\Phi$ the bulk Higgs field. In our set up we can ignore KK excitations of the fermions and gauge bosons and keep only the Higgs bosons and its excitations. For $\sigma \ll R$, the chiral zero mode of a fermion field $\Psi^a_i$ located at $y^a_i$ can be normalized to

$$\Psi^a_i(x,y) \sim \frac{1}{\pi \frac{1}{2} \sigma^2} \Psi^a_i(x)e^{-\frac{(y - y^a_i)^2}{2\sigma^2}}$$

(2)

The product of two fermion fields can be approximately replaced by

$$\overline{\Psi}^i(x,y) \Psi^b_j(x,y) \sim \exp \left( - \frac{(\Delta_{ij}^{ab})^2}{4\sigma^2} \right) \delta(y - \bar{y}^{ab}_{ij}) \overline{\Psi}^a_i(x) \Psi^b_j(x)$$

(3)
where \( \bar{y}_{ab}^{ij} = (y_i^a + y_j^b)/2 \) is their average positions and \( \triangle_{ab}^{ij} = y_i^a - y_j^b \). Note that the mass dimensions of various quantities are: \( [\Psi] = 2, [\Phi] = \frac{3}{2}, [\kappa] = 0 \) and \( [\lambda_{ij}] = 0 \).

The scalar field is taken to be even under \( Z_2 \) and we can write
\[
\Phi(x, y) = \frac{1}{\sqrt{\pi R}} \left( \frac{1}{\sqrt{2}} h_0(x) + \sum_{n=1}^{\infty} h_n(x) \cos \frac{ny}{R} \right).
\] (4)

The lowest mode \( h_0 \) is identified as the SM Higgs boson. A similar expression holds for the gauge fields but with \( R \) replaced by \( T \).

The fermion and gauge boson masses was generated by the vacuum expected value (VEV) of bulk Higgs which in the U gauge is
\[
\Phi(x, y) = \begin{pmatrix}
0 \\
\frac{1}{\sqrt{2}} \left( v_b^3 + h_b(x, y) \right)
\end{pmatrix}
\] (5)

The bulk VEV, \( v_b \), relates to \( M_W \) via \( \frac{1}{8} (g^2 \pi T v_b^3) = M_W^2 \). The \( n^{th} \)-KK gauge boson masses are: \( M_{g,n}^2 = \frac{n^2}{T^2} ; M_{W,n}^2 = M_W^2 + \frac{n^2}{T^2} ; M_{Z,n}^2 = M_Z^2 + \frac{n^2}{T^2} \). For the Higgs bosons one has \( M_H^2 = \kappa R v_b^3 \) and \( M_{\phi,n}^2 = M_H^2 + \frac{n^2}{R^2} \). Hence, in this model the KK Higgs bosons are lighter than their corresponding gauge modes.

The effective 4D Yukawa coupling is
\[
L_Y = -\frac{\lambda^{ij}_b}{\sqrt{2}} L^i(x) \left[ \sqrt{\pi R} v_b^3 + \frac{h_0}{\sqrt{2}} + \sum_{n=1}^{\infty} h_n \cos \frac{ny}{R} \right] E^j(x) + h.c.
\] (6)

where \( \lambda^{ij}_b = \lambda_{ij} \exp \left(-\frac{\gamma_{LR}^2}{4\sigma^2} \right) \). The mass matrix is readily seen to be
\[
\mathcal{M}_{ij} = \lambda_{ij} \sqrt{\frac{\pi R}{2}} v_b^3 e^{-\frac{\gamma_{LR}^2}{4\sigma^2}}
\] (7)

This is diagonalized by a bi-unitary transformation with
\[
L'(x) = V_L L(x), \quad E'(x) = V_R E(x)
\] (8)
and $L(x)$ and $E(x)$ are mass eigenstates.

