Superconformal Tensor Calculus on Orbifold in 5D

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Abstract

Superconformal Tensor Calculus on an orbifold $S^1/Z_2$ is given in five dimensional (5D) spacetime. Four-dimensional superconformal Weyl multiplet and various matter multiplets are induced on the boundary planes from the 5D supermultiplets in the bulk. We identify those induced 4D supermultiplets and clarify a general method for coupling the bulk fields to the matter fields on the boundaries in a superconformal invariant manner.

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§1. Introduction

It is a very interesting idea that our world may be a 3-brane embedded in a higher dimensional space-time. To investigate seriously various possibilities and problems in this framework of ‘brane world scenario’, \(^1\),\(^2\) we will need an off-shell formulation of supergravity in higher dimensions. Having this in mind, some groups have been developing supergravity \(^3\)-\(^5\) and superconformal \(^6\),\(^7\) tensor calculus in five dimensional (5D) space-time.

In this paper we report on the superconformal tensor calculus in 5D space-time in which the fourth spatial direction, \(x^4 \equiv y\), is compactified on an orbifold \(S^1/Z_2\). We clarify the 4D superconformal multiplets induced on the boundary planes from the 5D bulk fields. In particular, we show that the 4D superconformal Weyl multiplet is induced on the boundary planes from the 5D bulk Weyl multiplet. Similarly to the rigid supersymmetry case, \(^8\) 5D bulk Yang-Mills multiplets induce 4D gauge multiplets on the boundaries, if the vector components are assigned even \(Z_2\) parity. Hypermultiplet in 5D bulk produce a 4D chiral multiplet on the boundary. Linear multiplet in 5D bulk can also yield a 4D chiral multiplet on the boundary for a certain \(Z_2\) parity assignment, while, for the opposite \(Z_2\) parity assignment, it does not give 4D linear multiplet but a general-type multiplet.

Once we can identify the 4D superconformal multiplets induced on the boundary planes, it becomes immediately clear how to couple the 4D matter fields on the boundary to the bulk supergravity-Yang-Mills-matter fields in a superconformal invariant manner. Since the 4D compensating multiplet is also induced from the 5D bulk compensating multiplet, we can write down any 4D invariant action on the boundary planes using the known invariant action formulas of the 4D superconformal tensor calculus.\(^9\)-\(^14\)

Actually this type of tensor calculus on orbifold was first studied by Zucker.\(^15\) However his tensor calculus is not superconformal but supergravity one in which dilatation \(D\) and \(S\)-supersymmetry are already gauge-fixed. This fact (together with his choice of linear multiplet for the compensator in 5D bulk) has brought about an unwelcome situation that a quite unfamiliar non-minimal version\(^16\) of 4D Poincaré supergravity is induced on the boundary planes. In our case, 4D superconformal symmetry is fully realized on the boundary planes. Then the simplest 4D Poincaré supergravity, ‘old minimal’ version,\(^17\) can be clearly seen to be induced on the boundary planes if hypermultiplet is chosen as the compensator in 5D bulk.

This paper is organized as follows. We first, in §§2 and 3, respectively, recapitulate the 5D and 4D superconformal transformation rules of the Weyl multiplets and some matter multiplets. Comparing these transformation rules of 5D bulk multiplets with the 4D transformation rules as was done by Mirabelli and Peskin\(^8\) in the rigid supersymmetry case, we
identify in §4 all the basic 4D supermultiplets induced from the bulk field on the boundary. Then in §5 we identify the 4D compensator induced from the bulk hypermultiplet compensator and explains how the brane action can be written down generally which gives superconformal invariant couplings of the bulk supergravity-Yang-Mills-matter fields to an arbitrary set of 4D matter fields on the boundary. Finally in §6 we illustrate the procedure for writing invariant actions by taking the simplest case in which the bulk is the pure supergravity system with $U(1)_R$ gauged and the boundary planes contain only tension terms. For convenience of practical use of the present tensor calculus, we add three appendices: The notation and conventions are briefly explained in Appendix A. Explicit expressions for the curvatures both for 5D and 4D cases are given in Appendix B, which make manifest the structure functions of superconformal algebras for both cases. Old results of embedding and invariant action formulas in 4D superconformal tensor calculus are recapitulated in Appendix C in our present notation.

§2. 5D superconformal multiplets

Tensor calculus for 5D supergravity was first given by Zucker, $^3$ and later by the present authors $^4$ in a more complete form. $^*$ These formulations, however, do not contain conformal $S$ supersymmetry (nor dilatation $D$ symmetry in the former), which brings about much inconvenience when considering the general matter-coupled system; that is, we must do a very tedious field redefinitions in order to recover the canonical Einstein and Rarita-Schwinger fields. $^5$ These tedious field redefinitions can simply be bypassed by choosing improved gauge fixing conditions of $S$ supersymmetry and dilatation $D$ symmetries in the superconformal framework. $^9$ In view of this, Bergshoeff et al $^6$ have found the Weyl multiplets in the 5D superconformal tensor calculus for the first time, and almost at the same time Fujita and Ohashi have presented the full superconformal tensor calculus including matter multiplets and invariant action formulas in a paper $^7$ which we refer to as I henceforth. We here recapitulate the transformation laws of the Weyl and matter multiplets given in I. Taking account of the convenience in practical use, we here contain the explicit expressions for the superconformal covariant derivatives and curvatures, which were omitted in I.

2.1. 5D Weyl multiplet

5D Weyl multiplet consists of 32 Bose plus 32 Fermi fields

\[ \epsilon^a_{\mu}, \psi^i_{\mu}, V_{ij}^{\mu}, b_{\mu}, v^{ab}, \chi^i, D, \]  

$^*$ These are off-shell formulations of 5D supergravity. For the on-shell formulations of 5D supergravity, which have been known for a long time, see Refs. $^{18}$
whose properties are summarized in table I. We use $a, b, \ldots$ for the local lorentz indices, $\mu, \nu, \ldots$ for the world vector indices and $i, j = 1, 2$ for SU(2). The first four fields $e_\mu^a, \psi_i^\mu$, $V_{ij}^\mu$ and $b_\mu$ are the gauge fields for ‘translation’ $P_a$, supersymmetry $Q_i^j$, SU(2) $U_{ij}$ and dilatation $D$ transformations, respectively. The other gauge fields, $\omega_{\mu ab}$ for local Lorentz $M_{ab}$, $\phi_i^\mu$ for conformal supersymmetry $S_i$ and $f_\mu^a$ for special conformal boost $K_a$, are dependent fields given by functions of the above independent gauge fields, as a result of the imposition of the following constraints on the $P_a$, $Q_i$ and $M_{ab}$ curvatures, respectively:

$$
\hat{R}_{\mu a}(P) = 0 \quad \longrightarrow \quad \omega_{\mu ab} = \omega_{0 ab} + i(2\bar{\psi}_\mu \gamma^a \psi_b^0 + \bar{\psi}^a \gamma_\mu \psi^b) - 2e_{[a}^\mu b],
$$

with $\omega_{0 ab} \equiv -2e^\nu[a \partial_{\mu} e_{\nu b]} + e^\rho[a e^b]_{\sigma} e^\epsilon \partial_{\rho} e_{\sigma c}$,

$$
\gamma^\nu \hat{\hat{R}}_{\mu \nu} (Q) = 0 \quad \longrightarrow \quad \phi_i^\mu = \left( -\frac{1}{3} e^a_\mu \gamma^b + \frac{1}{24} \gamma_{\mu a} \gamma^{ab} \right) \hat{R}_{ab} (Q),
$$

$$
\hat{R}_{\mu}^a (M) = 0 \quad \longrightarrow \quad f_\mu^a = \left( \frac{1}{6} e^\nu a_b^\mu - \frac{1}{48} e^a_\nu e^\nu_b \right) \hat{R}_{\nu}^b (M),
$$

(2.2)

where $\hat{R}_{\mu}^a (M) \equiv \hat{R}_{\mu a} \partial (M) e^\nu_b$, and the primes on the curvatures imply $\hat{R}_{ab} (Q) = \hat{R}_{ab} (Q)|_{\phi_{\mu} = 0}$ and $\hat{R}_{\mu}^a (M) = \hat{R}_{\mu}^a (M)|_{f_\nu b = 0}$. A constraint-independent treatment for these dependent gauge fields were given in I, but here we prefer to imposing the constraints (2.2) explicitly since it is simpler in practice.

The full $Q, S, K$ transformation laws of the Weyl multiplet are given as follows. With

$$
\delta \equiv \bar{\psi}^i Q_i + \bar{\eta}^i S_i + \xi_K^a K_a \equiv \delta Q(\varepsilon) + \delta S(\eta) + \delta_K (\xi_K^a),
$$

$$
\delta e_\mu^a = -2i\bar{\varepsilon} \gamma^a \psi_\mu,
$$

$$
\delta \psi_i^\mu = \mathcal{D}_\mu \varepsilon^i + \frac{1}{2} \gamma^{ab} \gamma_{\mu ab} \varepsilon^i - \gamma_{\mu} \eta^i,
$$
\[ \delta b_\mu = -2i\varepsilon\phi_\mu - 2\bar{\eta}\psi_\mu - 2\xi_{K\mu}, \]
\[ \delta V^i_\mu = -6i\varepsilon^{i(i\phi_\mu)} + 4i\varepsilon^{i(\gamma\cdot\psi_\mu)} - \frac{i}{4}\varepsilon^{i(\gamma\mu\chi)} + 6i\bar{\eta}(i\psi_\mu), \]
\[ \delta v_{ab} = -\frac{i}{8}\varepsilon_{ab\chi} - \frac{3}{2}i\varepsilon\hat{R}_{ab}(Q), \]
\[ \delta \chi^i = D\varepsilon^i - 2\gamma\cdot\psi^i\bar{D}_{a}v_{bc} + \gamma\cdot\hat{R}(U)^i j\varepsilon_j - 2\gamma^a\varepsilon^i\varepsilon_{abcde}v^{bc}v^{de} + 4\gamma\cdot\eta\mu, \]
\[ \delta D = -i\varepsilon\hat{D}\chi - 8i\varepsilon\hat{R}_{ab}(Q)v^{ab} + i\bar{\eta}\chi, \]  
(2.3)

where the fermion bilinears like \( \bar{\eta}\psi_\mu, \varepsilon\gamma_{ab}\chi \) etc with their SU(2) spinor indices suppressed, always imply the northwest-southeast contraction \( \bar{\eta}\psi_{\mu i}, \varepsilon^i\gamma_{ab}\chi_i \). The dot \( \gamma\cdot T \) for a tensor \( T_{ab...} \) generally denotes the contraction \( \gamma^{ab...}T_{ab...} \). The transformation rules of dependent fields of course follow from those of independent fields and are found to be

\[ \delta \omega^{ab}_\mu = 2i\varepsilon\gamma^{ab}\phi_\mu - 2i\varepsilon\gamma[a\hat{\mu} b](Q) - i\varepsilon\gamma_\mu\hat{R}^{ab}(Q) \]
\[ -2i\varepsilon\gamma^{abc}\psi_\mu v_{cd} - 2i\varepsilon\gamma^{ab}\psi_\mu - 4\xi_{K[a}\varepsilon_{b]}, \]
\[ \delta \phi_\mu = D_\mu \varepsilon^i - \frac{1}{3}\gamma_{abc\eta}v^{bc} + \gamma^b\eta v_{ab} - \xi_{K[a} \varepsilon_{b]}, \]
\[ + \gamma_{abc}^i f_\mu^a - \frac{5}{3}\varepsilon\gamma^{abc}\psi_\mu \chi^i - \frac{5}{3}\varepsilon\gamma^{abc}\psi_\mu \gamma_{a} \chi^i + i\varepsilon^{i\gamma^{ab}\psi^i}(\hat{R}_{abj}(Q) - \frac{1}{32}\gamma_{a b j}), \]
\[ + \frac{1}{3}(\hat{R}_{ab}^j(U))\gamma^b - \frac{1}{8}\gamma_{a} \gamma^{i}(\hat{R}^j(U))\varepsilon^i \]
\[ - \frac{1}{12}\varepsilon_{a}^i (3\hat{D}_a\gamma^{-1}v_{i} + \gamma_{abc}\hat{D}_b v_{c}d^i e^i + \gamma_{ab}\hat{D}_c v_{b}^{c}d^i e^i - 2\gamma_{bc}e^i d^b v_{ab} - \gamma_{ab}d^c e^i v_{bc}v_{de} + 4v_{ab}v_{cd}
\]
(2.4)

Here the (unhatted) derivative \( D_\mu \) is covariant only with respect to the homogeneous transformations \( M_{ab}, D \) and \( U^{ij} \) (and \( G \) transformation for non-singlet fields under the Yang-Mills group \( G \)), while the hatted derivative \( \hat{D}_\mu \) denotes fully superconformal covariant derivative; that is, with \( h^A_\mu \) denoting the gauge fields of the transformation \( X_A \), we have

\[ \hat{D}_\mu \equiv \partial_\mu - \sum_{X_A=M_{ab},D,U^{ij}(G)} h^A_\mu X_A, \quad \hat{D}_\mu \equiv \partial_\mu - \sum_{X_A=Q,S,K} h^A_\mu X_A. \]  
(2.5)

The explicit form of the curvatures \( \hat{R}_\mu^a = e^b_a [\hat{D}_b, \hat{D}_a]^A \) is given in the Appendix. The covariant derivatives appearing in Eqs. (2.3) and (2.4) are given explicitly by

\[ \hat{D}_\mu \varepsilon^i = (\partial_\mu - \frac{1}{4}\omega^{ab}_\mu \gamma_{ab} + \frac{1}{2}\delta_\mu^i) \varepsilon^i - V_{\mu \mu i} \varepsilon^i, \]
Table II. Matter multiplets in 5D.

