HADRONIC LIGHT-BY-LIGHT SCATTERING
CONTRIBUTION TO THE MUON $g - 2$ IN LARGE-$N_C$ QCD

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We present our recent semi-analytical evaluation of the pion-pole contribution to hadronic light-by-light scattering, using a description of the pion-photon-photon form factor based on large-$N_C$ and short-distance properties of QCD. Inclusion of all light pseudoscalar states leads to $a_{\mu}^{\text{LbyL};\pi0} = (+8.3 (1.2) \times 10^{-10})$. We also sketch an effective field theory approach to hadronic light-by-light scattering. It yields the leading logarithmic terms that are enhanced by a factor $N_C$ and shows that the sign of $a_{\mu}^{\text{LbyL};\text{had}}$ cannot be inferred from a constituent quark-loop. In view of several problematic issues that still remain to be clarified, our estimate for the full hadronic light-by-light scattering contribution is $a_{\mu}^{\text{LbyL};\text{had}} = (+8 (4) \times 10^{-10})$.

1 Introduction

In early 2001, the Muon $g - 2$ Collaboration at the Brookhaven National Laboratory announced a deviation of 2.6 $\sigma$ between their measured value for $a_\mu$ from the prediction in the Standard Model (SM).

At the same time, we were working on a description of the pion-photon-photon transition form factor $F_{\pi\gamma\gamma}^{\mu}$ based on large-$N_C$ QCD, which incorporates short-distance constraints from the operator product expansion (OPE) 2. This form factor enters in the numerically dominant pion-pole contribution to the hadronic light-by-light scattering correction to $g - 2$, denoted by $a_{\mu}^{\text{LbyL};\text{had}}$. Since the form factors used in 3,4 did not fulfill all short-distance constraints, we found it worthwhile to reevaluate this pion-pole contribution $a_{\mu}^{\text{LbyL};\pi0}$. Furthermore, we wanted to push the analytical calculation further than was done in Refs. 3,4, which relied purely on numerical methods.

Our semi-analytical calculation 5 produced a result with roughly the same absolute size but with an opposite sign compared to the evaluations by three other groups 3,4,6. In an accompanying paper 7 we therefore provided a deriva-

tion of the coefficient of the leading log-square term, based on an effective field theory (EFT) approach and the renormalization group (RG), which lead again to a positive coefficient. The correctness of the positive result for $a_{\mu \pi L,L}^{\text{L-had}}$ was later confirmed by the authors of Refs. \textsuperscript{3,4} and by independent calculations, see Ref. \textsuperscript{8}. The new result for $a_{\mu \pi L,L}^{\text{L-had}}$ reduces the discrepancy between the experimental value and the SM prediction to about 1.6 $\sigma$.

Below I will present our evaluation \textsuperscript{5} of $a_{\mu \pi L}^{\text{L-pole}}$ (Sec. 2) and sketch an EFT approach \textsuperscript{7} to $a_{\mu \pi L,L}^{\text{L-had}}$ (Sec. 3). Section 4 contains some considerations about leading and next-to-leading logarithms and an argument why the sign of $a_{\mu \pi L,L}^{\text{L-had}}$ cannot be inferred from a constituent quark-loop. I will conclude with an estimate for $a_{\mu \pi L,L}^{\text{L-had}}$ with a conservative error bound which takes into account the still unsolved problems.

2 Pion-pole contribution

The quantity $a_{\mu \pi L,L}^{\text{L-had}}$ involves the fourth-rank vacuum polarization tensor $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3)$, built from the light-quark electromagnetic current $j_\mu = (2\bar{u}\gamma_\mu u - d\gamma_\mu d - s\gamma_\mu s)/3$. In this section we will concentrate on the contribution from the neutral pion intermediate state, see Fig. 1, which can be represented as follows\textsuperscript{5}

$$a_{\mu \pi L}^{\text{L-pole}} = -e \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \frac{1}{[(p + q_1)^2 - m^2][(p - q_2)^2 - m^2]} \times$$

$$\left[ \mathcal{F}_{\pi \gamma^* \gamma^*}(q_1^2, (q_1 + q_2)^2) \mathcal{F}_{\pi \gamma^* \gamma^*}(q_2^2, 0) T_1(q_1, q_2; p) - \mathcal{F}_{\pi \gamma^* \gamma^*}(q_1^2, q_2^2) \mathcal{F}_{\pi \gamma^* \gamma^*}((q_1 + q_2)^2, 0) T_2(q_1, q_2; p) \right].$$