It is easy to see that this diagonalization also rotates away the off-diagonal coupling of fermions to the Higgs zero mode; i.e. the SM Higgs to fermion couplings remain flavor diagonal. Furthermore, the SM gauge bosons fermion couplings are also flavor diagonal. However, the Higgs KK modes will couple different mass eigenstate fermions as displayed in

$$
\mathcal{L} = -\bar{L}_i \left( m_i + \frac{g m_i}{2 M_W} h_0 \right) E_i - \sum_{n=1} y_{n,ij} \bar{L}_i E_j h_n + h.c. \quad (9)
$$

where

$$
y_{n,ij} = \frac{1}{\sqrt{2}} V_{L,ki}^* \lambda_{kl} \exp \left( -\frac{\left( \Delta_{kl}^L \right)^2}{4 \sigma^2} \right) \cos \frac{n \bar{y}_{kl}^{LR}}{R} V_{R,ij}. \quad (10)
$$

In other words one cannot simultaneously diagonalize the fermion mass matrix and the Yukawa matrix of the KK Higgs-fermion couplings due to the presence of the cosine terms. This will induce tree level lepton flavor violation processes which we will discuss below.

3 Constraint on the fermion locations

3.1 Lepton Universality

In terms of mass eigenstate the effective 4D charge current interaction is

$$
\mathcal{L}_{eff} = -g L_i \left[ \gamma^\mu \frac{\tau^+}{\sqrt{2}} \left( \delta_{ij} W^+_{0,\mu} + \sum_{n=1} W^+_{n,\mu}(x) U_{ij}^{(n)} \right) \right] L_j + h.c. \quad (11)
$$

where $U_{kl}^{(n)} = \sum_i \sqrt{2} (V_L)^*_{ik} \cos \left( \frac{m_{kl}^+}{T} \right) (V_L)_{il}$. Since we only have one bulk Higgs field there is no mixing between physical $W$ boson, which is the zero mode, and its KK excitation. Therefore, it is universally coupled to the lepton families. We conclude
that lepton universality tested by ratios of the leptonic width of the $W$ boson will remain at the SM values $^1$.

On the other hand for the classic decay of $\mu \rightarrow e\nu\bar{\nu}$, where virtual $W$ KK modes also participate, information on the $y$-dependence can be gleamed. Explicitly we get

$$i\mathcal{M} \simeq -\frac{ig^2}{2M_W^2}(\bar{\nu}_\mu \gamma_\mu^\nu_L)(\bar{e}\gamma_{\mu L}\nu_e) \left[ 1 + \sum_{n=1} U^{(n)}_{22} U^{(n)*}_{33} \frac{M_W^2}{M_W^2 + \frac{m_e^2}{T^2}} + \cdots \right]$$  \hfill (12)

The square bracket gives the modification to the Fermi coupling constant, $G_F$ and also generalizes the usual KK result $^1$. This in turn leads to the following prediction for the decay of the $\tau$:

$$\frac{\Gamma(\tau \rightarrow \mu\bar{\nu}_\mu\nu_\tau)}{\Gamma(\tau \rightarrow e\bar{\nu}_e\nu_\tau)} \approx \frac{1 - 8 \frac{m_\mu^2}{m_\tau^2} - 12 \left( \frac{m_\mu^2}{m_\tau^2} \right)^2 \ln \frac{m_\mu^2}{m_\tau^2} + 2 M_W^2 T^2 \text{Re} \left\{ \sum_{n=1} \frac{t^{(n)*}_{22} t^{(n)}_{33}}{n^2} \right\}}{1 - 12 \left( \frac{m_\mu^2}{m_\tau^2} \right)^2 \ln \frac{m_\mu^2}{m_\tau^2} + 2 M_W^2 T^2 \text{Re} \left\{ \sum_{n=1} \frac{t^{(n)*}_{22} t^{(n)}_{33}}{n^2} \right\}}$$  \hfill (13)

Using the current experimental limit $^2$ we find

$$M_W^2 T^2 \sum_{n=1} \frac{1}{n^2} \text{Re} \left\{ U^{(n)*}_{22} U^{(n)}_{33} - U^{(n)*}_{11} U^{(n)}_{33} \right\} \leq .003$$  \hfill (14)

which is a constraint on chiral fermion geography. For illustration, we discuss a special charged lepton mass matrix$^3$:

$$M = \begin{pmatrix} \epsilon^5 & \epsilon^3 & 0 \\ \epsilon^3 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$$  \hfill (15)

as an example. Here $\epsilon$ is an arbitrary small number which can be related to the parameters of our model. To a good approximation, the rotation matrix $V_L$ and $V_R$ are