<table>
<thead>
<tr>
<th>field</th>
<th>type</th>
<th>remarks</th>
<th>SU(2)</th>
<th>Weyl-weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_\mu$</td>
<td>boson</td>
<td>real gauge field</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$M$</td>
<td>boson</td>
<td>real</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$Q^i$</td>
<td>fermion</td>
<td>SU(2)-Majorana</td>
<td>2</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$Y_{ij}$</td>
<td>boson</td>
<td>$Y^{ij} = Y^{ji} = (Y_{ij})^*$</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

**Hypermultiplet**

| $A_i^\alpha$ | boson | $A^\alpha_i = (\epsilon^{ij} A^\beta_j)_{\rho\alpha} = -(A^\alpha_i)^*$ | 2     | $\frac{3}{2}$ |
| $\zeta^\alpha$ | fermion | $\zeta^\alpha \equiv (\zeta_0)^T \gamma_0 = \zeta^\alpha T C$ | 1     | 2           |
| $F_i^\alpha$ | boson | $F^\alpha_i \equiv M Z A^\alpha_i, \ F^\alpha_i = -(F^\alpha_i)^*$ | 2     | $\frac{5}{2}$ |

**Linear multiplet**

| $L^{ij}$    | boson | $L^{ij} = L^{ji} = (L_{ij})^*$ | 3     | 3           |
| $\varphi^i$ | fermion | $SU(2)$-Majorana | 2     | $\frac{7}{2}$ |
| $E_a$       | boson | real, constrained | 1     | 4           |
| $N$         | boson | real                  | 1     | 4           |

\[
\mathcal{D}_\mu \eta^i = \left( \partial_\mu - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} - \frac{1}{2} b_\mu \right) \eta^i - V^{i}_{\mu} \eta^j, \\
\mathcal{D}_\mu \xi^a = (\partial_\mu - b_\mu) \xi^a - \omega_\mu^{ab} \xi^{Kb}, \\
\hat{D}_\mu v_{ab} = \partial_\mu v_{ab} + 2 \omega_{\mu}^{ac} v_{bc} + i \bar{\psi}_\mu \gamma_\mu \chi + \frac{3}{2} i \bar{\psi}_\mu \hat{R}_{ab}(Q), \\
\hat{D}_\mu \chi^i = \mathcal{D}_\mu \chi^i - D_{\psi^j i} 2 \gamma^c \epsilon^{abc} \psi^j_{\mu} \mathcal{D}_{a} v_{bc} - \gamma \hat{R}(U)^j_{i} \psi^j_{\mu} + 2 \gamma^a \psi^i_{\mu} \epsilon_{\alpha\beta\gamma\delta\epsilon} v^{bc} v^{de} - 4 \gamma \cdot v \psi^i_{\mu}, \\
\mathcal{D}_\mu \chi^i = \left( \partial_\mu - \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} - \frac{3}{2} b_\mu \right) \chi^i - V^{i}_{\mu} \chi^j. \quad (2.6)
\]

2.2. Matter multiplets in 5D

We here write the transformation rules of three kinds of matter multiplets: vector multiplet, hypermultiplet and linear multiplet. The components of these multiplets and their properties are listed in table II.

2.2.1. Vector multiplet

All the component fields of this multiplet are Lie-algebra valued, e.g., the first component scalar $M$ is a matrix $M^\alpha_\beta = M^I (t_I)^\alpha_\beta$, where the $t_I$ are (anti-hermitian) generators of the gauge group $G$: $[t_I, t_J] = -f_{IJ}^K t_K$. The $Q$ and $S$ transformation laws of the vector
defines an auxiliary field $F$. The central charge transformations are given by

$$
\delta W = -2i\bar{\varepsilon}\gamma_{\mu}\Omega + 2i\bar{\varepsilon}\psi_{\mu}M,
\delta M = 2i\bar{\varepsilon}\Omega,
\delta \Omega^i = -\frac{1}{4}\gamma^{\mu\nu}\hat{F}_{\mu\nu}(W)\varepsilon^i - \frac{1}{2}\hat{\Phi}M\varepsilon^i + Y^i_j\varepsilon^j - M\eta^i,
\delta Y^{ij} = 2i\bar{\varepsilon}(i\hat{\Phi}\Omega^j) - i\varepsilon(i\gamma\cdot\psi\Omega^j) - \frac{i}{4}\varepsilon(i\gamma^0)M - 2i\bar{\varepsilon}(i[M, \Omega^j]) - 2i\eta^{ij}\Omega^j. \quad (2.7)
$$

where the full covariant field strength $\hat{F}_{\mu\nu}(W)$ and covariant derivatives are given explicitly by

$$
\hat{F}_{\mu\nu}(W) = 2\partial_{[\mu}W_{\nu]} - g[W_{\mu}, W_{\nu}] + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega - 2i\bar{\psi}_{\mu}\psi_{\nu}M,
\hat{D}_{\mu}M = (\partial_{\mu} - b_{\mu})M - g[W_{\mu}, M] - 2i\bar{\psi}_{\mu}\Omega,
\hat{D}_{\mu}\Omega^i = \hat{D}_{\mu}\Omega^i + \frac{1}{4}i\hat{F}(W)\psi_{\mu}^i + \frac{1}{2}\hat{\Phi}M\psi_{\mu}^i - Y^i_j\psi_{\mu}^j + M\phi_{\mu}^i. \quad (2.8)
$$

Note that the gauge coupling constant $g$ used in this paper is a symbolic notation; it represents different values for different factor groups when $G$ is not a simple group.

2.2.2. Hypermultiplet

The hypermultiplet in 5D consists of scalars $A_i^\alpha$, spinors $\zeta^\alpha$ and auxiliary fields $F_i^\alpha$. They carry the index $\alpha (= 1, 2, \cdots, 2r)$ of a representation of a gauge group $G'$, which is lowered (or raised) with a $G'$-invariant tensor $\rho_{\alpha\beta}$ (and $\rho_{\beta\alpha}$ with $\rho_{\alpha\beta}\rho_{\gamma\beta} = \delta_{\alpha}^\beta$) like $A_\alpha = A_i^\beta\rho_{\beta\alpha}$. And this multiplet gives an infinite dimensional representation of a central charge gauge group $U_Z(1)$, which we regard as a subgroup of the full gauge group $G = G' \times U(1)_Z$.

The $Q$ and $S$ transformation rules of the $A_i^\alpha$ and $\zeta^\alpha$ are given by

$$
\delta A_i^\alpha = 2i\bar{\varepsilon}_i\zeta^\alpha,
\delta \zeta^\alpha = \hat{\Phi}A_j^\alpha\varepsilon^j - \gamma^j\cdot\psi\varepsilon^jA_j^\alpha - M_\alpha A_j^\alpha\varepsilon^j + 3A_j^\alpha\eta^j,
\delta F_i^\alpha = 2i\bar{\varepsilon}_i(\alpha Z\zeta^\alpha) + \frac{2i}{\alpha}\bar{\varepsilon}\Omega^0F_i^\alpha. \quad (2.9)
$$

where $\theta_* = M_*, \Omega_*, \cdots$ represent the $G$ transformations with parameters $\theta$ including the central charge transformation, $\delta_G(\theta) = \delta_{G'}(\theta) + \delta_Z(\theta^0)$; more explicitly, e.g.,

$$
M_\alpha A_i^\alpha = \delta_{G'}(M)A_i^\alpha + \delta_Z(M^0 = \alpha)A_i^\alpha = \sum_{I=1}^n g M^I(t_i)^\alpha_\beta A_i^\beta + \alpha Z A_i^\alpha. \quad (2.10)
$$

$Z$ denotes the generator of the $U_Z(1)$ transformation. The $U_Z(1)$ transformation of $A_i^\alpha$ defines an auxiliary field $F_i^\alpha \equiv \alpha Z A_i^\alpha$, where $\alpha$ is the scalar component of the $U_Z(1)$ vector.
multiplet $\mathbf{V}^0 = (M^0 \equiv \alpha, W^0, \Omega^0, \gamma^0)$. The $U(1)$ transformations of other components, $Z\zeta_\alpha$ and $Z\mathbf{F}_\alpha$, are defined by requiring
\begin{align*}
0 &= \hat{D}_\mu \zeta_\alpha + \frac{1}{2} \gamma^i \nu \zeta_\alpha - \frac{1}{8} \chi \zeta_\alpha^i + M_\zeta \zeta_\alpha^i - 2\Omega_\zeta \zeta_\alpha^i, \\
0 &= -\hat{D}_\alpha \hat{D}_\alpha \mathbf{A}_i^\alpha + M_\mathbf{A} \mathbf{A}_i^\alpha \\
&\quad + 4i\hat{\Omega}_s \zeta_\alpha - 2Y_{ij} \mathbf{A}_i^\alpha - \frac{i}{4} \zeta_\alpha \chi + \frac{1}{8} (D - 2v^2) \mathbf{A}_i^\alpha. \quad \text{(2.11)}
\end{align*}

Note that $\hat{D}_\mu \zeta_\alpha$ and $M_\zeta \zeta_\alpha$ contain the central charge transformation terms $-\mathbf{W}_\mu \zeta_\alpha$ and $\alpha \zeta_\alpha$, respectively, and that both $\hat{D}^\alpha \hat{D}_\alpha \mathbf{A}_i^\alpha$ and $M_\mathbf{A} \mathbf{A}_i^\alpha$ contain $Z\mathbf{F}_\alpha^\alpha = \alpha Z(\mathbf{Z}\mathbf{A}_i^\alpha)$. So these conditions (2.11) indeed determine the $Z\zeta_\alpha$ and $Z\mathbf{F}_\alpha^\alpha$.

2.2.3. Linear multiplet

The linear multiplet consists of the components listed in Table II and may generally carry a charge of the gauge group $G$.

The $Q$ and $S$ transformation laws of the linear multiplet are given by
\begin{align*}
\delta L^{ij} &= 2i\bar{\varepsilon}^{(i} \varphi^{j)} \\
\delta \varphi^i &= -\hat{D} L^{ij} \varphi_j + \frac{1}{2} \gamma^a \varepsilon^i E_a + \frac{1}{2} \varepsilon^i N + 2\gamma \cdot \nu \varepsilon_j L^{ij} + M_\mathbf{L} \varepsilon_j - 6L^{ij} \eta_j, \\
\delta E^a &= 2i\bar{\varepsilon}^{ab} \hat{D}_b \varphi - 2i\varepsilon^{abc} \varphi v_{bc} + 6i\bar{\varepsilon} \gamma_b \varphi v^{ab} + 2i\bar{\varepsilon} \gamma^a \mathbf{A} \varphi - 4i\bar{\varepsilon} \gamma^a \mathbf{O}_s L^{ij} - 8i\bar{\eta} \gamma_a \varphi, \\
\delta N &= -2i\bar{\varepsilon} \hat{D} \varphi - 3i\bar{\varepsilon} \gamma \cdot \nu \varphi + \frac{1}{2} \varepsilon^i \chi L_{ij} + 4i\bar{\varepsilon} \gamma^a \mathbf{O}_s L_{ij} - 6i\bar{\eta} \varphi, \quad \text{(2.12)}
\end{align*}

with $\theta_s$ defined above in Eq. (2.10). The closure of the algebra demands that $E^a$ satisfy the following $Q$- and $S$-invariant constraint:
\begin{align*}
\hat{D}_a E^a + M_\mathbf{N} + 4i\hat{\Omega}_s \varphi + 2Y_{ij} \mathbf{L}^{ij} = 0. \quad \text{(2.13)}
\end{align*}

§3. 4D superconformal tensor calculus

$N = 1$ 4D superconformal tensor calculus has been known for a long time. Here we cite the results following mainly Kugo and Uehara in the present notation. However, strangely enough, the transformation rules for the multiplets which carry gauge group charges have never been given in the literature. Our expressions given here are also valid for such cases.

