Scalar-tensor gravity and conformal continuations

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Global properties of vacuum static, spherically symmetric configurations are studied in various theories of scalar-tensor gravity (STT) in arbitrary dimensions. The conformal mapping between the Jordan and Einstein frames is used as a tool. Necessary and sufficient conditions are found for the existence of solutions admitting a conformal continuation (CC). The latter means that a singularity in the Einstein-frame manifold maps to a regular surface $S_{\text{trans}}$ in the Jordan frame, and the solution is then continued beyond this surface. $S_{\text{trans}}$ can be an ordinary regular sphere or a horizon; it is found that in the second case $S_{\text{trans}}$ connects two epochs of a Kantowski-Sachs type cosmology. It is shown that the list of possible types of global causal structure of vacuum space-times in any STT, with any potential function $U(\phi)$, is the same as in general relativity with a cosmological constant. This is even true for conformally continued solutions. A traversable wormhole is shown to be one of the generic structures created as a result of CC. Two explicit examples are presented: the known solution for a conformal field in general relativity, illustrating the emergence of singularities and wormholes due to CC, and a nonsingular 3-dimensional model with an infinite sequence of CCs.

1. Introduction

Scalar fields with various potentials are of great significance in various branches of theoretical physics and cosmology: it is sufficient to mention, e.g., the Higgs field in particle theory and numerous quintessence models in modern cosmology. It is thus highly desirable to know which kinds of gravitationally self-bound configurations can be formed by such fields.

This paper continues the study of global properties of static, spherically symmetric scalar-vacuum configurations of arbitrary dimension in various theories of gravity begun in Refs. [1–3]. We will here consider scalar-tensor theories (STT) belonging to the Bergmann-Wagoner-Nordtvedt family, where the Lagrangian depends on two essential arbitrary functions of the scalar field. It can be mentioned that STT are among the viable alternatives to general relativity (GR), and their different versions emerge in the field limits of the candidate “theories of everything”.

The field equations of an arbitrary STT are reduced by a conformal mapping to the equations of GR with a scalar field possessing a certain potential (the so-called Einstein frame). This paper will pay special attention to the properties of such mappings. The point is that, when a manifold $M[g]$ is conformally mapped to another manifold $\overline{M}[\overline{g}]$ (relating the metrics by $g_{\mu\nu} = F(x)\overline{g}_{\mu\nu}$), the global properties of both manifolds are the same as long as the conformal factor $F$ is everywhere smooth and finite. It can happen, however, that a singular surface in $\overline{M}$ maps to a regular surface $S_{\text{trans}}$ in $M$ due to a singularity in the conformal factor $F$. Then $M$ can be continued in a regular manner through this surface, and the global properties of $\overline{M}$ can be considerably richer than those of $M$: in the new region one can possibly find, e.g., new horizons or another spatial infinity. A known example of this phenomenon, to be called conformal continuation, is provided by the properties of the static, spherically symmetric solution for a conformally coupled scalar field in GR [4, 5] as compared with the corresponding solution for a minimally coupled scalar field — see Sec. 6.

The Einstein-frame action for vacuum configurations in STT reads

$$S_E = \int d^D x \sqrt{g} [R_E + (\partial\psi)^2 - 2V(\psi)],$$

i.e., coincides with the action of GR with a minimally coupled real scalar field $\psi$ possessing a potential $V(\psi)$.

The field equations due to (1) with nontrivial potentials can be integrated explicitly in very few cases, even for highly symmetric configurations such as cosmological or static, spherically symmetric ones. Nevertheless, rather much can be said about the nature of the solutions. Examples of such general statements for nonnegative potentials $V$ are the no-hair theorems [6] discarding nontrivial scalar field for asymptotically flat black holes and the generalized Rosen theorem [7] claiming that an asymptotically flat solution with a positive mass cannot have a regular centre.

It is also of interest what can happen if the asymptotic flatness and/or $V \geq 0$ assumptions are abandoned. Both assumptions are frequently violated in modern studies. Negative potential energy densities, in particular, the cosmological constant $\Lambda < 0$ giving rise to the anti–de Sitter ($\text{AdS}$) solution or $\text{AdS}$ asymptotic, do not lead to catastrophes (if bound below), are often treated in various aspects and quite readily appear from quantum effects like vacuum polarization.

Our previous papers [1, 2] have provided some essential restrictions on the possible behaviour of solutions of the theory (1) with arbitrary $V(\psi)$ in $D$ dimensions. It has been shown, in particular, that, whatever

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is the potential and irrespective of the asymptotic conditions, the variable scalar field adds nothing to the list of causal structures known for $\psi = \text{const}$. In the latter case $V$ becomes a cosmological constant, and the corresponding exact solutions are well known (Schwarzschild, Schwarzschild-de Sitter, Schwarzschild-anti-de Sitter and their multidimensional analogues) along with their causal structures.

The possibility of regular configurations without a centre (wormholes and horns) was also ruled out.

As was shown in [1, 2], the above results can be extended to (i) generalized scalar field Lagrangians in GR, with an arbitrary dependence on the $\psi$ field and its gradient squared, and (ii) to multiscalar field theories of sigma-model type in GR. To scalar-tensor theories, as is clear from the aforesaid, the same results can be extended only partly and only in the absence of conformal continuation (CC). This phenomenon is of particular interest since it widens the set of possible configurations. A study of possible CCs in the Jordan frames of STT was begun in [3] and is continued here more systematically and in more detail. Moreover, we will discuss the global properties of conformally continued space-times.