$^1$For models in which the gauge boson zero mode has a non-uniform shape in $y$ the lepton universality will be broken.
diagonal matrices $\text{diag}(-1, 1, 1)$. The mass matrix structure can be easily achieved by locating charged leptons at $y^L = \{y_1, y_2, (y_3 + \delta)\}$ and $y^R = \{-y_1, -y_2, (y_3 - \delta)\}$ with $y_3 > y_1 > y_2 > 0$ and a properly chosen $\delta$ to ensure a correct mass pattern. Note that not any arbitrary mass matrix pattern can be accommodated in the split fermion scenario, e.g. the one proposed in [14]. The solution is not unique, so we arrange $\bar{y}_{L/R}^{11} = 0$ to simplify and reduce the above limit to $(M_W^2 (y_1^2 - y_2^2)/2) < 0.003$ or $\sqrt{y_1^2 - y_2^2} < 1 \times 10^{-3}\text{GeV}^{-1}$. Note that after summing the series the $T$-dependence goes away.

### 3.2 $\mu$ and $\tau$ to three charged leptons

Due to its non-diagonal couplings the KK Higgs bosons will induce the following processes at tree level: $\mu \rightarrow 3e$, $\tau \rightarrow 3e$, $\tau \rightarrow \mu ee$, $\tau \rightarrow \mu \mu e$ and $\tau \rightarrow 3\mu$. The present upper branch ratio limit for the muon is around $10^{-12}$ and for the $\tau$ is about $10^{-6}$ [12]. To a good approximation we can neglect final state lepton masses and obtain the decay width ratio ($l_\alpha = \mu$ or $\tau$)

$$\eta_{ijk}^\alpha \equiv \frac{\Gamma(l_\alpha \rightarrow l_i l_j l_k)}{\Gamma(l_\alpha \rightarrow l_i \bar{\nu}_i \nu_\alpha)} \simeq \frac{R^4}{4G_F^2} \sum_{n=1}^\infty \frac{|y_{n,\alpha i}|^2 |y_{n,jk}|^2}{n^4} \quad (16)$$

Assuming a universal Yukawa coupling, i.e. $\lambda_{ij} = 1$, the same charged lepton mass matrix of Eq.(15), and also the previous fermion location setup, it predicts $\eta_{111}^3 \sim \eta_{211}^3 \sim \eta_{221}^3 \sim \eta_{222}^3 \sim 0$ and

$$\eta_{111}^2 \simeq \frac{R^4 \pi^4}{720G_F^2} \exp \left( -\frac{(y_1 + y_2)^2}{2\sigma^2} - \frac{2y_1^2}{\sigma^2} \right) < 1 \times 10^{-12}. \quad (17)$$

In other words, from experimental limit on $\text{Br}(\mu \rightarrow 3e)$, this setup requests

$$\frac{(y_1 + y_2)^2}{2\sigma^2} + \frac{2y_1^2}{\sigma^2} > 20.7. \quad (18)$$

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3.3 $\mu$ and $\tau \to e\gamma$

The flavor violating mechanism we have identified will lead to $\mu \to e\gamma$ at the 1-loop level from virtual KK Higgs diagrams. The effective $l_i \to l_j\gamma$ transition amplitude is

$$ T_{\text{eff}}(l_i \to l_j\gamma) = -\frac{\zeta_{ji}e}{16\pi^2m_H} l_j(p')[i\sigma^{\mu\nu}(p-p')\nu]l_i(p)\epsilon^* $$

$$ \zeta_{ji} = \sum_{\alpha \in \{e,\mu,\tau\}} \sum_{n=1}^{y_{n,j\alpha}^* y_{n,\alpha}^*} \frac{m_H}{m_{\phi,n}^2} \left( \alpha \left( \frac{3}{2} + \ln \frac{m_\alpha^2}{m_{\phi,n}^2} \right) - \frac{m_i + m_j}{6} \right) $$

Then $\mu e\gamma$ branch ratio is

$$ \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu \to e\nu\bar{\nu})} = \frac{3\alpha|\zeta_{12}|^2}{8\pi m_\mu^2 m_H^2 G_F^2} $$

Similar equations apply to the corresponding $\tau$ decays. Assuming $m_H = 110$ GeV, the present limits[12] translate into $|\zeta_{12}| < 1.5 \times 10^{-8}$, $|\zeta_{13}| < 1 \times 10^{-4}$ and $|\zeta_{23}| < 8 \times 10^{-5}$.