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1) We used notation $A_\mu$ to denote the gravi-photon field $W_\mu^0$ in the previous papers. However we here use $W_\mu^0$ since $A_\mu$ will be used to denote the U(1) gauge field of 4D superconformal group.
3.1. Weyl multiplet

The 4D superconformal group consists of usual bosonic conformal transformations, $P_a$, $M_{ab}$, $D$ and $K_a$, plus a bosonic $U(1)$ symmetry $A$ and fermionic Majorana $Q$ and $S$ supersymmetries. For simplicity of notations, we use the same symbols for the gauge fields, curvatures etc in 4D as those in 5D, although they, of course, denote different quantities. From now on, the world vector indices $\mu, \nu, \cdots$ and Lorentz indices $a, b, \cdots$ run only over 0, 1, 2, 3. We will attach the superscript of dimensions, (4) or (5), when the distinction is necessary. The Weyl multiplet in 4D consists of 12 Bose plus 12 Fermi gauge fields and no ‘matter’ fields:

$$e^a_\mu, \quad \psi_\mu, \quad A_\mu, \quad b_\mu, \quad (3.1)$$

where $A_\mu$ is the gauge field for the $U(1)$ transformation $A$. In this 4D case, the $M_{ab}$, $S$ and $K_a$ gauge fields $\omega^{ab}_\mu$, $\phi_\mu$, $f^a_\mu$ are also dependent fields by the usual constraints

$$\hat{R}_{ab}^c(P) = 0, \quad \gamma^b \hat{R}_{ab}(Q) = 0, \quad \tilde{R}_{ab}(M) - \frac{1}{2} \hat{F}_{ab}(A) = 0, \quad \text{(3.2)}$$

where the tilde denotes the dual tensor $\tilde{F}_{ab} \equiv \epsilon_{abcd} F^{cd}/2$. The solution of the spin connection $\omega^{ab}_\mu$ to the first constraint takes the same form as the 5D one in Eq. (2.2). The solutions of $\phi_\mu$ and $f^a_\mu$ to the latter two constraints have slightly different coefficients from 5D ones in Eq. (2.2) and are given by

$$\phi_\mu = -\frac{i}{3} \gamma^a \hat{R}_{(a}^{(\mu} \hat{R}_{\mu)b)} + \frac{i}{12} \gamma_{\mu ab} \hat{R}^{ab}(Q),$$
$$f^a_\mu = \frac{1}{4} \hat{R}_{(a}^{(\mu} (M) - \frac{1}{8} \hat{R}_{(a}^{(\mu} (A) - \frac{1}{24} \epsilon^{a}_{\mu} \hat{R}^{(a} (M)). \quad \text{(3.3)}$$

The $Q$, $S$, $K_a$ and $A$ transformation laws of the gauge fields are given as follows. With $\delta = \delta_Q(\varepsilon) + \delta_S(\eta) + \delta_K(\xi^a_K) + \delta_A(\theta)$,

$$\delta e^{a}_\mu = -2i \varepsilon \gamma^a \psi_\mu,$$
$$\delta \psi_\mu = D_\mu \varepsilon + i \gamma_\mu \eta + \frac{3}{4} \theta i \gamma_5 \psi_\mu,$$
$$\delta b_\mu = -2 \varepsilon \phi_\mu + 2 \eta \psi_\mu - 2 \xi_K \mu,$$
$$\delta A_\mu = 4 i \varepsilon \gamma_5 \phi_\mu - 4 i \eta \gamma_5 \psi_\mu + \partial_\mu \theta,$$
$$\delta \omega^{ab}_\mu = 2 \varepsilon \gamma^a \phi_\mu - 2i \varepsilon \gamma_\mu \hat{R}^{ab}_\mu (Q) + 2 \eta \gamma^{ab} \psi_\mu - 4 \xi^{[a}_{K} \epsilon^{b]}_\mu, \quad \text{(3.4)}$$

$$\delta \phi_\mu = D_\mu \eta + i \gamma_\mu f^a_\mu - i \xi^a_K \gamma_\mu \psi_\mu + \frac{i}{4} \gamma^b \varepsilon \hat{R}_{ab}(A) - \frac{1}{4} \gamma^b \gamma_5 \hat{R}_{ab}(A) - \frac{3}{4} \theta i \gamma_5 \phi_\mu,$$
$$\delta f^a_\mu = D_\mu \xi^a_K - 2i \eta \gamma^a \phi_\mu - i \varepsilon \gamma_\mu \hat{D}_\varepsilon \hat{R}^{bc}(Q),$$

where covariant derivatives on transformation parameters are defined by

$$D_\mu \varepsilon = \left( \partial_\mu - \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} + \frac{1}{2} b_\mu - \frac{3}{4} i \gamma_5 A_\mu \right) \varepsilon,$$
Table III. Weyl and Matter multiplets in 4D.

<table>
<thead>
<tr>
<th>field</th>
<th>type</th>
<th>remarks</th>
<th>Weyl-weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Wyle multiplet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e^a_\mu$</td>
<td>boson</td>
<td>real</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\psi_\mu$</td>
<td>fermion</td>
<td>Majorana</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$A_\mu$</td>
<td>boson</td>
<td>real</td>
<td>$0$</td>
</tr>
<tr>
<td>$b_\mu$</td>
<td>boson</td>
<td>real</td>
<td>$0$</td>
</tr>
</tbody>
</table>

|       | Complex (real) general multiplet |
| $C$   | boson | complex (real) | $w$         |
| $\zeta$ | fermion | Dirac (Majorana) | $w + \frac{1}{2}$ |
| $H, K, B_a$ | boson | complex (real) | $w + 1$ |
| $\lambda$ | fermion | Dirac (Majorana) | $w + \frac{3}{2}$ |
| $D$   | boson | complex (real) | $w + 2$ |

|       | gauge multiplet ($w = n = 0$) |
| $B^g_\mu$ | boson | adjoint rep. | $0$         |
| $\lambda^g$ | fermion | Majorana, adjoint rep. | $\frac{3}{2}$ |
| $D^g$   | boson | adjoint rep. | $2$         |

|       | chiral multiplet ($w = n$) |
| $A$   | boson | complex | $w$ |
| $\chi$ | fermion | Majorana | $w + \frac{1}{2}$ |
| $F$   | boson | complex | $w + 1$ |

|       | real linear multiplet ($w = 2, n = 0$) |
| $C^L$ | boson | real | $2$ |
| $\zeta^L$ | fermion | Majorana | $\frac{5}{2}$ |
| $B^L_a$ | boson | real, constrained | $3$ |

\[
\mathcal{D}_\mu \eta = \left( \partial_\mu - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{1}{2} b_\mu + \frac{3}{4} i \gamma_5 A_\mu \right) \eta, \\
\mathcal{D}_\mu \xi^a_K = (\partial_\mu - b_\mu) \xi^a_K - \omega_\mu^{ab} \xi_{Kb}.
\] (3.5)

3.2. Matter multiplets

3.2.1. Gauge multiplet

A multiplet which contains gauge field of a gauge group $G$ is a gauge multiplet $[B^g_\mu, \lambda^g, D^g]$. The $Q, S$ and $K$ transformation laws are given by

\[
\delta B^g_\mu = -i \varepsilon \gamma_\mu \lambda^g,
\]
\[
\delta \lambda^g = -\frac{1}{2} \gamma \cdot \hat{F}(B^g) \varepsilon + i \gamma_5 \varepsilon D^g + \frac{3}{4} \theta i \gamma_5 \lambda^g,
\]
\[
\delta D^g = \varepsilon \gamma_5 \hat{D} \lambda^g,
\]
(3.6)

where \( \hat{F}_{\mu\nu}(B^g) = \hat{R}_{\mu\nu}(G) \) is a super-covariantized field strength given by
\[
\hat{F}_{\mu\nu}(B^g) = 2 \partial_{[\mu} B_{\nu]}^g - [B^g_{\mu}, B^g_{\nu}] + 2 i \bar{\psi}_{[\mu} \gamma_{\nu]} \lambda^g,
\]
\[
\hat{D}_{\mu} \lambda^g = D_{\mu} \lambda^g + \frac{1}{2} \gamma \cdot \hat{F}(B^g) \psi_{\mu} - i \gamma_5 \psi_{\mu} D^g,
\]
with \( D_{\mu} \lambda^g = (\partial_{\mu} - \frac{3}{4} \omega_{\mu}^{ab} \gamma^{ab} - \frac{3}{2} b_{\mu} - \frac{3}{4} A_{\mu} i \gamma_5) \lambda^g \).
(3.7)

As is well-known in rigid supersymmetry case, this gauge multiplet is embedded into a superfield strength multiplet \( W_{\alpha} \), a chiral multiplet with an external spinor index \( \alpha \), whose first component is \( \lambda^g_{\alpha} \). [See Ref. 12) for superconformal multiplets with external Lorentz indices.]

### 3.2.2. Complex, or real, general multiplet \( \Phi \)

A maximal unconstrained multiplet whose first component is a complex scalar \( C \), is called a complex general multiplet, and its full components are listed in table III. The dilatation and \( U(1) \) transformations of the first component \( C \) define the Weyl and chiral weights, \( w \) and \( n \), of the multiplet,
\[
\delta_D(\lambda_D)C = wC, \quad \delta_A(\theta)C = \frac{i}{2} nC,
\]
(3.8)

and the multiplet is characterized by these weights \( w \) and \( n \). When the chiral weight vanishes \((n = 0)\), the complex general multiplet decomposes into two irreducible real general multiplets, whose components are all real or Majorana fields. The transformation laws of the complex general multiplet are given as follows. (Those of the real general multiplet is simply given by setting \( n = 0 \).)

\[
\delta C = i \bar{\varepsilon} \gamma_5 \zeta + \frac{1}{2} i n \theta C,
\]
\[
\delta \zeta = \left( i \bar{\gamma}_5 H - K + i B + \hat{D} C \gamma_5 \right) \varepsilon
\]
\[
+ 2i (n + w \gamma_5) C \eta + \left( \frac{1}{2} i n - \frac{3}{2} i \gamma_5 \right) \theta \zeta,
\]
\[
\delta H = \varepsilon \gamma_5 \hat{D} \zeta - \bar{\varepsilon} \gamma_5 \lambda - \bar{\eta} \gamma_5 \zeta + \left( \frac{1}{2} i n H + \frac{3}{2} K \right) \theta,
\]
\[
\delta K = i \varepsilon \hat{D} \zeta - \varepsilon \lambda - \bar{\eta} \gamma_5 \zeta + \left( \frac{1}{2} i n K - \frac{3}{2} H \right) \theta,
\]
\[
\delta B_a = -\varepsilon \hat{D}_a \zeta - i \varepsilon \gamma_5 \lambda
\]
\[
- i \bar{\eta} \left( (w + 1) + n \gamma_5 \right) \gamma_a \zeta + \frac{1}{2} i n \theta B_a + 2ni \xi_{Ka} C,
\]
\[
\delta \lambda = -\frac{1}{2} \gamma \cdot \hat{F} \varepsilon + i \gamma_5 \varepsilon D
\]
\[ + \left( i\gamma_5 H + K - iB - \hat{\Phi} C \gamma_5 \right) (w + n\gamma_5)\eta \\
+ \left( \frac{1}{2} i n + \frac{3}{4} i\gamma_5 \right) \theta \lambda \gamma_a \gamma_5 \zeta \]

\[
\delta D = \varepsilon \gamma_5 \hat{\Phi} \lambda + \frac{1}{2} \varepsilon \theta D \]

\[-\bar{\eta}(w\gamma_5 + n)\hat{\Phi} \zeta - 2i\bar{\eta}(w\gamma_5 + n)\lambda + 2w\xi^a_K \hat{D}_a C + 2ni\xi^a_K B_a, \] (3.9)

where \( \hat{F}_{ab} \) is a field-strength like quantity given by

\[
\hat{F}_{ab} = 2\hat{D}_{\{a}B_{b\}} + \frac{1}{2} \varepsilon_{abcd}[\hat{D}^c, \hat{D}^d]C. \] (3.10)

Up to this point, this general multiplet has tacitly been assumed to carry no extra charges. If this multiplet carries charges of the gauge group \( G \), the transformation rules are slightly modified; First the \( G \)-covariantization term \(-\delta G(B^a)\) should also be included in the full-covariant derivative \( \hat{D}_\mu \). Second the following terms should be added to the above transformation laws (3.9):

\[
\Delta[\delta B_a] = \varepsilon \gamma_5 \gamma_a \lambda^5 C, \\
\Delta[\delta \lambda] = -\varepsilon D^a \gamma^a + \gamma^a \varepsilon \frac{1}{2} \hat{\lambda}^a \gamma_a \zeta + \gamma^a \gamma_5 \varepsilon \frac{1}{2} \hat{\lambda}^a \gamma_a \gamma_5 \zeta, \\
\Delta[\delta D] = \varepsilon \gamma_5 \gamma^a \lambda^a B_a + i\varepsilon \Phi (\lambda^a C) + \varepsilon D^a \zeta, \] (3.11)

where \( \theta = (\lambda^a, D^a) \) denotes the \( G \) transformation; \( \theta \Phi = \delta G(\theta) \Phi \).