We will here avoid a detailed discussion of which conformal frame (Jordan or Einstein) in STT should be regarded as the physical one, referring to the paper [8] and references therein. Only one comment is in order: when an STT emerges in a weak-field or low-energy limit and references therein. Only one comment is in order: when an STT emerges in a weak-field or low-energy limit of some more fundamental theory, its Lagrangian generally contains the scalar curvature with a $\phi$-dependent coefficient before $\phi$, thus leading to one of numerous possible Jordan frames (e.g., the string metric in models of string origin). So, by origin, it is this formulation of the theory that should be used for studying such fundamental issues as topology, singularities, causal properties, etc., although a comparison with observations may require a different formulation.

The paper is organized as follows. Sec. 2 presents the field equations. Sec. 3 reviews the known results on scalar vacuum structures in GR and configurations described by generic STT solutions. We begin with a brief description of purely vacuum structures in $D$-dimensional GR with a cosmological constant and then reproduce the no-go theorems of Refs. [1, 2] on the properties of scalar vacuum in GR and mention some other known theorems and examples. Two theorems, providing the necessary and sufficient conditions under which a given STT contains a CC, are formulated and proved in Sec. 4. Sec. 5 discusses the global properties of Jordan-frame space-times in the present of CCs. It turns out, in particular, that even the presence of CCs does not enlarge the number of possible horizons and hence the above list of global causal structures. It is shown that one of generic structures created by CCs is a traversable wormhole. The whole space-time is then globally regular and static. Some particular kinds of singularities can also be created beyond a CC surface. Sec. 6 contains two explicit examples of STT solutions with CCs. One of them represents the well-known solution for a conformally coupled scalar field in GR, which, in addition to singular cases, contains a family of traversable wormhole solutions [5, 9]. The other is a nonsingular model containing an infinite sequence of CCs in 3-dimensional gravity with a conformally coupled scalar field having a certain nonnegative potential.

To sum up, with all theorems and examples at hand, we now have, even without solving the field equations, rather a clear picture of what can and what cannot be expected from static scalar-vacuum configurations in a general class of STT of gravity with various scalar field potentials.

Throughout the paper all relevant functions are assumed to be sufficiently smooth, unless otherwise indicated. The symbol $\sim$, as usual, connects quantities of the same order of magnitude. The ends of theorem proofs are marked with $\square$.

2. Field equations

The general STT action in a $D$-dimensional pseudo-Riemannian manifold $M_J[g]$ is

$$ S_{\text{STT}} = \int d^Dx \sqrt{|g|} f(\phi) R + h(\phi)(\partial\phi)^2 - 2U(\phi) + L_m, $$

where $g_{\mu\nu}$ is the metric, $R = R[g]$ is the scalar curvature, $g = |\det g_{\mu\nu}|$, $f$, $h$ and $U$ are functions of the real scalar field $\phi$, $(\partial\phi)^2 = g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi$, and $L_m$ is the matter Lagrangian. The manifold $M_J[g]$ with the metric $g_{\mu\nu}$ comprises the so-called Jordan conformal frame. The vacuum ($L_m = 0$) field equations due to (2) read

$$ \nabla_{\alpha}(h\nabla^\alpha)\phi - \frac{1}{2} R f_{\phi} = -\frac{dU}{d\phi}, $$

$$ f(\phi)(R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R) = h(\phi)(-\phi_{,\mu}\phi_{,\nu} + \frac{1}{2}\delta^\nu_\mu \phi\phi_{,\alpha}) $$

$$ -\delta^\nu_\mu U(\phi) + (\nabla_\mu\nabla_\nu - \delta^\nu_\mu \Box)f_{\mu}(m), $$

where $\Box = \nabla^\alpha\nabla_\alpha$ is the d’Alembert operator, $R^\nu_\mu$ is the Ricci tensor, and the last term in (4) is the energy-momentum tensor of matter. (The usual constant factor $8\pi G$, where $G$ is the gravitational constant, can be restored by proper re-definition of the variables).

The standard transition to the Einstein frame, which generalizes Wagoner’s [10] 4-dimensional transformation,

$$ g_{\mu\nu} = F(\psi)\overline{g}_{\mu\nu}, \quad F = |f|^{-2/(D-2)}, $$

$$ \frac{d\psi}{d\phi} = \pm \sqrt{\frac{\overline{|f(\phi)|}}{f(\phi)}}, \quad l(\phi) \overset{\text{def}}{=} fh + \frac{D-1}{D-2}(\frac{df}{d\phi})^2, $$

removes the nonminimal scalar-tensor coupling expressed in the $\phi$-dependent coefficient before $R$. Putting $L_m = 0$ (vacuum), one can write the action (2) in the new manifold $M_E[\overline{g}]$ with the new metric $\overline{g}_{\mu\nu}$ and the
new scalar field $\psi$ as follows (up to a boundary term):

$$S_E = \int d^D x \sqrt{\mathcal{g}} \left( \text{sign} \, f [\mathcal{R} + (\text{sign} \, l)(\partial \psi)^2] - 2V(\psi) \right),$$

(7)

where the determinant $\mathcal{g}$, the scalar curvature $\mathcal{R}$ and $(\partial \psi)^2$ are calculated using $\mathcal{g}_{\mu\nu}$ and

$$V(\psi) = |f|^{-D/(D-2)}(\psi) U(\phi).$$

(8)

Note that sign $l = -1$ corresponds to the so-called anomalous STT, with a wrong sign of scalar field kinetic energy, while sign $f = -1$ means that the effective gravitational constant in the Jordan frame (which can be defined as $1/f$ up to a constant factor) is negative. So the normal choice of signs is sign $l = \text{sign} \, f = 1$, when the scalar-vacuum action takes the form (1). We will adhere to theories with $l > 0$ in the whole paper, but we shall see that the continuations to be discussed in Sec. 4–6 lead to $f < 0$ in some regions of $M_J$.