The pion-photon-photon transition form factor $\mathcal{F}_{\pi \gamma^* \gamma^*}$ is defined by

$$i \int d^4 x e^{ix \cdot \ell} \langle \Omega | T \{ j_\mu(x) j_\nu(0) \} | \pi^0(p) \rangle = \varepsilon_{\mu\nu\alpha\beta} q_\alpha p_\beta \mathcal{F}_{\pi \gamma^* \gamma^*}(q^2, (p - q)^2).$$

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that we studied in Ref. 2 with one photon on shell behaves like 1

In the following, we consider the form factors that are obtained by truncation of the infinite sum (3) to one (lowest meson dominance, LMD), respectively, two (LMD+V), vector resonances per channel:

\[
F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2 + q_2^2 - c_V}{(q_1^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2)},
\]

\[
F_{\pi^0 \gamma^* \gamma^*}(q_1^2, q_2^2) = \frac{F_\pi}{3} \frac{q_1^2 q_2^2 (q_1^2 + q_2^2) + h_1(q_1^2 + q_2^2)^2 + h_2 q_1^2 q_2^2 + h_5(q_1^2 + q_2^2) + h_7}{(q_1^2 - M_{V_1}^2)(q_1^2 - M_{V_2}^2)(q_2^2 - M_{V_2}^2)(q_2^2 - M_{V_2}^2)}.
\]

with \(c_V = N_C M_{V_1}^2 / (4\pi^2 F_\pi^2)\), \(h_7 = -N_C M_{V_1}^4 M_{V_2}^4 / (4\pi^2 F_\pi^2)\). Not all parameters in the LMD+V form factor are fixed by the normalization and the leading term in the OPE. We have therefore determined the coefficients \(h_1, h_2, \) and \(h_5\) phenomenologically. According to Refs. 10,11 the form factor \(F_{\pi^0 \gamma^* \gamma^*}(-Q^2, 0)\) with one photon on shell behaves like \(1/Q^2\) for large spacelike momenta, \(Q^2 = -q^2\). Whereas the LMD form factor does not have such a behavior, it can be reproduced with the LMD+V ansatz, provided that \(h_1 = 0\). A fit of the LMD+V form factor to the CLEO data 11, with \(h_1 = 0\), yields
$h_5 = 6.93 \pm 0.26 \text{ GeV}^4$, see Ref. 2. Analyzing the experimental data\textsuperscript{12} for the decay $\pi^0 \to e^+e^-$ in the context of Ref. \textsuperscript{13}, leads to $|h_2| \lesssim 20 \text{ GeV}^2$.

The WZW form factor is given by $F_{\pi^0\gamma^*\gamma}^\text{WZW}(q_1^2, q_2^2) = -N_C/(12\pi^2 F_\pi)$ and the usual VMD form factor reads

$$F_{\pi^0\gamma^*\gamma}^\text{VMD}(q_1^2, q_2^2) = -\frac{N_C}{12\pi^2 F_\pi} \frac{M_V^2}{(q_1^2 - M_V^2)} \frac{M_V^2}{(q_2^2 - M_V^2)}.$$ (7)

Note that this form factor does not correctly reproduce the OPE in Eq. (4).