3.2.3. Chiral multiplet \( \Sigma \)

Chiral multiplet \( \Sigma = [A, \mathcal{P}_R \chi, \mathcal{F}] (P_R \equiv \frac{1}{2}(1 + \gamma_5)) \) can exist when \( w = n \), and anti-chiral multiplet \( \Sigma^* = [A^*, \mathcal{P}_L \chi, \mathcal{F}^*] (P_L \equiv \frac{1}{2}(1 - \gamma_5)) \) when \( w = -n \). Their embedding into the complex general multiplet is

\[
\Phi(\Sigma) = [A, -iP_R \chi, -\mathcal{F}, i\mathcal{F}, i\hat{D}_a A, -2iP_L \lambda^a A, -iD^a A], \\
\Phi(\Sigma^*) = [A^*, iP_L \chi, -\mathcal{F}^*, -i\mathcal{F}, -i\hat{D}_a A^*, 2iP_R \lambda^a A^*, iD^a A^*]. \] (3.12)

The transformation laws of these multiplet can be read from those of \( \Phi \) as follows.

\[
\delta A = \varepsilon P_R \chi + \frac{i}{2} w \theta A, \\
\delta (P_R \chi) = P_R \left( -2i\hat{\mathcal{P}} A\varepsilon + 2F \varepsilon - 4w A\eta + \frac{i}{2} (w - \frac{3}{2}) \theta \chi \right), \\
\delta F = -i\varepsilon \hat{\mathcal{P}} (P_R \chi) + 2\varepsilon P_L \lambda^a A + 2(w - 1)\eta P_R \chi + \frac{i}{2} (w - 3) \theta F. \] (3.13)

3.2.4. Real linear multiplet \( L \)

This multiplet, which is denoted by \( L = [C^L, \zeta^L, B^L_a] \), can exist only in the case of the weight \( w = 2, n = 0 \). The vector component \( B^L_a \) is subject to the constraint

\[
0 = \hat{D}^a B^L_a - D^a \xi^L + \hat{\lambda}^a \zeta^L. \] (3.14)
Interestingly, this constraint is solvable in the case if this multiplet is $G$-inert or the matrix $D^g$ is invertible. This multiplet is also embedded into the real general multiplet in the form

$$\Phi(L) = [C^L, \zeta^L, 0, 0, B^L_a, i\vec{\Phi}\zeta^L, \bar{D}^a\bar{D}_a C^L - i\bar{\lambda}_a^g\gamma_5\zeta^L], \quad (3.15)$$

The transformation laws of the components can also be read from those of $\Phi$.

§4. Identification of $N = 1, d = 4$ supermultiplets at the boundary

We must treat both 5D and 4D fields simultaneously from here. We use the vector indices $\mu, \nu, \cdots$ and $a, b, \cdots$ to denote 4D ones running over $0, 1, 2, 3$, and write the fifth component by $y$ for world vector and by 4 for Lorentz vector. For instance a 5D vector is $(V_\mu, V_y)$ or $(V_a, V_5)$.

From the viewpoint of four-dimensional boundary plane, any supermultiplet in the 5D bulk is reducible to an infinite number of supermultiplets of 4D superconformal algebra. We here identify all the basic 4D supermultiplets which each contain at least one bulk field on the boundary without derivative $\partial_4$ (with respect to the transverse direction $y$) as their member.

4.1. 4D Weyl multiplet from 5D one

The fields $\phi$ are classified into even and odd fields under the $Z_2$ parity transformation $y \equiv x^4 \rightarrow -y$. The $Z_2$ parity eigenvalue $\Pi(\phi) = \pm 1$ is defined by

$$\phi(-y) = \Pi(\phi) \phi(y). \quad (4.1)$$

For SU(2)-Majorana spinor fermions $\psi^i$ ($i = 1, 2$), however, the $Z_2$ parity transformation mixes the two components $\psi^1$ and $\psi^2$ as

$$\psi^i(-y) = \Pi(\psi) \gamma_5 M^i_j \psi^j(y) \quad (4.2)$$

where $M^i_j$ is a $2 \times 2$ matrix satisfying $M^* = -\sigma_2 M \sigma_2$, which we here take $M = \sigma_3$ as our convention. We therefore define the following combinations of spinors $\psi^i$

$$\psi_+(y) \equiv \psi_R^1(y) + \psi_L^2(y) \quad \psi_-(-y) \equiv i(\psi_L^1(y) + \psi_R^2(y)) \quad \left(\psi_R^\pm \equiv \frac{1 \pm \gamma_5}{2}\psi\right), \quad (4.3)$$

which give the $Z_2$ parity eigenstates

$$\psi_\pm(-y) = \pm \Pi(\psi) \psi_\pm(y) \quad (4.4)$$
Table IV. \( Z_2 \) parity eigenvalues\(^{21} \)

<table>
<thead>
<tr>
<th>( \Pi = +1 )</th>
<th>( e^a_\mu, \ e^4_\mu, \ \psi^+<em>\mu, \ \psi^-</em>\mu, \ \varepsilon_+, \ \eta_+, \ b^i_\mu, \ V^3_\mu, \ V^{1,2}<em>y, \ \psi^{4a}, \ \chi</em>+, \ D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pi = -1 )</td>
<td>( e^4_\mu, \ e^a_\mu, \ \psi^-<em>\mu, \ \psi^+</em>\mu, \ \varepsilon_-, \ \eta_+, \ b^i_\mu, \ V^3_\mu, \ V^{1,2}<em>y, \ \psi^{ab}, \ \chi</em>-, \ )</td>
</tr>
</tbody>
</table>

**Weyl multiplet**

\( \Pi_V \)  \( M, \ W_y, \ Y^{1,2}, \ \Omega_- \)

\( -\Pi_V \)  \( W^i_\mu, \ Y^3, \ \Omega_+ \)

**Linear multiplet**

\( \Pi_L \)  \( L^{1,2}, \ N, \ E^4, \ \varphi_+ \)

\( -\Pi_L \)  \( L^3, \ E^a, \ \varphi_- \)

**Hypermultiplet**

\( \Pi_A \)  \( A^{2\hat{\alpha}}_{i = 1}, \ A^{2\hat{\alpha}}_{i = 2}, \ F^{2\hat{\alpha}}_{i = 1}, \ F^{2\hat{\alpha}}_{i = 2}, \ \zeta^{\hat{\alpha}} \)

\( -\Pi_A \)  \( A^{2\hat{\alpha}}_{i = 2}, \ A^{2\hat{\alpha}}_{i = 1}, \ F^{2\hat{\alpha}}_{i = 1}, \ F^{2\hat{\alpha}}_{i = 2}, \ \zeta^{\hat{\alpha}} \)

satisfying also the 4D Majorana property \( \bar{\psi}_\pm \equiv (\psi_\pm)^\dagger \gamma^0 = \psi_\pm C_4 \).

The \( Z_2 \) parity eigenvalues are assigned to fields by demanding the invariance of the action and the consistency of both sides of the superconformal transformation rules. We list in Table IV the \( Z_2 \) parity eigenvalues for the Weyl multiplet fields and \( Q \)- and \( S \)-transformation parameters \( \varepsilon \) and \( \eta \),\(^{21} \) where the notation of ‘isovector’ \( \vec{t} = (t^1, t^2, t^3) \) is used which we generally define for any symmetric SU(2) tensor \( t^{ij} \) [satisfying hermiticity \( t^{ij} = (t_{ij})^\ast \)] as

\[
\vec{t}^i = (t_{ik} \varepsilon_{kj}) \equiv i \vec{t} \cdot \vec{\sigma}^j.
\] (4.5)

The even parity fields are non-vanishing on the 4D boundary planes at \( y = 0 \) and \( y = \tilde{y} \) and can form 4D superconformal multiplets there. In four dimensions the parameters of \( Q \) and \( S \) supersymmetry transformations are both 4-component Majoranas. In accordance with this, half of the parameters of 5D \( Q \) and \( S \) supersymmetries vanish on the boundaries, and only the other halves \( \varepsilon_+ \) and \( \eta_- \), respectively, remain nonvanishing which are indeed Majorana spinors.

First of all the 4D superconformal Weyl multiplet is induced on the boundary planes from the 5D bulk Weyl multiplet, and the multiplet members can be identified as follows comparing the superconformal transformation laws in both dimensions; that is, the following 5D fields on the right-hand sides can be seen to transform exactly the same way as the 4D Weyl multiplet obeying the superconformal transformation rule (3.4). (The 5D fields are
always understood to be those evaluated at the boundary, \( y = 0 \) or \( \tilde{y} \), in the relations between 4D and 5D.)

\[
e^{(4)a}_{\mu} = e_{\mu}^a, \quad \psi^{(4)}_{\mu} = \psi_{\mu+}, \quad b^{(4)}_{\mu} = b_{\mu},
\]
\[
o^{(4)ab}_{\mu} = \omega^{ab}_{\mu}, \quad A^{(4)}_{\mu} = \frac{4}{3} \left( V^3_{\mu} + v_{\mu} \right),
\]
\[
\phi^{(4)}_{\mu} = \phi_{\mu} - \gamma_5 \gamma^a v_{a4} \psi_{\mu+} + \Delta \phi_{\mu},
\]
\[
f^{(4)a}_{\mu} = f_{\mu}^a - \bar{\psi}_{\mu+} \Delta \phi^a + \Delta f^a_{\mu} 
\]

(4.6)

with \( \Delta \phi_{\mu} \) and \( \Delta f^a_{\mu} \) given by

\[
\Delta \phi_{\mu} \equiv \frac{1}{2} i \gamma_5 \hat{R}^{(4)}_{\mu 4}(Q)_-, \\
\Delta f^a_{\mu} \equiv - \frac{1}{6} \epsilon^{abc} \left( \hat{\mathcal{D}}_b v_{c4} + \frac{1}{2} \hat{R}^{(4)}_{bc3} (V) \right) + \frac{1}{4} \hat{R}^{(4)4a4}(M). 
\]

(4.7)

Note, however, that the 4D \( Q \) supersymmetry transformation \( \delta_Q^{(4)}(\epsilon) \) here is identified with the following linear combination of 5D transformations at the boundaries:

\[
\delta_Q^{(4)}(\epsilon = \epsilon_+) = \delta_Q(\epsilon_+) + \delta_S(\gamma_5 \gamma^a v_{a4} \epsilon_+) + \delta_K(\epsilon_+ \Delta \phi^a),
\]

(4.8)

and the other 4D superconformal transformations, \( U(1) \) transformation \( \delta_A^{(4)}(\theta) \), \( S \) supersymmetry transformation \( \delta_S^{(4)}(\eta) \), etc, are identified as

\[
\delta_A^{(4)}(\theta = \frac{4}{3} \theta^3) = \delta_U(\theta^3), \\
\delta_S^{(4)}(\eta = \eta_-) = \delta_S(\eta_-), \\
\delta_D^{(4)}(\rho) = \delta_D(\rho), \\
\delta_M^{(4)}(\lambda^{ab}) = \delta_M(\lambda^{ab}), \\
\delta_K^{(4)}(\xi^a_K) = \delta_K(\xi^a_K). 
\]

(4.9)

With these identifications of the fields and superconformal transformations, we have the following relation between the superconformal covariant derivatives in 4D and 5D:

\[
\hat{\mathcal{D}}^{(4)}_a = \hat{\mathcal{D}}_a - \delta_{U_3}(v_{a4}) - \delta_S(\Delta \phi_a) - \delta_K(\Delta f_a^b). 
\]

(4.10)

If we use this equation and \([\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] = -\hat{R}_{ab}^c X_A \) holding in any dimensions, we can most easily find the expressions (4.7) for \( \Delta \phi_{\mu} \) and \( \Delta f^a_{\mu} \). Indeed, comparing the coefficients of \( X_A = Q^a, M_{ab} \) and \( U^3 \) on both sides of the commutators of Eq. (4.10), we straightforwardly find the relations between curvatures in both dimensions:

\[
\hat{R}^{(4)}_{ab}(Q) = \hat{R}_{ab}(Q)_+ - 2 \gamma_{[a} \Delta \phi_{b]}, \\
\hat{R}^{(4)cd}_{ab}(M) = \hat{R}_{ab}^{cd}(M) + 8 \Delta f_a^{[c} [d] \delta_{b]}, \\
\frac{3}{4} \hat{R}^{(4)}_{ab}(A) = \hat{R}_{ab}^3 (U) + 2 \hat{\mathcal{D}}_a v_{b4}. 
\]

(4.11)

Applying to these relations the constraints on the \( Q \) and \( M_{ab} \) curvatures in both dimensions, we immediately find the above expressions (4.7) for \( \Delta \phi_{\mu} \) and \( \Delta f^a_{\mu} \).
In addition to this 4D Weyl multiplet, the 5D Weyl multiplet also induces a 4D ‘matter’ multiplet. Indeed, the extra dimensional component $e_y^4$ of the fünfein is also non-vanishing on the boundaries and are $S$- and $K$-inert, so that it can be the first component of a superconformal multiplet. It turns out to be a general multiplet $W_y$ with Weyl and chiral weights $(w, n) = (-1, 0)$. The identification of the multiplet members is given by

$$W_y = (C, \zeta, H, K, B_a, \lambda, D)$$

$$= \left( e_y^4, -2\psi_{y-}, -2V_y^2, 2V_y^1, -2\nu_{y}, \frac{i}{4}\gamma_5\chi + e_y^4 + 2\phi_{y+} + 2\gamma_5\gamma^5\nu_{44}\psi_{y-}, \right.$$  

$$\left. \left( \frac{1}{4}D - (\nu_{a4})^2 \right) e_y^4 - 2f_y^4 + \frac{i}{4}\bar{\chi} + \gamma_5\psi_{y-} \right). \quad (4.12)$$