Among the three functions of $\phi$ entering into (2), only two are independent since there is a freedom of transformations $\phi = \phi(\phi_{\text{new}})$. We assume $h \geq 0$ and use this freedom, choosing in what follows $^2 h(\phi) = 1$.

From the viewpoint of the field equations, the transformation (5), (6) is merely a simplifying substitution. Instead of Eqs. (3) and (4) (assuming $f > 0$), we deal in $M_E$ with simpler equations due to (1):

$$\Box \psi + dV/d\psi = 0,$$

(9)

$$\mathcal{R}_{\mu} - \frac{1}{2} \delta_{\mu}^{\nu} \mathcal{R} = - \psi_{,\mu} \psi_{,\nu} + \frac{1}{2} \psi_{,\mu} \psi_{,\nu} (\partial \psi)^2 - \delta_{\mu}^{\nu} V(\psi).$$

(10)

with the Ricci tensor $\mathcal{R}_{\mu}$ and the d’Alembert operator $\Box$ corresponding to $\mathcal{g}_{\mu\nu}$.

Consider static, spherically symmetric configurations, so that the metric in $M_E$ is written as

$$ds_{2}^{2} = A(\rho) dt^{2} - \frac{d\rho^{2}}{A(\rho)} - r^{2}(\rho) d\Omega^{2},$$

(11)

where $d\Omega^{2}$ is the linear element on the sphere $S^{2}$ of unit radius, and the scalar field is $\psi = \psi(\rho)$.

Then Eq. (9) and some combinations of the Einstein equations (10) have the form

$$A r^{2} \psi'' = r^{2} V \psi;$$

(12)

$$A' r^{2} \psi' = - (4/\mathcal{d}) r^{2} V;$$

(13)

$$\mathcal{R}' / r = - \psi^{2};$$

(14)

$$A r^{2} \psi'' - r^{2} A' + (\mathcal{d} - 2) r' \left( 2 A r - A' r \right) = 2(\mathcal{d} - 1);$$

(15)

$$\frac{\mathcal{d} (\mathcal{d} - 1) (1 - A r^{2})}{2} - \mathcal{A} r r' = - A r^{2} \psi' / 2 + 2 r^{2} V,$$

(16)

where the prime denotes $d/d\rho$. Only three of these five equations are independent: the scalar equation (12) follows from the Einstein equations, while Eq. (16) is a first integral of the others. Given a potential $V(\psi)$, this is a determined set of equations for the unknowns $r$, $A$, $\psi$.

This choice of the radial coordinate according to the condition $\mathcal{g}_{tt} \mathcal{g}_{\rho\rho} = -1$ is preferable for considering Killing horizons, which correspond to zeros of the function $A(\rho)$, since such zeros are regular points of Eqs. (12)–(16), and therefore one can jointly consider regions at both sides of a horizon; moreover, in a close neighbourhood of a horizon, the coordinate $\rho$ defined in this way varies (up to a positive constant factor) like manifestly well-behaved Kruskal-like coordinates used for an analytic continuation of the metric [11]. Therefore this coordinate frame can be called quasiglobal.

The corresponding metric in $M_J$ reads

$$ds_{3}^{2} = F(\phi) \left[ A(\rho) dt^{2} - \frac{d\rho^{2}}{A(\rho)} - R^{2} d\Omega_{2}^{2} \right]$$

$$= A(q) dt^{2} - \frac{d\rho^{2}}{A(q)} - R^{2} d\Omega_{2}^{2},$$

(17)

where we have introduced the quasiglobal coordinate $q$ in $M_J$, similar to $\rho$ in (11), such that $g_{tt} g_{\rho\rho} = -1$. The quantities in (17) and (11) are related by

$$\pm dq = F d\rho, \quad A(q) = FA(\rho), \quad R(q) = \sqrt{Fr}(\rho).$$

(18)

With our convention $h(\phi) \equiv 1$, three independent field equations in $M_J$ can be written as follows:

$$f \left( A_{qq} + 2 A_{\rho} R_{\rho} \right) + \frac{D}{d} A_{\rho} f_{\rho} + 2 A R_{\rho} f_{\rho}$$

$$+ \frac{2}{d} A f_{qq} + \frac{4}{d} U = 0,$$

(19)

$$7 d f_{\rho q} R_{\rho q} + \delta_{\rho}^{2} f_{qq} = 0,$$

(20)

$$\frac{f}{R^{2}} \left[ - (\mathcal{d} - 1) + A \mathcal{R} f_{qq} + A R_{\rho}^{2} + \frac{1}{2} (\mathcal{d} - 2) R_{\rho} (2 A \mathcal{R} - R A_{\rho}) \right]$$

$$+ \left( \frac{A R_{\rho}^{2}}{R} - \frac{A_{\rho}}{2} \right) f_{\rho} = 0.$$  

(21)

where the subscript $q$ denotes $d/dq$.

3. Properties of generic STT solutions

3.1. Some known results for the Einstein frame

Let us enumerate some consequences of Eqs. (12)–(16) valid in $M_E$.