Very similar calculations can be performed for rare $Z$ decays. Using the above numbers, we conclude that $Br(Z \to e\mu) < 1.4 \times 10^{-20}$, $Br(Z \to e\tau) < 6.4 \times 10^{-13}$ and $Br(Z \to \tau\mu) < 4.1 \times 10^{-13}$ which are way below experimental capabilities in the near future.

We note in passing that the same flavor violating Yukawa coupling can give new contribution to leptons’ anomalous magnetic moment. With the limits derived above we find no significant shift of the SM value to $(g-2)_\mu$.

4 $W$ boson universality in the SM

As we have seen previously that the LFV mechanism predicts that universality holds in $W$ boson decays but not $\mu$ or $\tau$ decays. Thus, it is important to establish the
SM values for these processes. We calculate the $W$ boson branching ratios to 1-loop order in the on-shell scheme and used the unitary gauge which was done in [6]. It is well known the width of $W \to l\nu$ is infrared finite only after including the radiative mode $W \to l\nu\gamma$ [15]. After a laborious calculation we find the leptonic decay width including undetected photon is

$$\frac{\Gamma}{\Gamma^0} = \left[ 1 + \frac{\alpha}{2\pi} \left\{ 2 \left( 2 + \frac{1 + \beta}{1 - \beta} \ln \beta \right) \left( \ln \frac{M_W}{2\Delta E_l} + 2\ln(1 - \beta) \right) \right. \right.$$

$$\left. - \frac{3}{2} (1 - \beta) \ln \beta + f_H \beta \right\} \right]$$

where $\Gamma^0$ is the lowest order width and $\beta \equiv m_l^2/M_W^2$. The quantity $\Delta E_l$ is the finite energy resolution of the charged lepton and is determined by a given experiment. $f_H$ is a complicate function dependent on the Higgs mass. The exact form of $f_H$ is not very illuminating and to a good approximation it is

$$f_H \approx 7.72 + 0.78 \ln \frac{M_H^2}{M_H^2}$$

Numerically the values of $f_H$ are $\{7.36, 6.32, 5.84, 5.34\}$ corresponding to Higgs masses of $M_H = \{110, 180, 250, 400\}$ GeV respectively. Assuming that energy resolutions are the same for all charged leptons the $W \to \nu e, \nu\mu, \nu\tau$ decay width ratio is $1 : 1.067 : 1.103$ for $\Delta E = 2$ GeV and $1 : 1.038 : 1.057$ for $\Delta E = 5$ GeV. With the expected large number of $W$ bosons to be produced at the LHC[16] we can expect this prediction to be tested in the near future.

5 Conclusion

We have shown that in the split fermion scenario with a bulk Higgs boson it is not possible to diagonalize simultaneously the lepton mass matrix and the Yukawa
matrix of the KK Higgs modes. This leads to interesting new mechanism for rare $\mu$ and $\tau$ decays without affecting lepton universality as probe by $W$ boson decays. On the other hand leptonic universality as probed by $\tau$ lepton decays is altered by the virtual KK gauge boson exchanges. This gives a upper limit on the separation of different families of leptons. In contrast rare LFV effects if seen are to be understood as measuring the relative distances of a left-handed fermion of one family to the right-handed fermion of a different family in the extra dimension. If no signals are found in the next round of experiments they give a lower bound on the fermion separations. Similarly the fermion masses sets the relative distances between fermions of opposite chiralities in the same family.

The above considerations can easily be extended to the quark sector. The universality test of pion leptonic decay will set a limit of (see Eq.(13))

$$M_W^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Re} \left[ U_{ud}^{(n)} \left( U_{11}^{(n)*} - U_{22}^{(n)*} \right) \right] \leq 10^{-7}$$

where $U_{ud}^{(n)}$ is an obvious generalization to the quark sector.

Our study has shown the importance of low energy precision tests in covering the parameter space for these models. While it is too early to do complete phenomenological analysis of even the minimal model due to the scarcity of data at the same time we feel that more studies involving similar rare processes are crucial. They are complementary to direct collider searches for the KK excitations of the SM particles.

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References


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