These fields transform just according to the general multiplet transformation rule (3.9), provided that the covariant derivatives $\hat{D}^{(4)}$ appearing there are understood to be given by

$$\hat{D}^{(4)}_{\mu} e_y^4 = e_{\mu a} \omega^a_{y} = D^{(4)}_{\mu} e_y^4 + 2i\bar{\psi}_{\mu} + \gamma_5\psi_{y-} - \bar{\psi}_y e_{\mu}^4$$

$$\hat{D}^{(4)}_{\mu} \psi_{y-} = D^{(4)}_{\mu} \psi_{y-} + [-V_y^1 - \gamma_5V_y^2 - \gamma_5\psi_{y+} + \frac{1}{2}(\hat{D}^{(4)}_{\mu} e_y^4)\gamma_5]\psi_{y+} - ie_y^4\gamma_5\phi_{\mu} - \partial_y \psi_{y-}$$

$$= \hat{R}_{\mu y}(Q)_{y-} + i\gamma_5\phi_{+} + i\gamma_5\gamma^5\nu_{a4}\gamma_{\mu}\psi_{y-} \quad (4.13)$$

with ‘homogeneous covariant derivative’ $D^{(4)}_{\mu} = \partial_{\mu} - \delta_M(\omega^{ab}_{\mu}) - \delta_D(b_{\mu}) - \delta_A(A_{\mu})$ covariant only with respect to the homogeneous transformations $M_{ab}$, $D$ and $A$. The last terms, $-\partial_y e_{\mu}^4$ in $\hat{D}^{(4)}_{\mu} e_y^4$ and $-\partial_y \psi_{y-}$ in $\hat{D}^{(4)}_{\mu} \psi_{y-}$, are unusual which appear as a result of the fact that $e_y^4$ and $\psi_{y-}$ carry strange ‘new’ charges, as we explain now.

Generally, if a 5D local transformation parameter $A(x, y)$ is $Z_2$-odd, it itself vanishes on the boundary. However, its first derivative in $y$, $\partial_y A(x, y)$ is $Z_2$-even and gives a non-vanishing 4D gauge transformation parameter $\partial_y A(x, 0) \equiv A^{(1)}(x)$ on the boundary. Therefore, for the $Z_2$-odd parts of the 5D superconformal transformation parameters, there exist the corresponding 4D gauge transformations with parameters given as follows:

$$\xi^y \text{ of GC transformation } P \rightarrow \xi^{(1)}(x) \equiv \partial_y \xi^y(x, 0)$$

$$\varepsilon_{-} \text{ of Q supersymmetry } \rightarrow \varepsilon^{(1)}(x) \equiv \partial_y \varepsilon_{-}(x, 0)$$

$$\theta_{1, 2} \text{ of SU(2) transformation } U \rightarrow \theta^{(1)}_{1, 2}(x) \equiv \partial_y \theta_{1, 2}(x, 0)$$

$$\lambda_{a}^{4} \text{ of local Lorentz } M \rightarrow \lambda_{a}^{(1)}(x) \equiv \partial_y \lambda_{a}^{4}(x, 0)$$

$$\eta_{+} \text{ of S supersymmetry } \rightarrow \eta^{(1)}(x) \equiv \partial_y \eta_{+}(x, 0)$$

$$\xi_{K}^{4} \text{ of special conformal } K \rightarrow \xi_{K}^{(1)}(x) \equiv \partial_y \xi_{K}^{4}(x, 0) \quad (4.14)$$

The general multiplet $W_y$ in Eq. (4.12) transforms non-trivially under these transformations. Under the first $\xi^{(1)}$ transformation, every members of $W_y$ receive a common scale
transformation
\[ \delta W_y = \xi^{(1)} W_y, \] (4.15)
and many members of \( W_y \) are shifted inhomogeneously as Nambu-Goldstone fields under the other transformations:
\[ \delta W_y = (0, -2\varepsilon^{(1)}, -2\theta_2^{(1)}, 2\theta_1^{(1)}, 0, 2\eta^{(1)}, -2\xi_K^{(1)} + \frac{i}{4} \varepsilon(1) \gamma_5 \chi_+). \] (4.16)

We can find the gauge fields for these transformations:
\[
\begin{align*}
E^{(1)}_{\mu} &\equiv (e^4_y)^{-1} \partial_y e^4_{\mu} \quad \text{for } \xi^{(1)} \\
\psi^{(1)}_{\mu} &\equiv \partial_y \psi_{\mu} - E^{(1)}_{\mu} \psi_{\mu} \quad \text{for } \varepsilon^{(1)} \\
V_{\mu}^{(1)1,2} &\equiv \partial_y V_{\mu}^{1,2} - E^{(1)}_{\mu} V_{\mu}^{1,2} \quad \text{for } \theta^{(1)}_{1,2} \\
&\vdots \\
&\vdots \\
&\vdots
\end{align*}
\] (4.17)

The last terms \( -\partial_y e^4_{\mu} \) in \( \hat{D}^{(4)}_{\mu} e^4_y \) and \( -\partial_y \psi_{\mu} \) in \( \hat{D}^{(4)}_{\mu} \psi_y \) in Eq. (4.13) can be understood to be just the terms which appear as the gauge covariantization \( -\delta_{\xi^{(1)}}(E^{(1)}_{\mu}) - \delta_{\varepsilon^{(1)}}(\psi^{(1)}_{\mu}) \) using these gauge fields \( E^{(1)}_{\mu} \) and \( \psi^{(1)}_{\mu} \).

Since the general multiplet \( W_y \) transforms non-trivially under these gauge transformations (4.14), the utility of the multiplet \( W_y \) is rather limited. If we wish to use it in constructing 4D invariant actions on the brane with the other multiplets, we must satisfy the gauge invariance also under the transformations (4.14), which seems a non-trivial task.

4.2. from vector multiplet

We define the \( Z_2 \)-parity \( \Pi_V \) of vector multiplet \( V = (M, W_{\mu,y}, \Omega^i, Y^{ij}) \) to be that of the first scalar component \( M \). The \( Z_2 \)-parity quantum numbers of the other members are given as shown in Table IV.

If a vector multiplet \( V \) is assigned odd \( Z_2 \)-parity \( \Pi_V = -1 \), then the components \( W_{\mu}, \Omega_+ \) and \( Y^3 \) are even and non-vanishing on the brane, and gives a 4D gauge multiplet \( (B^g_{\mu}, \lambda^g, D^g) \) defined in Eq. (3.6) with \( (w, n) = (0, 0) \) with the following identification:
\[ (B^g_{\mu}, \lambda^g, D^g) = (W_{\mu}, 2\Omega_+, 2Y^3 - \hat{D}_4 M). \] (4.18)
This implies that, if \( \Pi_V = -1 \), the bulk Yang-Mills multiplets can also couple to the matter multiplets on the brane as the 4D Yang-Mills multiplets.

If a vector multiplet \( V \) has even \( Z_2 \)-parity \( \Pi_V = +1 \), then we can identify the following real general-type multiplet with weight \( (w, n) = (1, 0) \) whose first component is \( M \):
\[ (C, \zeta, H, K, B_a, \lambda, D) \]
\begin{align*}
&= \left( M, -2i\gamma_5\Omega_- + 2Y^1, 2Y^2, \hat{F}_{\alpha 4}(W) + 2v_{\alpha 4}M, \\
&\quad -2\hat{D}_4\Omega_+ + 2i\gamma^0v_{\alpha 4}\Omega_- - \frac{i}{4}\gamma_5\chi M, \\
&\quad \hat{D}_4^2 M - 2\hat{D}_4 Y^3 - \frac{1}{4}DM + v_{\alpha 4}(2\hat{E}_{\alpha 4}(W) + v_{\alpha 4}M) + \frac{1}{2}\hat{\chi} \Omega_+ \right) \quad (4.19)
\end{align*}

The field $D$ in the term $-\frac{1}{4}DM$ is the auxiliary field $D$ in the 5D Weyl multiplet.

In the latter case of $\Pi_V = +1$, we can construct a 4D chiral multiplet with weight $(w, n) = (0, 0)$:

\begin{align*}
A &= \frac{1}{2}(W_y + ie^y_4M), \\
\chi &= 2\psi_1 - M + 2ie^y_4\gamma_5\Omega_, \quad (4.20)
\end{align*}

However, this multiplet is also of limited utility because of its non-trivial transformation property under the gauge transformation $A^{(1)}(x) = \partial y A(x, 0)$ as well as the above $\xi^{(1)}, \varepsilon^{(1)}, \theta_1^{(1)}$ and $\theta_2^{(1)}$ transformations.

### 4.3. from hypermultiplet

Hypermultiplet $H^\alpha = (A^\alpha_i, \zeta^\alpha, \mathcal{F}^\alpha_i)$ ($\alpha = 1, 2, \cdots, 2r$) generally splits into $r$ pairs $(H^{\hat{1}2\alpha-1}, H^{\hat{2}2\alpha})$ ($\hat{\alpha} = 1, 2, \cdots, r$) in the standard representation, in which $\rho_{\alpha\beta} = \epsilon \otimes 1_r$. Then the following $2 \times 2$ matrix of $(A^{2\hat{1}\alpha-1}_i, A^{2\hat{2}\alpha}_i)$ for each $\hat{\alpha}$ satisfies the same reality structure as a quaternion $q = q^0 + iq^1 + jq^2 + kq^3$ mapped to the $2 \times 2$ matrix:

\begin{equation}
\begin{pmatrix}
A^{2\hat{1}\alpha-1}_{i=1} & A^{2\hat{1}\alpha-1}_{i=2} \\
A^{2\hat{2}\alpha}_{i=1} & A^{2\hat{2}\alpha}_{i=2}
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
q^0 - iq^3 \\
-q^1 - q^2
\end{pmatrix} \quad (4.21)
\end{equation}

The matrix element fields of this matrix as well as those of the similar matrix $(\mathcal{F}^{2\hat{1}\alpha-1}_i, \mathcal{F}^{2\hat{2}\alpha}_i)$ for the auxiliary fields turn out to give the $Z_2$ parity eigenstates. For spinors, the pair $(\zeta^{2\hat{1}\alpha-1}, \zeta^{2\hat{2}\alpha})$ for each $\hat{\alpha}$ satisfies the same reality condition as SU(2)-Majorana spinor $\psi^i = (\psi^1, \psi^2)$, so that we can define the $Z_2$-parity eigenstate 4D-Majorana spinors $\zeta^\alpha_L$ in the same way as in Eq. (4.3); $\zeta^\alpha_L = \zeta^{2\hat{1}\alpha-1}_L + \zeta^{2\hat{2}\alpha}_L$ and $\zeta^\alpha \equiv i(\zeta^{2\hat{1}\alpha-1}_R + \zeta^{2\hat{2}\alpha}_R)$.

Then, if the 1-1 component $A^{2\hat{1}\alpha-1}_{i=1}$ has $Z_2$-parity $\Pi_{\hat{\alpha}}$, the $Z_2$-parity quantum numbers of the other hypermultiplet members are given as shown in Table IV. For either choice of the $Z_2$ parity assignment $\Pi_{\hat{\alpha}} = \pm 1$, we can find the following 4D chiral multiplet with weight $(w, n) = (3/2, 3/2)$:

\begin{align*}
(A, \chi_R, \mathcal{F}) &= (A^{2\hat{1}\alpha}_{i=2}, -2i\zeta^{2\hat{1}\alpha}_R, (iM_\alpha A + \hat{D}_4 A)_{i=1}^{2\hat{1}\alpha}), \\
(A, \chi_R, \mathcal{F}) &= (A^{2\hat{1}\alpha-1}_{i=2}, -2i\zeta^{2\hat{1}\alpha-1}_R, (iM_\alpha A + \hat{D}_4 A)_{i=1}^{2\hat{1}\alpha-1}), \quad (4.22)
\end{align*}
for $\Pi_\alpha = \pm 1$, respectively. Since $M_\alpha \mathcal{A}_{\mu}^\alpha = g M^I(t_I)^\alpha\beta \mathcal{A}_{\mu}^\beta + \mathcal{F}_{\mu}^\beta$, the $\mathcal{F}$-components of these chiral multiplets contain the $\mathcal{F}_{\mu}^\beta$ components of the hypermultiplet.

4.4. from linear multiplet

The $Z_2$ parity quantum numbers for linear multiplet $L = (L_{ij}, \varphi^i, E^a, N)$, are shown in Table IV.