The first important restriction is the nonexistence of regular configurations having no centre ($r = 0$), namely, wormholes, horns and flux tubes [1, 2].

For the metric (11), a (traversable, Lorentzian) wormhole is, by definition, a configuration with two
asymptotics at which \( r(\rho) \to \infty \), hence with \( r(\rho) \) having at least one regular minimum. A horn is a region where, as \( r \) tends to some finite value, \( \frac{\partial}{\partial \rho} A \) remains finite whereas the length integral \( l = \int dp/\sqrt{A} \) diverges. In other words, a horn is a configuration ending with a regular, infinitely long \( (d+1) \)-dimensional “tube” of finite radius. Such “horned particles” were discussed as possible remnants of black hole evaporation [12]. Lastly, a flux tube is a configuration with \( r = \text{const} \), a “cylindrical” space.

**Theorem 1.** The field equations due to (1) for \( D \geq 4 \) do not admit (i) solutions where the function \( r(\rho) \) has a regular minimum, (ii) solutions describing a horn, and (iii) flux-tube solutions with \( \psi \neq \text{const} \).

The formulation of the theorem and its proof [1, 2], which essentially rests on Eq. (14), do not refer to any kind of asymptotic, therefore wormhole throats or horns are absent in solutions having any large \( r \) behaviour — flat, de Sitter or any other, or having no large \( r \) asymptotic at all.

For \( D = 3 \) items (i) and (iii) of Theorem 1 hold, but solutions with a horn can exist; though, a horn can only appear at a maximum of \( r(\rho) \), so that horned configurations have no spatial asymptotic.

The global causal structure of space-time is unambiguously determined (up to identification of isometric surfaces, if any) by the disposition of static \( (A > 0) \) and nonstatic, homogeneous \( (A < 0) \) regions, separated by horizons [13]–[16]. The following two theorems severely restrict such possible dispositions.

**Theorem 2.** Consider solutions of the theory (1), \( D \geq 4 \), with the metric (11) and \( \psi = \psi(\rho) \). Let there be a static region \( a < \rho < b \leq \infty \). Then:

(i) all horizons are simple;

(ii) no horizons exist at \( \rho < a \) and at \( \rho > b \).

**Theorem 2a.** A static, circularly symmetric configuration in the theory (1), \( D = 3 \), has either no horizon or one simple horizon.

The proof of these theorems [1, 2] employs the properties of Eq. (15), which can be rewritten in the form

\[
rg^2B'' + (d + 2)rg' B' = -2(d - 1)
\]  
(22)

where \( B(\rho) = A/\rho^2 \). This equation shows that \( B \) cannot have a regular minimum, therefore, having once become negative while moving to the left or to the right along the \( \rho \) axis, \( B(\rho) \) (and hence \( A(\rho) \)) cannot return to zero or positive values.

Theorems 2 and 2a show that the possible disposition of zeros of the function \( A(\rho) \) is the same as in the case of vacuum with a cosmological constant. Therefore the list of possible global causal structures is also the same.

Let us, for reference purposes, enumerate these structures. The metric satisfying Eqs. (13)–(16) with \( \psi' = 0, \ V = \Lambda = \text{const} \) is

\[
ds^2 = A(r)dt^2 - \frac{dr^2}{A(r)} - r^2 d\Omega_d^2
\]  
(23)

where \( A(r) \) is defined by

\[
A(r) = 1 - \frac{2m}{r^{d-1}} - \frac{2\Lambda r^2}{d(d+1)}
\]  
(24)

This is the multidimensional Schwarzschild-de Sitter solution. Its special cases correspond to the Schwarzschild \( (d = 2, \ \Lambda = 0) \) and Tangherlini (any \( d, \ \Lambda = 0) \) solutions and the de Sitter solution in arbitrary dimension when \( m = 0 \), called anti-de Sitter (AdS) in case \( \Lambda < 0 \). Different qualitative behaviours of \( A(r) \) for different values of \( \Lambda \) and \( m \) correspond to the following structures [17]:

1. \( \Lambda = 0, m \leq 0 \): curves 1a and 1b in Fig. 1, diagram 1 in Fig. 2 (Minkowski and \( m < 0 \) Schwarzschild, respectively).
2. \( \Lambda < 0, m \leq 0 \): curves 2a and 2b in Fig. 1, diagram 2 in Fig. 2 (AdS and \( m < 0 \) Schwarzschild-AdS).
3. \( \Lambda < 0, m > 0 \): curve 3 in Fig. 1, diagram 3 in Fig. 2 (Schwarzschild-AdS).
4. \( \Lambda = 0, m > 0 \): curve 4 in Fig. 1, diagram 4 in Fig. 2 (Schwarzschild).
5. \( \Lambda > 0, m \leq 0 \): curves 5a and 5b in Fig. 1, diagram 5 in Fig. 2 (de Sitter and \( m < 0 \) Schwarzschild-de Sitter).
6. \( \Lambda > 0, m > 0 \): curves 6a, 6b and 6c in Fig. 1, and the corresponding diagrams in Fig. 2 (Schwarzschild-de Sitter in case 6a and Kantowski-Sachs homogeneous cosmologies in cases 6b and 6c).
The centre } r = 0 \text{ is regular for } m = 0 \text{ and singular for } m \neq 0.

In case 6, given a particular value of } \Lambda > 0 \text{, the solution behaviour depends on the mass parameter } m.