All form factors above can be written in the following way

$$F_{\pi^0\gamma^*\gamma}^\text{WZW}(q_1^2, q_2^2) = \frac{F_\pi}{3} \left[ f(q_1^2) - \sum_{M Vi} \frac{1}{q_2^2 - M Vi} g_{M Vi}(q_1^2) \right],$$ (8)

where the explicit expressions for $f(q_1^2)$ and $g_{M Vi}(q_1^2)$ for the different form factors can be found in\textsuperscript{5}. This crucial observation allows one to cancel all dependences on $q_1, q_2$ in the numerators in $F_{\pi^0\gamma^*\gamma}^\text{WZW}(q_1^2, (q_1 + q_2)^2)$ in Eq. (1) and to perform all angular integrations in the two-loop integrals analytically using the method of Gegenbauer polynomials\textsuperscript{14}. The pion-exchange contribution to $a_\mu$ can then be written as a two-dimensional integral representation as follows, where the integration runs over the moduli of the Euclidean momenta:

$$a_{\mu \text{toy}}^\text{toy} = \left( \frac{\alpha}{\pi} \right)^3 \left[ a_{\mu \text{toy}}^\text{toy}(1) + a_{\mu \text{toy}}^\text{toy}(2) \right],$$ (9)

$$a_{\mu \text{toy}}^\text{toy}(1) = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \left[ w_{f_1}(Q_1, Q_2) f(1)(Q_1^2, Q_2^2) \right.$$

$$+ \sum_{M Vi} w_{g_1}(M Vi, Q_1, Q_2) g_{M Vi}^{(1)}(Q_1^2, Q_2^2) \left. \right],$$ (10)

$$a_{\mu \text{toy}}^\text{toy}(2) = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \sum_{M=M_0, M Vi} w_{g_2}(M, Q_1, Q_2) g_M^{(2)}(Q_1^2, Q_2^2).$$ (11)

The dependence on the form factors is contained in the functions $f(1), g_{M Vi}^{(1)}$ and $g_M^{(2)}$, whereas the universal [for the class of form factors with the decomposition (8)] weight functions $w_{f_1}, w_{g_1}, w_{g_2}$ are rational functions, square roots, and logarithms\textsuperscript{5}. We have plotted them in Fig. 2. The functions $w_{f_1}$ and $w_{g_1}$ are positive and concentrated around momenta of the order of 0.5 GeV. Note, however, the tail in $w_{f_1}$ in the $Q_1$ direction for $Q_2 \sim 0.2$ GeV, which produces for the constant WZW form factor a divergence of the form $\sim C \ln^2 \Lambda$ for some UV-cutoff $\Lambda$. From the analytical expression for $w_{f_1}$, we obtain\textsuperscript{5} $C = 3(N_C m/(12\pi F_\pi))^2$. On the other hand, the function $w_{g_2}$ has positive and
negative contributions in the low-energy region, which will lead to a strong cancellation in the corresponding integrals.

In Table 1 we present the numerical results for the different form factors. One observes that all form factors (apart from the unrealistic constant WZW

Table 1. Results for the terms $a_{\mu \pi^0}^{\text{LbyL}(1)}$, $a_{\mu \pi^0}^{\text{LbyL}(2)}$, and $a_{\mu \pi^0}^{\text{LbyL}(0)}$ according to Eq. (9) for the different form factors. In the WZW model we used a cutoff of 1 GeV in the first contribution, whereas the second term is ultraviolet finite. In the LMD+V ansatz we have set $h_1 = 0$ GeV and $h_5 = 6.93$ GeV.

<table>
<thead>
<tr>
<th>Form factor</th>
<th>$a_{\mu \pi^0}^{\text{LbyL}(1)}$</th>
<th>$a_{\mu \pi^0}^{\text{LbyL}(2)}$</th>
<th>$a_{\mu \pi^0}^{\text{LbyL}(0)} \times 10^{10}$</th>
</tr>
</thead>
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<tr>
<td>WZW</td>
<td>0.095</td>
<td>0.0020</td>
<td>12.2</td>
</tr>
<tr>
<td>VMD</td>
<td>0.044</td>
<td>0.0013</td>
<td>5.6</td>
</tr>
<tr>
<td>LMD</td>
<td>0.057</td>
<td>0.0014</td>
<td>7.3</td>
</tr>
<tr>
<td>LMD+V ($h_2 = -10$ GeV$^2$)</td>
<td>0.049</td>
<td>0.0013</td>
<td>6.3</td>
</tr>
<tr>
<td>LMD+V ($h_2 = 0$ GeV$^2$)</td>
<td>0.045</td>
<td>0.0013</td>
<td>5.8</td>
</tr>
<tr>
<td>LMD+V ($h_2 = 10$ GeV$^2$)</td>
<td>0.041</td>
<td>0.0013</td>
<td>5.3</td>
</tr>
</tbody>
</table>

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form factor, which only serves illustrative purposes) lead to similar results, given mainly by \( a_\mu^{LbyL,\pi^0(1)} \). It is important to correctly reproduce the slope of the form factor at the origin and the data at intermediate energies. On the other hand, the asymptotic behavior at large \( Q_\perp \) seems not very relevant.