In case of $\Pi_L = +1$, we can identify the following 4D chiral multiplet with weight $(w, n) = (3, 3)$ on the brane:

$$ (A, \chi, \mathcal{F}) = (-L^1 + iL^2, 2\varphi_+, \frac{1}{2}(N + iE_4) - \hat{D}_4 L^3 - iM_\ast L^3). $$ (4.23)

If $\Pi_L = -1$, the scalar component $L_3$ is non-vanishing on the brane. Since it is $S$- and $K$-inert and carries Weyl and chiral weights $(w, n) = (3, 0)$, there is a 4D general-type real multiplet with weight $(w, n) = (3, 0)$ starting with $L_3$. We identify here the components except for the last $D$ component:

$$(C, \zeta, H, K, B_a, \lambda, D)$$

$$= (L^3, -\varphi_-, -M_\ast L^2 + \hat{D}_4 L^1, M_\ast L^1 + \hat{D}_4 L^2, -\frac{1}{2}E_4 + 2v_a L^3, -i\hat{D}^{(4)} \varphi_- + i\gamma_5 \hat{D}_4 \varphi_+ + M_\ast \varphi_+ - \gamma_5 \gamma^a v_a \varphi_- - 2\Omega_- L^1 + 2i\gamma_5 \Omega_- L^2 - \frac{i}{4}\gamma_5 \chi_+ L^3, \cdots).$$ (4.24)

§5. Compensator and general brane action

As is well-known by now, we need not only the Weyl multiplet but also a special matter field called ‘compensator’ in order to write down superconformal invariant supergravity actions. The compensator multiplet is used to fix the extraneous gauge freedoms of the superconformal symmetries like $D$, $A$ and $S$ as well as to saturate the required Weyl and chiral weights in applying the invariant action formulas.

The most popular formulation in 4D case is called ‘old minimal’ supergravity, where the compensator used is a chiral multiplet $\Sigma$ with weight $w = n = 1$. General superconformal invariant action in 4D is given by

$$ S_{\text{brane}} = \int d^5x \delta(y) \left( \left[ \Sigma \Sigma e^{K(S, \bar{S})} \right]_D + \left[ f_{IJ}(S) W^{I\alpha} W^J_\alpha \right]_F \right. $$

$$+ \left. [\Sigma^3 W(S)]_F \right) + \int d^5x \delta(y - \bar{y}) \left( \text{the same} \right). $$ (5.1)
superconformal tensor calculus,\textsuperscript{11)} explained in Appendix C. $S_i$ are the 4D chiral matter multiplets with weight $w = n = 0$ and $W_\alpha^I$ denotes the superfield strength of 4D Yang-Mills multiplets $V^I$. Both of these chiral and gauge multiplets, $S$ and $V$, may be genuine 4D multiplets living solely on the brane or induced multiplets on the boundary from the 5D bulk multiplets.

The 4D Weyl multiplet used in writing down the action formulas in Eq. (5.1) should of course be the induced Weyl multiplet found in the previous section. Since gravity is unique, we cannot add genuine 4D Weyl multiplet on the brane in addition to the induced one. In the same sense, we cannot add genuine 4D compensator on the brane in addition to that induced from the bulk compensator fields.

Let us now identify the 4D compensator induced from the 5D bulk compensator. The most useful and common choice of the 5D compensator is hypermultiplet which we have discussed in Ref. 5). Consider the simplest case of a single-quaternion compensator ($p = 1$); namely, the hypermultiplet compensator is given by $H^a = (A_a^i, \zeta^a, F_a^i)$ ($a = 1, 2$). Then, as we have seen in the previous section, this hypermultiplet gives a 4D chiral multiplet $\Sigma_c = (A_c, \chi_c, F_c)$ with weight $(w, n) = (3/2, 3/2)$ on the brane, assuming the $Z_2$ parity assignment $\Pi_a = +1$:

$$
\Sigma_c : \begin{cases} 
A_c = A^a_{i=2}, \\
\chi_c = -2i\zeta^a_{i=2} \\
F_c = iF^a_{i=1} + ig(MA)^a_{i=1} + {\hat D}_4 A^a_{i=1}
\end{cases}. \tag{5.2}
$$

Since this multiplet $\Sigma_c$ carries Weyl and chiral weights $w = n = 3/2$, we should identify $\Sigma_c^{2/3}$ with the 4D chiral compensator $\Sigma$ with $w = n = 1$ induced on the brane. Note that, if the 5D superconformal gauges are fixed, for example, by the conditions

$$
D, \quad U^{ij} : \quad A^a_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
S^a : \quad \zeta^a = 0 \tag{5.3}
$$

in the bulk, then, these yield the conditions on $\Sigma \equiv \Sigma_c^{2/3} = (A, \chi, F)$ on the boundary,

$$
D, \quad A : \quad A = 1, \quad S : \quad \chi = 0, \tag{5.4}
$$

the same 4D superconformal gauge fixing conditions as imposed on the usual chiral compensator in the case of pure supergravity system.\textsuperscript{11)}

This brane action (5.1) gives an superconformal invariant coupling between the 4D matters on the brane and the 5D bulk fields through the induced Weyl, Yang-Mills and compensating multiplets.
§6. Altendorfer-Bagger-Nemeschansky approach

We illustrate how the action is given for the bulk plus brane system by taking the simplest case where the bulk is pure supergravity system with $U(1)_R$ gauged and the brane contains only the tension term. This is the system which was first constructed by Altendorfer, Bagger and Nemeschansky \cite{23} when trying to supersymmetrize the Randall and Sundrum \cite{24} scenario.

In our off-shell formulation, the system contains 5D Weyl multiplet, the hypermultiplet compensator $H^a = (A^a_i, \zeta^a, F^a_i) \ (a = 1, 2)$ and a vector multiplet $V_0 = (M^0 = \alpha, W^0_\mu, \Omega^{0i}, Y^{0ij})$ coupling to the central charge $Z$ of the hypermultiplet. Note here that this central charge vector multiplet $V_0$ is simultaneously the $U(1)_R$ gauge multiplet coupling to the $U(1)_R$-charge which is in general represented by a $2 \times 2$ matrix $(t_R)^{a_b}$ acting on the group index $a = 1, 2$ of $H^a$.

$$t_R = i\tilde{Q} \cdot \tilde{\sigma} = i(Q_1 \sigma_1 + Q_2 \sigma_2 + Q_3 \sigma_3), \quad |\tilde{Q}| = 1. \quad (6.1)$$

$V_0$ can be made to play such double roles if we add a ‘mass term’ $m\eta^{ab}\mathcal{L}_{VL}(V_0 \cdot L(H_a, H_b))$ to the Lagrangian with a symmetric tensor $\eta^{ab}$ related with the $U(1)_R$ generator $t_R$ by $g_R(t_R)^{a_b} = m\eta^{ab}/2$. (Here $\mathcal{L}_{VL}(V \cdot L)$ is the invariant $V$-$L$ action formula [Eq. (4.5) in I] and $L(H_a, H_b)$ is the embedding formula [Eq. (4.3) in I] of two hypermultiplets $H$ and $H'$ into a linear multiplet.) The bulk lagrangian is given in the form

$$-2\mathcal{L}_{VL}(V_0 \cdot L(H^a, ZH_b + \frac{1}{2}m\eta^{bc}H_c)) + \mathcal{L}_{VL}(V_0 \cdot L(-cV_0^2)) \quad (6.2)$$

where $ZH^a$ is the central-charge transformed hypermultiplet whose first component is $ZA^a_i = F^a_i/\alpha$, and $L(\frac{1}{2}f_{IJ}V^IV^J)$ denotes generally the embedding formula [Eq. (4.1) in I] of vector multiplets $V^I$ into a linear multiplet based on the homogeneous quadratic function $f(M) = \frac{1}{2}f_{IJ}M^IM^J$. The $\mathcal{L}_{VL}(V_0 \cdot L(-cV_0^2))$ here thus corresponds to the choice of $f(\alpha) = -c\alpha^2$ and to the action with ‘norm function’ \cite{25} $N(\alpha) = c\alpha^3$, where $c$ is a constant coefficient. The explicit component form of the action (6.2) can be read form the general expression given in Ref. 5). The extraneous gauges are generally fixed by the gauge conditions

$$D: \quad N = 1, \quad S: \quad \Omega^{Ii}N_I = 0, \quad K: \quad \delta_{\alpha}N = 0. \quad (6.3)$$

The first $D$ gauge-fixing condition $N = c\alpha^3 = 1$ may equivalently be replaced by $A^a_iA^i_a = -2$ thanks to the auxiliary field equation $\delta S/\delta D \propto A^a_iA^i_a + 2N = 0$. Similarly, if we use the equation of motion $\delta S/\delta \chi^i \propto A^a_i\zeta_a + N_I\Omega^I_i = 0$, the $S$-gauge condition $N_I\Omega^I_i = 0$ is equivalent to $A^a_i\zeta_a = 0$. Then, imposing also $A^a_i \propto \delta^a_i$ as the $U^{ij}$ gauge-fixing conditions, these conditions reproduce the previous gauge-fixing conditions (5.3). The kinetic term $-(1/4)F_{\mu\nu}(W^0)$ of the gravi-photon field $W^0_\mu$ has coefficient $-(1/2)N(\partial^2 \ln N/\partial \alpha^2)$ in the
action$^5$) so that it is properly normalized if the constant $c$ is chosen to satisfy $c \alpha = 2/3$. Together with the $D$ gauge condition $N = c \alpha^3 = 1$, this determines $\alpha = \sqrt{3/2}$ and $c = (\sqrt{2/3})^3$. The cosmological constant in the bulk is found to be $-(8/3)g_R^2 \alpha^2 = -4g_R^2$.  

Note here that the consistency of the $U(1)_R$ symmetry with the $Z_2$ parity requires $Q^3 = 0$:  

$^{20}$ This can be seen if we look at the two terms in the covariant derivative, 

$$\mathcal{D}_\mu A^a_i = \partial_\mu A^a_i - g R W^0_\mu (t_R)^a b A^b_i + \cdots \quad (6.4)$$

Recall that the scalar component $M^0 = \alpha$ of the central charge vector multiplet is $Z_2$-even so that the vector component $W^0_\mu$ is $Z_2$-odd as seen in Table IV. Since $W^0_\mu$ is $Z_2$-odd and $A^a_i = 1$ and $A^a_i = -2$ carry opposite $Z_2$-parity, the first and the second terms in Eq. (6.4) can have the same $Z_2$ even-oddness if and only if $t_R$ possesses only off-diagonal components. Namely, $t_R = i (Q^1 \sigma_1 + Q^2 \sigma_2)$ with no $\sigma_3$ component. (If the coupling constant $g_R$ were lifted to the $Z_2$-odd field as in the Gherghetta-Pomarol$^{26}$ and Falkowski-Lalak-Pokorski (GPFLP) approach,$^{26,27,20}$ then the $g_R W^0_\mu$ part became $Z_2$-even and $t_R$ should be diagonal so that $t_R = i \sigma_3$.) After the SU(2) gauge is fixed in the bulk by the condition $A^a_i \propto \delta^a_i$, the $U(1)_R$ gauge transformation is modified into the combined (diagonal) $U(1)$ transformation of the original $U(1)_R$ and SU(2); e.g., $\delta_{U(1)_R}(\theta) \mathcal{F}^a_i = i \theta [Q^1 \sigma_1 + Q^2 \sigma_2, \mathcal{F}]^a_i$. However, the SU(2) is explicitly broken by the $Z_2$ parity assignment down to a $U(1)$ with generator $\sigma_3$. This breaking is manifest only at the boundaries, since the SU(2) is a local symmetry while $Z_2$ parity transformation relates the fields at $y$ only with those at $-y$. Since the $U(1)$ transformation of the generator $Q^1 \sigma_1 + Q^2 \sigma_2$ in SU(2) is broken, the $U(1)_R$ symmetry is explicitly broken in this Altendorfer-Bagger-Nemeschansky approach.