When } m \text{ is smaller than the critical value

\begin{equation}
m_{\text{cr}} = \frac{1}{d+1} \left[ \frac{d(d-1)}{2\Lambda} \right]^{(d-1)/2},
\end{equation}

there are two horizons, the smaller one being interpreted as a black hole horizon and the greater one as a cosmological horizon. If } m = m_{\text{cr}}, \text{ the two horizons merge, and one has two homogeneous } T \text{ regions separated by a double horizon. Lastly, the solution with } m > m_{\text{cr}} \text{ is purely cosmological and has no Killing horizon.}

In (2+1)-dimensional gravity (} d = 1\text{), according to Theorem 2a, the list is even shorter: the structures corresponding to the curves 6a and 6b are absent.

Let us also mention, for completeness, some results known for } D = 4 \text{ and most probably admitting a generalization to other dimensions.

**No-hair theorems** state that (1) if } V \geq 0 \text{, an asymptotically flat black hole cannot have a nontrivial scalar field [6, 18, 19]; (2) if } V \geq 0 \text{ and } d^2V/d\psi^2 \geq 0 \text{ (a convex potential), an asymptotically anti-de Sitter black hole cannot have a nontrivial scalar field [20].

**The generalized Rosen theorem** states that, provided } V \geq 0 \text{, a particlelike solution with a regular centre, a flat asymptotic and positive mass does not exist [7].

These theorems cannot be directly extended to STT in the Jordan frame and will not be discussed any more, though an attempt to formulate additional conditions able to provide such extensions may be of interest.

**Explicit examples** have been obtained, confirming the existence of some kinds of solutions admitted by the above theorems. Thus, there exist: (1) black holes possessing nontrivial scalar fields (scalar hair), with } V \geq 0 \text{, but with non-flat and non-de Sitter asymptotics [21]; (2) black holes with scalar hair and flat asymptotics, but partly negative potentials [2]; (3) configurations with a regular centre, a flat asymptotic and positive mass, but also with partly negative potentials [2].

Thus black holes with scalar hair are not excluded in general, but such objects as regular black holes [22], possessing a regular centre and a global structure coinciding with that of Reissner-Nordstrom or Reissner-Nordstrom-de Sitter space-time, are ruled out.

### 3.2. Generic solutions in the Jordan frame

It should be, above all, noted that when a space-time manifold } M_{\text{E}} [7] \text{ (the Einstein frame) with the metric (11) is conformally mapped into another manifold } M_{\text{J}}[g] \text{ (the Jordan frame), equipped with the same coordinates, according to the law

\begin{equation}
g_{\mu\nu} = F(\rho)g_{\mu\nu}',
\end{equation}

it is easily verified that a horizon } \rho = h \text{ in } M_{\text{J}} \text{ passes into a horizon of the same order in } M_{\text{E}}, \text{ a centre } (r = 0) \text{ and an asymptotic } (r \to \infty) \text{ in } M_{\text{J}} \text{ pass into a centre and an asymptotic, respectively, in } M_{\text{E}} \text{ if the conformal factor } F(\rho) \text{ is regular (i.e., finite, at least } C^2\text{-smooth and positive) at the corresponding values of } \rho. \text{ A regular centre passes to a regular centre and a flat asymptotic to a flat asymptotic under evident additional requirements.

The validity of Theorems 1, 2 and 2a in the Jordan frame depends on the nature of the conformal mapping (5) that connects } M_{\text{J}}[g] \text{ with } M_{\text{E}} [7]). \text{ There are four variants:

I. } M_{\text{J}} \longleftrightarrow M_{\text{E}},

II. } M_{\text{J}} \longleftrightarrow (M_{\text{J}}' \subset M_{\text{E}}),

III. } (M_{\text{J}}' \subset M_{\text{J}}) \longleftrightarrow M_{\text{E}},

IV. } (M_{\text{J}}' \subset M_{\text{J}}) \longleftrightarrow (M_{\text{E}}' \subset M_{\text{E}}),

where } \longleftrightarrow \text{ is a diffeomorphism preserving the metric signature. The last three variants are possible if the

Figure 2: Carter-Penrose diagrams for different cases of the metric (23), (24), labelled according to Fig. 1. The R and T letters correspond to R and T space-time regions; } T_+ \text{ and } T_- \text{ denote expanding and contracting } T \text{ region (i.e., with } r \text{ increasing and decreasing with time, respectively). Single lines on the border of the diagrams denote } r = 0, \text{ double lines } -\infty < r < \infty. \text{ Diagrams 6b and 6c are drawn for the case of expanding Kantowski-Sachs cosmologies; to obtain diagrams for contracting models, one should merely interchange } r = 0 \text{ and } r = \infty \text{ and replace } T_+ \text{ with } T_-. \text{ Diagrams 6a and 6b admit identification of isometric timelike sections.
conformal factor $F$ vanishes or blows up at some values of $\rho$, which then mark the boundary of $M_{J'}$ or $M_{E'}$.

A situation of the kind III or IV can be called a conformal continuation (CC) from $M_{E}$ into $M_{J}$.

One can notice that such continuations can only occur for special solutions: to admit a CC, the singularity in $M_{E}$ should be removable by a conformal factor, i.e., be, in a sense, isotropic. Moreover, the factor $F$ should have precisely the behaviour needed to remove it.

Thus generic situations are I and II, the latter meaning that the factor $F$ “spoils” the geometry and creates a singularity. In these cases Theorem 2 (or 2a for $D = 3$) on horizon dispositions is obviously valid in $M_{J}$.