All these observations can easily be understood from the plots of the weight functions. For a fixed value of \( h_2 \) in the LMD+V form factor, our results are rather stable under the variation of the other parameters. If we vary \( M_{V1} \), \( M_{V2} \), and \( h_5 \) by \( \pm 20 \) MeV, \( \pm 25 \) MeV, and \( \pm 0.5 \) GeV\(^4 \), respectively, the result for \( a_\mu^{LbyL,\pi^0(1)} \) changes by \( \pm 0.2 \times 10^{-10} \). In contrast, if all other parameters are kept fixed, our result depends for \( |h_2| < 20 \) GeV\(^2 \) almost linearly on \( h_2 \). In this range \( a_\mu^{LbyL,\pi^0(1)} \) changes by \( \pm 0.9 \times 10^{-10} \) from the value for \( h_2 = 0 \).

Thus, using the LMD+V form factor, our estimate reads
\[
a_\mu^{LbyL,\pi^0} = + 5.8 (1.0) \times 10^{-10},
\]
where the error includes the variation of the parameters and the intrinsic model dependence. A similar short-distance analysis in the framework of large-\( N_C \) QCD and including quark mass corrections for the form factors for the \( \eta \) and \( \eta' \) was beyond the scope of Ref. 5. We therefore used VMD form factors fitted to the relevant CLEO data\(^1\) to obtain our final estimate
\[
a_\mu^{LbyL,P8} \equiv a_\mu^{LbyL,\pi^0} + a_\mu^{LbyL,\eta}\big|_{\text{VMD}} + a_\mu^{LbyL,\eta'}\big|_{\text{VMD}} = +8.3 (1.2) \times 10^{-10}. \quad (13)
\]
We think that an error of 15 % for the pseudoscalar pole contribution is reasonable, since we impose many theoretical constraints from long and short distances on the form factors. Furthermore, we use experimental information whenever available, e.g. the CLEO data for \( \mathcal{F}_{\pi^0\gamma\gamma}(Q^2,0) \) and the decay rate \( \pi^0 \to e^+e^- \). A better measurement of the latter decay could considerably reduce the error within the LMD+V ansatz, i.e. the bounds on the parameter \( h_2 \). Our two-dimensional integral representation moreover shows that there are no dangerous cancellations, at least in the main contribution \( a_\mu^{LbyL,\pi^0(1)} \).

### 3 EFT approach to hadronic light-by-light scattering

In Ref. 7 we discussed an EFT approach to \( a_\mu^{LbyL,\text{had}} \) based on an effective Lagrangian that describes the physics of the Standard Model well below 1 GeV, see also 15. It includes photons, light leptons, and the pseudoscalar mesons and obeys chiral symmetry and \( U(1) \) gauge invariance. All pieces of \( \mathcal{L}_{\text{eff}} \) are available in the literature\(^1\) and generalizing the chiral power counting by treating \( e, m, \) and fermion bilinears as order \( p \), one can write \( \mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots \), where the relevant terms are given by (for the notation, see Ref. 16)
\[
\mathcal{L}^{(2)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not D - m)\psi + e^2 C \langle QU^+ QU \rangle
\]
The leading contribution to $a^{\text{hyb, had}}_{\mu}$, of order $\mathcal{O}(p^6)$, is given by a loop of charged pions with point-like electromagnetic vertices. It is finite with the value $-4.5 \times 10^{-10}$. Since this contribution involves a loop of hadrons, it is subleading in the large-$N_C$ expansion.