The brane tension terms are supplied by 

$$S_{\text{brane}} = \int d^5 x \left( A_1 \delta(y) + A_2 \delta(y-\bar{y}) \right) \left[ \Sigma^3 = \Sigma^2_c \right] F.$$ \hspace{1cm} (6.5)

Here $A_1$ and $A_2$ are assumed to be real for simplicity and the F-term action formula (C-3) reads for our chiral compensator $\Sigma^3_c = (A_c = 1, \chi_c = 0, F_c)$ as 

$$\left[ \Sigma^2_c \right] F = e_4 \left( 2 (F_c + \bar{F}_c) - 2 \bar{\psi}_i \gamma_{\mu
u} \psi^i \right)$$ \hspace{1cm} (6.6)

where $e_4$ is the four dimensional determinant of the vierbein, $e_4 = e/e_y = e \cdot e'_y$. Note now 

$$\hat{\mathcal{D}}_4 A^a_i = (\partial_i - \frac{3}{2} b_4 A^a_i) - V_{4ij} A^{aj} - \frac{W^0_4}{\alpha} \mathcal{F}^a_i - 2i \bar{\psi}_i \zeta^a$$

$$\rightarrow \hat{\mathcal{D}}_4 A^a = - \frac{W^0_4}{\alpha} \tilde{F}^2 + (V_4)^2 - g R W^0_4 (t_R)^2, \quad (6.7)$$

where we have used $A^a_i = \delta^a_i$ and $\zeta^a = 0$ by the superconformal gauge fixing (5.3), and $\tilde{F}^a_i$ is defined by 

$$\tilde{F}^a_i \equiv F^a_i - \frac{1}{2} m \alpha \eta^a_b A^b_i = F^a_i - g_R \alpha (t_R)^a b A^b_i.$$ \hspace{1cm} (6.8)
It is this field $\tilde{F}^a_i$ that vanishes in the absence of the brane term $A_1 = A_2 = 0$. Then we find the real part of the F-component $\mathcal{F}_c$ of $\Sigma_c$ in (5·2) as

$$\text{Re}\mathcal{F}_c = \text{Re}\left\{i(1 + \frac{W^0_0}{\alpha})\tilde{F}^2_1 + ig_R\alpha(t_R)^2_1 + (V_4 - g_RW^0_4t_R)^2_1\right\}$$

$$= \tilde{F}^1 - \frac{W^0_4}{\alpha}\tilde{F}^2 - g_R\alpha Q^1 - \tilde{V}^2_4. \quad (6·9)$$

Here $Q^1$ is the first component of the direction vector $\tilde{Q} \equiv (Q^1, Q^2, Q^3)$ of the $U(1)_R$ generator (6·1) in SU(2), $\mathcal{F}^1$ and $\mathcal{F}^2$ are the 1 and 2 components of the ‘quaternion’ $\mathcal{F}^a_i$: $\mathcal{F}^a_i = \mathcal{F}^01_2 - i\mathcal{F}^1\sigma_1 - i\mathcal{F}^2\sigma_2 - i\mathcal{F}^3\sigma_3$, and $\tilde{V}^k_4$ are defined by

$$(\tilde{V}^k_4)_j \equiv (V_4 - g_RW^0_4t_R)^i_j \equiv \sum_{k=1}^3 i\tilde{V}^k_4(\sigma_k)^i_j. \quad (6·10)$$

Since the auxiliary fields $\tilde{F}^1$, $\tilde{F}^2$ and $\tilde{V}^2_4$ appear in the bulk action in the form

$$e\left[-2\left(1 + \frac{(W^0_4)^2 - W^0_\mu W^0_\mu}{\alpha^2}\right)((\tilde{F}^1)^2 + (\tilde{F}^2)^2) + 2(\tilde{V}^2_4)^2\right] \quad (6·11)$$

with opposite signs and $W^0_\mu$ vanishes on the brane ($\Pi(W^0_\mu) = -1$), the solution to these auxiliary field equations of motion are given by

$$\tilde{F}^1 = \left(1 + \frac{(W^0_4)^2}{\alpha^2}\right)^{-1} e^y_4(A_1\delta(y) + A_2\delta(y - \bar{y})),$$

$$\tilde{F}^2 = \left(1 + \frac{(W^0_4)^2}{\alpha^2}\right)^{-1} (-\frac{W^0_4}{\alpha}) e^y_4(A_1\delta(y) + A_2\delta(y - \bar{y})),$$

$$\tilde{V}^2_4 = e^y_4(A_1\delta(y) + A_2\delta(y - \bar{y})). \quad (6·12)$$

Elimination of these auxiliary fields by substituting these solutions back to the action (6·11) plus (6·5), could potentially yield singular square terms of delta function. However, in fact, we see that the contributions from $\tilde{F}^1$ and $\tilde{F}^2$ and from $\tilde{V}^2_4$ exactly cancel each other.

After eliminating these auxiliary fields, the brane action becomes

$$S_{\text{brane}} = \int d^5x \left(A_1\delta(y) + A_2\delta(y - \bar{y})\right) e_4 \left(-4g_RQ^1\alpha - 2\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right), \quad (6·13)$$

The scalar $\alpha$ of the $U(1)_R$ gauge multiplet is nonvanishing. If the $U(1)_R$ gauging is done with $Q^1 = 1$, i.e., $t_R = i\sigma_1$, then this reproduces essentially the same brane action as that given by Altendorfer, Bagger and Nemeschansky. The point here is, however, that the parameters $A_1$ and $A_2$ remain arbitrary and are not determined by the supersymmetry requirement at all. Therefore, despite that the bulk cosmological constant is given by the parameter $g_R\alpha$, the brane tensions are $A_1$ or $A_2$ times $-4g_R\alpha$ and thus have no relation with the bulk...
cosmological constant $-(8/3)g_R^2\alpha^2 = -4g_R^2$.\(^5\) Zucker noted the same thing in his off-shell Poincaré supergravity formulation based on a linear multiplet compensator.\(^15\)

Let us comment on the Killing spinor on the Randall-Sundrum background:\(^24\)

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2.$$ (6.14)

The Killing spinor is found by demanding that the $Q$ and $S$ transformation $\delta = \delta_Q(\varepsilon) + \delta_S(\eta)$ of the gravitino $\psi_i^\mu$, $\psi_i^y$ and the fermion components $\Omega^{\mu i}$ of $V_0$ and $\zeta^a$ of $H^a$ vanish. The $S$-transformation parameter $\eta$ is given by a function of $\varepsilon$ by the condition $\delta \Omega^{\mu i} = 0$ (and then $\delta \zeta^a = 0$ is automatically satisfied.) Assuming the Killing spinor $\varepsilon(y)$ depends only on extra dimension coordinate $y$, one can show that such a Killing spinor can exist only when $A_1 = -A_2 = 2$, $Q^1 = 1$ ($Q^2 = Q^3 = 0$) and $k = 2\alpha g_R/3$. This implies that the brane tension $\pm \tau$ of the two boundary planes should be $\pm \tau \equiv \pm 4g_R\alpha Q^1 A_1 = \pm 12k$ while the bulk cosmological term is $-4g_R^2 = -6k^2$. However, this value of the brane tension is twice as large as the Randall-Sundrum’s value $\pm \tau = \pm 6k^2$. Zucker also noted this fact in his formulation.\(^15\) Since the effective four-dimensional cosmological term vanishes only when the Randall-Sundrum’s case, this implies that there exists no Killing spinor and therefore the (global) supersymmetry is \textit{spontaneously broken} on the Randall-Sunroom background. Note that this conclusion is in the framework of Altendorfer-Bagger-Nemeschansky approach. In fact in the GPFLP approach whose off-shell formulation is given in Ref.\(^{21}\), the same Randall-Sunroom background is shown\(^{20}\) to allow the existence of a Killing spinor.

Acknowledgements

The authors owe a lot to Tomoyuki Fujita who collaborated with them at the early stage of this work. The authors would like to thank Tony Gherghetta, David E. Kaplan, Tatsuo Kobayashi, Hiroaki Nakano and Stefan Vandoren for their encouragement and interest in this work. They also appreciate the Summer Institute 2001 held at Fuji-Yoshida, the discussions at which were valuable. T. K. is supported in part by Grants-in-Aid for Scientific Research No. 13640279 from Japan Society for the Promotion of Science and the Grants-in-Aid for Scientific Research on Priority Areas No. 12047214 from the Ministry of Education, Science, Sports and Culture, Japan.

Appendix A

\textbf{Conventions}

The gamma matrices $\gamma^a$ ($a = 0, 1, 2, \cdots, d - 1$) in $d$ dimensions satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $(\gamma_a)^\dagger = \eta^{ab}\gamma_b$, where $\eta^{ab} = \text{diag}(+, -,-, \cdots)$. $\gamma^{a-b}$ is the antisymmetrized product of
gamma matrices:

\[ \gamma^{a \cdots b} = \gamma^{[a \cdots} \gamma^{b]} \]  

where the square bracket \([\cdots]\) implies complete antisymmetrization of the indices with weight 1. Similarly \((\cdots)\) implies complete symmetrization with weight 1.

In five dimensions, we choose the Dirac matrices to satisfy

\[ \gamma^{a_1 \cdots a_5} = \epsilon^{a_1 \cdots a_5}, \]  

where \(\epsilon^{a_1 \cdots a_5}\) is a totally antisymmetric tensor with \(\epsilon^{01234} = 1\).

SU(2) index \(i\) \((i=1,2)\) is raised and lowered with antisymmetric \(\epsilon_{ij}\) tensor \((\epsilon_{12} = \epsilon^{12} = 1)\) according to the northwest-to-southeast (NW-SE) contraction convention:

\[ A^i = \epsilon_{ij} A_j, \quad A_i = A^j \epsilon_{ji}. \]  

The charge conjugation matrix \(C_5\) in 5D has the properties

\[ C_5^T = -C_5, \quad C_5^\dagger C_5 = 1, \quad C_5 \gamma_a C_5^{-1} = \gamma_a^T. \]  

Our five-dimensional spinors \(\psi^i\) satisfy the SU(2)-Majorana condition

\[ \bar{\psi}^i \equiv \left(\psi_i\right)^\dagger \gamma^0 = \psi^T C_5 \]  

where spinor indices are omitted. When SU(2) indices are suppressed in bilinear terms of spinors, NW-SE contraction is understood, e.g. \(\bar{\psi} \gamma^{a_1 \cdots a_n} \lambda = \bar{\psi}^i \gamma^{a_1 \cdots a_n} \lambda_i\).

In four dimensions, the Dirac matrices satisfy

\[ \gamma^{a_1 \cdots a_4} = -i \epsilon^{a_1 \cdots a_4} \gamma_5, \quad \epsilon^{0123} = 1, \]  

with the chirality matrix \(\gamma_5\). The fifth Dirac matrix in 5D, \(\gamma^4\), is anti-hermitian and related with \(\gamma_5\) as \(\gamma^4 = i \gamma_5\). The Majorana-condition is defined by

\[ \bar{\psi} \equiv \psi^\dagger \gamma^0 = \psi^T C_4, \]  

where the charge conjugation matrix \(C_4\) in 4D has the properties

\[ C_4^T = -C_4, \quad C_4^\dagger C_4 = 1, \quad C_4 \gamma_a C_4^{-1} = -\gamma_a^T. \]  

In the text, we take as our convention the relation \(C_5 = -C_4 \gamma_5\) between the charge conjugation matrices in 5D and 4D.
Table V. The transformation operators and the gauge fields.

<table>
<thead>
<tr>
<th>in 5 dimensions</th>
<th>(X_A \text{ in 4 dimensions} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_A | P_a(= \hat{D}<em>a) | Q^i | M</em>{ab} | b_\mu | U_{ij} | S^i | K_a | G(Z) | )</td>
<td>(e_\mu^a | \psi_\mu | \omega_{\mu}^{ab} | D | V_{ij} | \phi_\mu | f_{\mu}^a | W_\mu | )</td>
</tr>
</tbody>
</table>

**Appendix B**

Curvatures \(\hat{R}_{\mu \nu}^A\) are defined by \(\hat{R}_{\mu \nu}^A X_A \equiv [\hat{D}_a, \hat{D}_b]\) in terms of full superconformal covariant derivative \(\hat{D}_a \equiv \partial_a - \sum_{A \neq P} h_A^A X_A\) and are written by means of the structure function in the form

\[
\hat{R}_{\mu \nu}^A = \epsilon^b_P e^a_P f_{ab}^A = 2\partial_{[\mu} h_{\nu]}^A + h_C^A h_B^P f_{BC}^A .
\]

(B.1)

Here \(X_A\) and \(h^A_\mu\) denote the transformation operators and the corresponding gauge fields, respectively, whose explicit contents in 5D and 4D are listed in Table V. \(f_{AB}^C\) is a ‘structure function’, defined by \([X_A, X_B] = f_{AB}^C X_C\), which depends on the fields generally. The primed \(f_{BC}^A\) is zero when \(B = \bar{P}_b\) and \(C = P_c\), and otherwise \(f_{BC}^A = f_{BC}^A\). The explicit expression of (B.1) for the curvature \(\hat{R}_{\mu \nu}^A\) can most easily be read off from the gauge field transformation law \(\partial(\delta \varepsilon) h_\mu^A\) of the same generator \(X_A\):

\[
\varepsilon^B X_B h_\mu^A \equiv \delta(\varepsilon) h_\mu^A = \partial_\mu \varepsilon^A + \varepsilon^C h_\mu^B f_{BC}^A .
\]

(B.2)

That is, we can obtain \(\hat{R}_{\mu \nu}^A\) simply by replacing \(\varepsilon^B\) by \(h_\nu^C\) in \(\delta(\varepsilon) h_\mu^A\).