The manifolds $M_{J}$ then cannot have other causal structures than those depicted in Fig. 2. This is manifestly true for STT with $f(\phi) > 0$.

Theorem 1 cannot be directly transferred to $M_{J}$ in any case except the trivial one, $F = \text{const}$. In particular, minima of $g_{\theta \theta}$ (wormhole throats) can appear. It is only possible to assert without specifying $F(\psi)$ that wormholes as global entities are impossible in $M_{J}$ in case I if the conformal factor $F$ is finite in the whole range of $\rho$, including the boundary values. Indeed, if we suppose that there is such a wormhole, it will immediately follow that there are two large $r$ asymptotics and a minimum of $r(\rho)$ between them even in $M_{E}$, in contrast to Theorem 1 which is valid there. Wormholes are also absent in case II since we then have a singularity instead of at least one of the asymptotics.

The above-mentioned examples of black holes with scalar hair when the potential is not positive-definite or the asymptotic is non-flat are also directly transferred to $M_{J}$ provided $F(\psi)$ is regular at least outside the horizon. Given a particlelike solution in $M_{E}$, with a regular centre and positive mass, the condition that a solution with similar properties occurs in $M_{J}$ can also be easily formulated, but we will not concentrate on this question here.

Conformal continuations, if any, can in principle lead to other, maybe more complex structures. In what follows we will try to answer two questions: (1) under which conditions the mapping (26) creates a conformal continuation in STT and (2) what can be the nature of conformally continued solutions in the Jordan frame.

4. Conformal continuation conditions

4.1. Preliminaries

A CC from $M_{E}$ into $M_{J}$ can occur at such values of the scalar field $\phi$ that the conformal factor $F$ in the mapping (5) is singular while the functions $f$, $h$ and $U$ in the action (2) are regular. This means that at $\phi = \phi_{0}$, corresponding to a possible transition surface $S_{\text{trans}}$, the function $f(\phi)$ has a zero of a certain order $n$. We then have in the transformation (6) near $\phi = \phi_{0}$ in the leading order of magnitude

$$f(\phi) \sim \Delta \phi^{n}, \quad n = 1, 2, \ldots, \quad \Delta \phi \equiv \phi - \phi_{0}. \quad (27)$$

One can notice, however, that $n > 1$ leads to $l(\phi_{0}) = 0$ (recall that by our convention $h(\phi) \equiv 1$). This generically leads to a curvature singularity in $M_{J}$, as can be seen from the trace of Eqs. (4):

$$l(\phi)R = -\left(1 + 2\frac{D-1}{D-2}f_{\phi\phi}\phi_{\alpha}\phi_{\alpha}ight)\phi^\alpha\phi_{\alpha} + \frac{2}{D-2} \left[DU + (D-1)f_{\phi}U_{\phi} \right]. \quad (28)$$

(the subscript $\phi$ denotes $d/d\phi$). If the right-hand side of (28) is nonzero at $\phi = \phi_{0}$ at which $l = 0$, the scalar curvature $R$ is infinite. There can be special choices of $f$ and $U$ such that this singularity is avoided, but we will ignore this possibility and simply assume $l > 0$ at $S_{\text{trans}}$.

Thus, according to (6), we have near $S_{\text{trans}}(\phi = \phi_{0})$:

$$f(\phi) \sim \Delta \phi \sim e^{-\psi \sqrt{d/(d+1)}}, \quad (29)$$

where without loss of generality we choose the sign of $\psi$ so that $\psi \to \infty$ as $\Delta \phi \to 0$.

In the CC case, the metric $\mathcal{g}_{\mu\nu}$ specified by (11) is singular on $S_{\text{trans}}$ while $g_{\mu\nu} = F(\psi)\mathcal{g}_{\mu\nu}$ is regular. There are two opportunities. The first one, to be called CC-I for short, is that $S_{\text{trans}}$ is an ordinary regular surface in $M_{J}$, where both $\mathcal{g}_{tt} = A = FA$ and $-g_{\theta \theta} = R^{2} = Fr^{2}$ (squared radius of $S_{\text{trans}}$) are finite. (Here $\theta$ is one of the angles that parametrize the sphere $S^{d}$.) The second variant, to be called CC-II, is that $S_{\text{trans}}$ is a horizon in $M_{J}$. In the latter case only $g_{\theta \theta}$ is finite, while $g_{tt} = 0$. We will consider these two kinds of CC separately.

In both cases some necessary conditions for CC are easily obtained using the field equations in $M_{E}$, but these equations describe the system only on one side of $S_{\text{trans}}$. Another Einstein frame may be built for the region beyond $S_{\text{trans}}$, but in this case there arises the problem of matching the solutions obtained in two non-intersecting regions. Therefore to prove sufficient conditions for the existence of solutions in $M_{J}$ which are regular on and near $S_{\text{trans}}$ we have to deal with the field equations in $M_{J}$.

4.2. Continuation through an ordinary sphere (CC-I)

Given a metric $\mathcal{g}_{\mu\nu}$ of the form (11) in $M_{E}$, a CC-I can occur if

$$F(\psi) = |f|^{-2/7} \sim 1/r^{2} \sim 1/A \quad (30)$$

as $\psi \to \infty$, while the behaviour of $f$ is specified by (29). The surface $S_{\text{trans}}$, being regular in the Jordan frame, is singular in $M_{E}$ ($r^{2} \sim A \to 0$): it is either a singular centre if the continuation occurs in an R-region, or a cosmological singularity in the case of a T-region.