At leading order in $N_C$ and at order $p^8$ in $a_{\mu}$, we encounter the divergent pion-pole contribution, diagrams (a) and (b) of Fig. 3, involving two WZW vertices from Eq. (15). Note that the diagram (c) is actually finite. The divergences of the triangular subgraphs in the diagrams (a) and (b) are removed by inserting the counterterm $\chi$ from Eq. (16), see the one-loop diagrams (d) and (e). Finally, there is an overall divergence of the two-loop diagrams (a) and (b) that is removed by a local counterterm denoted by $\kappa$, diagram (f). Note that at order $p^8$ there will also be additional contributions, if we replace the point-like vertices in the charged pion-loop by vertices from $\mathcal{L}^{(4)}$, e.g. involving the low-energy constants $L_9$ and $L_{10}$. These contributions will be subleading in $N_C$, although they may be enhanced by logarithms.

Figure 3. The graphs contributing to $a^{\text{hyb, had}}_{\mu}$ at lowest order in the effective field theory.

Since the EFT involves a local contribution $\kappa$ to $a_{\mu}$, we will not be able to give a numerical prediction for $a^{\text{hyb, had}}_{\mu}$. Nevertheless, we can learn a few things by considering the leading logarithms that are in addition enhanced by a factor $N_C$ and which can be calculated using the RG. The expression of the renormalized contribution to $a_{\mu}$ arising from the graphs of Fig. 3 reads

$$a^{\text{hyb, had}}_{\mu} = H(m/\mu) + \chi(\mu)J(m/\mu) + \kappa(\mu).$$

The dependence on the subtraction scale $\mu$ in the two-loop function reads $H(m/\mu) = \sum_{p=0,1,2} h_p \ln^p(m/\mu)$, and in the one-loop function $J(m/\mu) =$
\[ \sum_{q=0,1} j_q \ln^q (m/\mu). \] The physical quantity \( a^L_{\mu} \) satisfies \( \mu \frac{da^L_{\mu}}{d\mu} = 0. \) Since the dimensionless coefficients \( h_p \) and \( j_q \) do not depend on \( \mu \), but are functions of the ratios \( M_{\pi}/m \) and \( F_\pi/m \), we have \( D h_p = 0 \) and \( D j_q = 0 \), where \( D \equiv m \partial / \partial m + M_{\pi}/m \partial / \partial M_{\pi} + F_\pi / \partial F_\pi \). Finally, the conditions \( D \chi(\mu) = 0 \) and \( D \kappa(\mu) = 0 \) hold. We thus obtain the two RG equations

\[ h_2 = \frac{1}{2} \gamma \chi j_1, \quad \mu \frac{d \chi(\mu)}{d \mu} = -\gamma \chi j_0 + \chi(\mu) j_1 + h_1, \quad (18) \]

with \( \gamma \chi \equiv \mu \frac{d \chi(\mu)}{d \mu} = N_C \), see Ref. \(^1\). The coefficient \( j_1 \) of the logarithm in the sum of the one-loop graphs (d) and (e) can easily be worked out \(^7\). In this way we obtain from Eq. (18)

\[ h_2 \equiv \left( \frac{\alpha}{\pi} \right)^3 C, \quad C = +3 \left( \frac{N_C}{12 \pi} \right)^2 \left( \frac{m^2}{F_\pi^2} \right), \quad (19) \]

in agreement with the result given in Section 2.

4 Further comments and conclusions

The EFT and large-\( N_C \) analysis tells us that we can write

\[ a^L_{\mu, \text{had}} = \left( \frac{\alpha}{\pi} \right)^3 \left\{ f \left( \frac{m_{\pi^\pm}}{m}, \frac{m_{K^\pm}}{m} \right) \right. \]

\[ + N_C \left( \frac{m^2}{16 \pi^2 F_\pi^2} \right) \left[ \ln^2 \frac{\mu_0}{m} + c_1 \ln \frac{\mu_0}{m} + c_0 \right] \]

\[ + \mathcal{O} \left( \frac{m^2}{\mu_0^2} \times \log's \right) + \mathcal{O} \left( \frac{m^4}{\mu_0^4} \times N_C \times \log's \right) \left\} \right. \]