The explicit forms of the curvatures \(\hat{R}_{\mu \nu}^A\) in 4D are given by \(11), 13)\)

\[
\hat{R}_{\mu \nu}^a(P) = 2\partial_{[\mu} e_{\nu]}^a - 2\omega_{[\mu}^b e_{\nu]}^b + 2b_{[\mu} e_{\nu]}^a + 2\bar{\psi}_\mu^a \gamma_\nu^a \psi_\nu,
\]

\[
\hat{R}_{\mu \nu}^{ab}(M) = 2\partial_{[\mu} \omega_{\nu]}^{ab} - 2\omega_{[\mu}^c \omega_{\nu]}^b c + 8f_{[\mu}^a e_{\nu]}^b - 4\bar{\psi}_{[\mu}^a \gamma_\nu^{ab} \phi_\nu + 4i\bar{\psi}_{[\mu}^a \gamma_\nu \psi_\nu \hat{R}_{\nu}^{ab}(Q),
\]

\[
\hat{R}_{\mu \nu}(D) = 2\partial_{[\mu} b_{\nu]} + 4f_{[\mu}^a e_{\nu]}^a + 4\bar{\psi}_{[\mu} \gamma_\nu \phi_\nu,
\]

\[
\hat{R}_{\mu \nu}(A) = 2\partial_{[\mu} A_{\nu]} - 8i\bar{\psi}_{[\mu} \gamma_5 \phi_\nu,
\]

\[
\hat{R}_{\mu \nu}(K) = 2\partial_{[\mu} f_{\nu]} - 2\omega_{[\mu}^{ab} f_{\nu]}^b - 2b_{[\mu} f_{\nu]}^a + 2i\bar{\psi}_{[\mu} \gamma_\nu \phi_\nu + i\bar{\psi}_{[\mu} \gamma_\nu \psi_\nu \hat{D}_{\nu} \hat{R}_{\nu}^{ab}(Q),
\]

\[
\hat{R}_{\mu \nu}(Q) = 2\partial_{[\mu} \psi_{\nu]} - \frac{1}{2} \omega_{[\mu}^{ab} \gamma_{ab} \psi_{\nu]} + b_{[\mu} \psi_{\nu]} - \frac{3}{2}i A_{[\mu} \gamma_5 \psi_{\nu]} + 2i e_{[\mu} \gamma_\phi \phi_{\nu]},
\]

\[
\hat{R}_{\mu \nu}(S) = 2\partial_{[\mu} \phi_{\nu]} - \frac{1}{2} \omega_{[\mu}^{ab} \gamma_{ab} \phi_{\nu]} - b_{[\mu} \phi_{\nu]} + \frac{3}{2} i A_{[\mu} \gamma_5 \phi_{\nu]} + 2i f_{[\mu} \gamma_\alpha \phi_{\nu]} + \frac{i}{2} \gamma^a \left(\hat{R}_{[\mu a}(A) + i\gamma_5 \hat{R}_{[\mu a}(A)\right) \psi_{\nu]},
\]
\[ \hat{F}_{\mu
u}(B^g) = 2\partial_{[\mu}B^g_{\nu]} - g[B^g_{\mu}, B^g_{\nu}] + 2i\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^g. \]  

(B.3)

By the help of Bianchi identities \([\hat{D}_a, [\hat{D}_b, \hat{D}_c]] = 0\), one can show that the constraints (3.2) imply the following useful equalities:

\[
\begin{align*}
\hat{R}_{ab}(D) &= -\hat{R}_{[ab]}(M) = -\frac{1}{2}\hat{R}_{ab}(A), \\
\hat{R}_{ab}(Q) &\equiv \frac{1}{2}\epsilon_{abcd}\hat{R}^{cd}(Q) = -i\gamma_5\hat{R}_{ab}(Q), \\
\hat{D}_{[a}\hat{R}_{bc]}(Q) &= -i\gamma_{[a}\hat{R}_{bc]}(S), \\
\hat{D}^{b}\hat{R}_{ab}(Q) &= -i\gamma^{b}\hat{R}_{ab}(S), \\
\gamma\cdot\hat{R}(S) &= 0, \\
i\gamma_5\hat{R}_{ab}(S) &\equiv i\frac{2}{2}\epsilon_{abcd}\gamma_5\hat{R}^{cd}(S) = \hat{R}_{ab}(S) - i\hat{D}\hat{R}_{ab}(Q),
\end{align*}
\]

(B.4)

where \(\hat{R}_{a}^{b}(M) \equiv \hat{R}_{ac}^{cb}(M)\) and \(\hat{R}_{ab} \equiv (1/2)\epsilon_{abcd}\hat{R}^{cd}\).

The curvatures \(\hat{R}_{\mu\nu}^{A}\) in 5D are given explicitly by \(7, 6\):

\[
\begin{align*}
\hat{R}_{\mu\nu}^{a}(P) &= 2\partial_{[\mu}e_{\nu]}^{a} - 2\omega_{[\mu}^{ab}e_{\nu]}^{b} + 2b_{[\mu}e_{\nu]}^{a} + 2i\bar{\psi}_{[\mu}\gamma^{a}\psi_{\nu]}, \\
\hat{R}_{\mu\nu}^{i}(Q) &= 2\partial_{[\mu}\psi_{\nu]}^{i} - \frac{1}{2}\omega_{[\mu}^{ab}\gamma_{\nu]}\psi_{\nu]}^{i} + b_{[\mu}\psi_{\nu]}^{i} - 2V^{i}_{[\mu}V^{j}_{\nu]} + \gamma_{\nu][a}\psi_{\nu]}^{a} + 2\gamma_{[\mu}\phi_{\nu]^{i}}, \\
\hat{R}_{\mu\nu}^{ab}(M) &= 2\partial_{\mu}\omega_{\nu]^{acb}} - 2\omega_{\mu}^{a}\epsilon_{\nu]\cd} + 4i\bar{\psi}_{[\mu}\gamma_{[\nu}\phi_{\nu]}^{i} - 2i\bar{\psi}_{[\mu}\gamma_{[\nu}\phi_{\nu]}^{i}V^{cd} + \
+ 4i\bar{\psi}_{[\mu}\gamma_{[\nu}\phi_{\nu]}^{i}V^{cd} + 4f_{[\mu}[a\bar{e}_{\nu]}^{b]}, \\
\hat{R}_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} + 4i\bar{\psi}_{[\mu}\phi_{\nu]} + 4f_{[\mu}], \\
\hat{R}_{\mu\nu}^{ij}(U) &= 2\partial_{[\mu}V_{\nu]}^{ij} - 2V_{[\mu}^{k}V_{\nu]}^{kj} + 12i\bar{\psi}_{[\mu}\gamma^{i}\psi_{\nu]} - 8i\bar{\psi}_{[\mu}\gamma_{[\nu}\phi_{\nu]}^{i} + \frac{i}{2}\bar{\psi}_{[\mu}\gamma_{[\nu}\lambda^{j]}, \\
\hat{R}_{\mu\nu}^{i}(S) &= 2\partial_{\mu}\phi_{\nu]}^{i} - \frac{1}{2}\omega_{[\mu}^{ab}\gamma_{\nu]}\phi_{\nu]}^{i} - b_{[\mu}\phi_{\nu]}^{i} - 2V^{i}_{[\mu}\phi_{\nu]}^{j} - 2f_{[\mu}\gamma_{\nu]}^{i} + \cdots, \\
\hat{R}_{\mu\nu}^{a}(K) &= 2\partial_{\mu}f_{\nu]}^{a} - 2\omega_{[\mu}^{ab}f_{\nu]}^{b} - 2b_{[\mu}f_{\nu]}^{a} + 2i\bar{\phi}_{[\mu}\gamma_{\nu]}^{i} \cdots, \\
\hat{F}_{\mu\nu}(W) &= 2\partial_{\mu}W_{\nu] - g[W_{\mu}, W_{\nu]} + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}^{i}O - 2i\bar{\psi}_{[\mu}\gamma_{\nu]}^{i}M. 
\end{align*}
\]

(B.5)

Here the dots in the \(S^a\) and \(K^a\) curvature expressions denote terms containing the other curvatures. The constraints (2.2) in 5D also imply the following equalities, and the \(S^a\) and \(K^a\) curvatures turn to be written in terms of the other curvatures:

\[
\begin{align*}
\hat{R}_{ab}(D) &= -\frac{2}{3}\hat{R}_{ab}(M) = 0, \\
\hat{R}_{[abc]d}(M) &= \hat{R}_{a[bc]d]}(M) = 0, \\
\hat{R}_{ab}^{i}(S) &= \hat{D}\hat{R}_{ab}^{i}(Q) + \gamma_{[a}\hat{D}^{c}\hat{R}_{bc]}^{i}(Q) + \hat{R}_{[c}^{i}(Q)\psi_{bc}^{i} \\
&\quad + \frac{1}{4}\gamma\cdot\nu\hat{R}_{ab}^{i}(Q) + \frac{1}{12}\gamma_{ab}\hat{R}_{cd}^{i}(Q)\nu^{cd}, \\
\hat{R}_{ab}^{c}(K) &= \frac{1}{4}\hat{D}_{d}\hat{R}_{ab}^{cd}(M) + \frac{1}{2}\hat{R}_{d[a}(Q)\gamma_{c}\hat{R}_{d]}^{b}(Q) + \frac{1}{2}\hat{R}_{d[a}(Q)\gamma_{b]}\hat{R}_{d]c}(Q). 
\end{align*}
\]

(B.6)
Appendix C

Embedding and Invariant Action Formulas in 4D

A product of chiral multiplets also forms a chiral multiplet. More generally, for an arbitrary set of chiral multiplets \( \Sigma^I = [A^I, \mathcal{P}_R^I, \mathcal{F}^I] \), we can have a new chiral multiplet \( g(\Sigma) \) whose first component is given by a general function \( g(A) \) of \( \{A^I\} \) carrying a homogeneous degree in the Weyl-weight:\(^{11}\)

\[
g(\Sigma) = [g(A), \mathcal{P}_R^I g_I(A), \mathcal{F}^I g_I(A) - \frac{1}{4} \chi^I \mathcal{P}_R^I g_{IJ}(A)], \tag{C.1}
\]

where \( g_I(A) \equiv \partial g(A) / \partial A^I \) and \( g_{IJ}(A) \equiv \partial^2 g(A) / \partial A^I \partial A^J \).

Similarly, for an arbitrary set of general multiplets \( \Phi^I = [C^I, \zeta^I, \cdots] \) and an arbitrary function \( f(C) \) homogeneous in the Weyl weight, we can have a new general multiplet \( \Phi' = f(\Phi) \) whose components are given by\(^{12}\)

\[
C' = f(C), \quad \zeta' = \zeta^I f_I, \\
H' = H^I f_I - \frac{1}{4} \tilde{\zeta}^I \zeta^J f_{IJ}, \quad K' = K^I f_I + \frac{1}{4} \tilde{\zeta}^I \gamma_5 \zeta^J f_{IJ}, \\
B'_a = B_a^I f_I + \frac{1}{4} \tilde{\zeta}^I \gamma_a \gamma_5 \zeta^J f_{IJ}, \\
\lambda' = \lambda^I f_I - \frac{1}{2} i \gamma_5 (i \gamma_5 H + K + i B + \hat{\Phi} C \gamma_5)^I \zeta^J f_{IJ} - \frac{1}{4} \tilde{\zeta}^I \tilde{\zeta}^J \zeta^K f_{IJK}, \\
D' = D^I f_I + \frac{1}{2} (H^I H^j + K^I K^j + B_a^I B_a^j + \hat{\Phi}_a C^I \hat{\Phi}_a C^j + i \tilde{\zeta}^I \tilde{\Phi} \zeta^J - 2 \tilde{\zeta}^I \lambda^j) f_{IJK} \\
+ \frac{1}{4} \tilde{\zeta}^I i \gamma_5 (i \gamma_5 H + K + i B)^j \zeta^K f_{IJK} + \frac{1}{16} \zeta^I \tilde{\zeta}^J \zeta^K \zeta^L f_{IJKL}, \tag{C.2}
\]

with \( f_I \equiv \partial f(C) / \partial C^I \), etc. This formula is also valid for complex general multiplets, provided that the spinor conjugate \( \bar{\psi} \) is understood to be \( \psi^T C_4 \) but not \( \psi^T \gamma^0 \).

For a chiral multiplet \( \Sigma^{(w=3)} = [A, \mathcal{P}_R \chi, \mathcal{F}] \) with weight \( w = n = 3 \), we have the following superconformal-invariant \( F \)-term action formula:\(^9\)

\[
I_F = \int d^4 x \left[ \Sigma^{(w=3)} \right]_F = \int d^4 x \left[ \mathcal{F} - i \bar{\psi} \cdot \gamma \mathcal{P}_R \chi - 2 \bar{\psi}_a \gamma^{ab} \mathcal{P}_L \psi_b A + \text{h.c.} \right]. \tag{C.3}
\]

For a real general multiplet \( \Phi^{(w=2, n=0)} = [C, \zeta, H, K, B_a, \lambda, D] \) with weight \( w = 2, n = 0 \), the invariant \( D \)-term action formula is given by\(^{11}\)

\[
I_D = \int d^4 x \left[ \Phi^{(w=2, n=0)} \right]_D \\
= \int d^4 x \left[ D - \bar{\psi} \cdot \gamma_5 \lambda + i e^{abcd} \bar{\psi}_a \gamma_b \psi_c (B_d - \bar{\psi}_d \zeta) \\
+ \frac{1}{3} (R(\omega) + 4 i \bar{\psi}_a \gamma^{\mu \nu} D_\mu \psi_\lambda) C + \frac{2}{3} i \zeta_5 \gamma^{\mu \nu} D_\mu \psi_\lambda \right], \tag{C.4}
\]

where \( R(\omega) \) is the scalar curvature constructed from the spin connection \( \omega^{\mu \nu} \) and \( D_\mu \) is the covariant derivative with respect to the homogeneous transformations \( M_{ab}, D, \) and \( A \).
References

25) M. Günaydin, G. Sierra and P.K. Townsend, in Ref. 18)