The following theorem is valid:

Theorem 3. Consider scalar-vacuum configurations with the metric (17) and $\phi = \phi(q)$ in the theory (2)

\begin{align*}
\frac{dU}{d\phi} & = 0, \\
\mathcal{E}(\phi) & = 0, \\
\mathcal{J}(\phi) & = 0.
\end{align*}
with \( h(\phi) \equiv 1 \) and \( l(\phi) > 0 \). Suppose that \( f(\phi) \) has a simple zero at some \( \phi = \phi_0 \), and \( |U(\phi_0)| < \infty \). Then:

(i) there exists a solution in \( \mathbb{M}_1 \), smooth in a neighbourhood of the surface \( S_{\text{trans}}(\phi = \phi_0) \), which is an ordinary regular surface in \( \mathbb{M}_1 \); (ii) in this solution the ranges of \( \phi \) are different on different sides of \( S_{\text{trans}} \).

**Proof.** Let us begin with item (ii): given a CC-I, we will show that \( d\phi/dq \neq 0 \) at \( \phi = \phi_0 \), so that \( \phi_0 \) is not a maximum or minimum of \( \phi(q) \).

Indeed, it can be deduced from the conditions (29) and (30) and Eq. (14) that near \( S_{\text{trans}} \)

\[
\tau(\rho) \sim (\rho - \rho_0)^{1/D}
\]

where \( \rho = \rho_0 \) is the location of \( S_{\text{trans}} \). It then follows that \( q_0 = q(\rho_0) \) is finite on \( S_{\text{trans}} \) and both \( \Delta \phi \) and \( q - q_0 \) behave as \( r^d \) in its neighbourhood, hence \( d\phi/dq \) is finite.

With this necessary condition, we can prove item (i), seeking a solution to Eqs. (19)–(21) in an appropriate form. Here, the unknowns are \( \mathcal{A}(q), R(q), \phi(q) \), while \( f(\phi) \) and \( U(\phi) \) are prescribed by the choice of the theory. However, since \( f(\phi_0) = 0 \) and \( df/d\phi \neq 0 \) at \( \phi = \phi_0 \), we can treat \( \phi(f) \) as a known function in a certain neighbourhood of \( f = 0 \) and consider \( f(q) \) as an unknown instead of \( \phi(q) \).

Let \( S_{\text{trans}} \) be located (without loss of generality) at \( q = 0 \). It is sufficient to find a solution in the form of a power series in \( q \) near \( q = 0 \). Since \( R \) and \( A \) should be finite at \( S_{\text{trans}} \), while \( f \) and \( df/d\phi \neq 0 \), we seek a solution in the form

\[
\begin{align*}
\mathcal{A}(q) &= \sum_{n=0}^{\infty} A_n q^n/n! = A_0 + A_1 q + \frac{1}{2} A_2 q^2 + \ldots, \\
R(q) &= \sum_{n=0}^{\infty} R_n q^n/n! = R_0 + R_1 q + \frac{1}{2} R_2 q^2 + \ldots, \\
f(q) &= \sum_{n=1}^{\infty} f_n q^n/n! = f_1 q + \frac{1}{2} f_2 q^2 + \ldots
\end{align*}
\]

with nonzero \( A_0, R_0, f_1 \).

Substituting (32) into the equations, we see that in the senior order of magnitude, \( O(1) \), the coefficients \( A_1, R_1, f_2 \) are expressed in terms of \( A_0, R_0, f_1 \) and \( U(\phi_0) \). Next powers of \( q \) express the further expansion factors in terms of the previous ones. Namely, in every order of magnitude \( O(q^n) \), \( n > 0 \), Eq. (20) gives

\[
(n-1)dR_n/R_0 + f_{n+1}/f_1 = \ldots,
\]

where the dots on the right mean various combinations of coefficients of the previous orders as well as power expansion factors of the known functions \( U(\phi) \) and \( \phi(f) \).

Then, excluding \( f_{n+1} \) from the other two equations in the order \( O(q^n) \), we obtain a set of two linear algebraic equations for \( A_n/A_0 \) and \( R_n/R_0 \):

\[
\begin{align*}
A_n/A_0 - 2R_n/R_0 &= \ldots, \\
(n + 2)A_n/A_0 - 2d(n - 2)R_n/R_0 &= \ldots
\end{align*}
\]

whose determinant is equal to \( 4(d+1) \) for any \( n \). We conclude that all the expansion factors in (32) are uniquely expressed in terms of \( A_0, R_0, f_1 \) and the expansion factors of the known functions. This proves the existence of the solution in \( \mathbb{M}_1 \) near \( S_{\text{trans}} \).

The order of smoothness of the solution obtained depends on the smoothness of the original functions \( f, h, U \). If they are \( C^\infty \), as is natural for a field theory, then the metric functions and \( \phi \) are also \( C^\infty \).

The existence of such a solution automatically implies the existence of the corresponding solutions on different sides of \( S_{\text{trans}} \) in two different Einstein frames. These solutions are special, being restricted by Eq. (30).

As follows from the proof, the solution in \( \mathbb{M}_1 \), and hence its counterparts in both \( \mathbb{M}_E \), contain two essential integration constants \( (R_0 \text{ and } f_1, \text{ whereas } A_0 \text{ determines } \) the time scale on \( S_{\text{trans}} \) and can be chosen arbitrarily).

It is of interest that, under the CC-I conditions, the potential \( V(\psi) \) in \( \mathbb{M}_E \) (although it may even blow up) is inessential: the solution is close to Fisher’s scalar-vacuum solution [23] for \( D = 4 \) or its modification in other dimensions.