where \(^3 f(m_{\pi^\pm}/m, m_{K^\pm}/m) = -0.038 \) represents the charged pion and kaon-loop that is formally of order one in the chiral and \( N_C \) counting. We have denoted by \( \mu_0 \) some hadronic scale, e.g. \( M_\rho \). The coefficient of the log-square term in the second line is universal and of order \( N_C \), since \( F_\pi = \mathcal{O}(\sqrt{N_C}) \). The remaining parts of the coefficient \( c_1 \) have recently been calculated in Ref. \(^{18}\) in the \( \overline{\text{MS}} \)-scheme: \( c_1 = -[2 \chi(\mu_0)/3 - 0.904] \). The low-energy constant \( \chi \) can be determined from the decay \( \pi^0 \to e^+ e^- \). However, due to the big experimental error, this leads to a large uncertainty in \( \chi \) and thus in \( a^L_{\mu, \text{had}} \). Assuming lepton universality, one can obtain \( \chi \) also from the decay \( \eta \to \mu^+ \mu^- \), with better precision \( \chi(M_\rho) = 1.75^{+1.25}_{-1.00} \). One observes that although the logarithms are sizeable, \( \ln(M_\rho/m) = 1.98 \), and give the ballpark of the final result, there is some cancellation between the log-square and the log-term in \( a^L_{\mu, \text{had}} \). Note that the form factors based on large-\( N_C \) QCD that we discussed

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in Section 2 give values for the constant $c_0$ in Eq. (20) that are of order one, i.e. of natural size. Furthermore, in addition to $\pi^0 \to e^+ e^-$, we have also taken other experimental and theoretical constraints on $F_{\pi^0 \gamma^* \gamma^*}$ into account.

It has been argued that $a_{\mu}^{\text{LbyL;had}}$ has to be positive, because of quark-hadron duality, since also the (constituent) quark-loop leads to a positive result. We do not agree with this argument for the following reason. Equation (20) tells us that at leading order in $N_C$ any effective theory or model of QCD has to show the behavior $a_{\mu}^{\text{LbyL;had}} \sim (\alpha/\pi)^3 N_C [N_C m^2/(48\pi^2 F_{\pi}^2)] \ln^2 \Lambda$, with a universal coefficient, if one sends the cutoff $\Lambda$ to infinity. From the analytical result given in Ref. 20 for the quark-loop one obtains the behavior $a_{\mu}^{\text{LbyL;CQM}} \sim (\alpha/\pi)^3 N_C (m^2/M_Q^2) + \ldots$, for $M_Q \gg m$, if we interpret the constituent quark mass $M_Q$ as a hadronic cutoff. Note that the quark-loop is also leading in large $N_C$. Even though one may argue that $N_C/16\pi^2 F_{\pi}^2$ in $C$ can be replaced by $1/M_Q^2$, the log-square term is not correctly reproduced with this model. Therefore, the constituent quark model cannot serve as a reliable description for the dominant contribution to $a_{\mu}^{\text{LbyL;had}}$. On the other hand, the coefficient $C$ could, a priori, have any sign, since it is determined purely within the EFT, where it is related to the decay amplitude $\pi^0 \to e^+ e^-$.

In conclusion, the pseudoscalar exchange contribution $a_{\mu}^{\text{LbyL;PS}}$ seems to be under control at the 15% level. In addition, the EFT and large-$N_C$ analysis shows a systematic approach to $a_{\mu}^{\text{LbyL;had}}$ and yields the leading and next-to-leading logarithmic terms, enhanced by a factor $N_C$. On the other hand, these terms tend to cancel each other to some extent. Furthermore there remains the issue of the other contributions to $a_{\mu}^{\text{LbyL;had}}$. For instance, the charged pion-loop is the leading term in the low-energy expansion, however, after dressing the couplings to the photons with form factors, there is a huge suppression, by a factor of 3 to 10, although the model-dependent results after the dressing differ by about $1 \times 10^{-10}$. Similar statements apply to the dressed quark-loop, that appears in these models. Taking all these uncertainties into account by adding the model-dependent errors linearly, my conservative estimate for the full hadronic light-by-light scattering contribution to $a_{\mu}$ is as follows:

$$a_{\mu}^{\text{LbyL;had}} = + 8 (4) \times 10^{-10}.$$

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