In case \( D = 3 \), as follows from Eq. (15), a necessary condition for CC is \( A f/2r^2 = \) const.

One can also notice that no restriction other than regularity is imposed on the potential \( U,^3 \text{ in particular, } U \) may vanish in some region or in the whole space. The latter case will be used as an explicit example of CC in Sec. 6.1.

### 4.3. Continuation through a horizon in \( \mathbb{M}_1 \) (CC-II)

Let us suppose that in the metric (17) a certain value of \( q \) (without loss of generality, \( q = 0 \)) corresponds to a horizon of order \( k \geq 1 \). This means that \( q = 0 \) is a zero of order \( k \) of the function \( \mathcal{A}(q) \).

Suppose now that this horizon is \( S_{\text{trans}} \), a transition sphere in a CC. In other words, in the vicinity of \( q = 0, f(\phi) \sim \Delta \phi \). One can directly verify that in the corresponding value \( \rho_0 \) of the coordinate \( \rho \) in the Einstein-frame metric (11) is finite, and we can choose for convenience \( \rho_0 = 0 \). We thus have near \( q = 0 \)

\[
\mathcal{A}(q) = AF \sim q^k, \quad R^2(q) = FR^2 = O(1).
\]

Now, the question is: under which requirements to the original theory a horizon in \( \mathbb{M}_1 \), described by (35), can be a transition sphere \( S_{\text{trans}} \). An answer is given by the following theorem.

**Theorem 4.** Consider scalar-vacuum configurations with the metric (17) and \( \phi = \phi(q) \) in the theory (2) with \( h(\phi) \equiv 1 \) and \( l(\phi) > 0 \). Suppose that \( f(\phi) \) has a simple zero at some \( \phi = \phi_0 \). There exists a solution

---

3The conclusion that \( U(\phi) = 0 \) on \( S_{\text{trans}} \), obtained in [3], appeared there due to an additional assumption on the form of the expansion of \( A(\rho) \) in powers of \( \rho \). An inspection shows that this assumption is unnecessary.
in $M_1$, smooth in a neighbourhood of the surface $S_{\text{trans}}$ ($\phi = \phi_0$), which is a Killing horizon in $M_1$, if and only if:

(a) $D \geq 4$,
(b) $\phi_0$ is a simple zero of $U(\phi)$,
(c) $dU/df > 0$ at $\phi = \phi_0$.

Then, in addition,

(d) $S_{\text{trans}}$ is a second-order horizon, connecting two T-regions in $M_1$;
(e) the ranges of $\phi$ are different on the two sides of $S_{\text{trans}}$.

**Proof.** Necessity. Given a CC-II, we will prove items (a)–(e).

Let us use Eqs. (13)–(15) in $M_E$. In particular, Eq. (15), which contains only $A(\rho)$ and $r(\rho)$, can be rewritten in the form

$$[r^D(A/r^2)']' + 2(\overline{A} - 1)r^{D-2} = 0. \quad \text{(36)}$$

Suppose CC-II at $\rho = 0$ ($q = 0$) which is a horizon of order $k$ in $M_2$. Let us put $\overline{A} > 1$ and assume for certainty, without generality loss, that $q > 0$ as $\rho \to +0$. We have $dq/d\rho \sim F \sim 1/r^2$ and $A/r^2 \sim q^k$. Therefore the first term in (36) at small $\rho$ behaves as

$$(\text{sign } A)[r^{D-1}q^{k-1}].$$

Since both $r(\rho)$ and $q$ are growing functions of $\rho$, this derivative is nonnegative, whereas the second term in (36) is manifestly positive. The only way to satisfy (36) is to put $A < 0$. In other words, the horizon is approached from a T-region, where we deal with a Kantowski-Sachs type cosmological model.

Such a reasoning applies to approaching the surface $q = 0$ from either side, therefore the horizon $S_{\text{trans}}$ connects two T-regions and is of even order.

We must also ascertain that the orders of magnitude of the two terms in (36) are the same. This is only true if $k = 2$, as can be easily verified using (31), which now reads $r \sim \rho^{D/2}$. So item (d) is proved.

Eqs. (29) and (31) can be used to show that the derivative $d\phi/d\rho$ is finite at $q = 0$, leading to item (e).

The behaviour of $U(\phi)$ can be determined using Eq. (13), taking into account the relation (8) between $U$ and $V$ and the conditions (35) with $k = 2$. We find in this way that $U(\phi) \sim D\phi$ (i.e., the potential has a first-order zero) and that $dU/d\rho > 0$ at $\phi = \phi_0$, so items (b) and (c) hold.

It remains to rule out $D = 1$ (3-dimensional gravity). In this case (36) leads to $(A/r^2) = c_1/r^3$, $c_1 = \text{const.}$ The value $c_1 = 0$ is excluded since we must have $A/r^2 \sim q^k$. For $c_1 \neq 0$ we obtain $(q^k)' = kq^{k-1}dq/d\rho$, whence due to (35) $q^{k-1} \sim 1/r \to \infty$, whereas $q \to 0$ and $k \geq 1$. This contradiction proves item (a).

**Sufficiency.** As in Theorem 3, the existence of a solution in $M_E$, smoothly continued across $S_{\text{trans}}$, is proved using Eqs. (19)–(21) and a power expansion for a sought-for solution. It is again convenient to treat $\phi(\rho)$ as a known function and $f(q)$ as unknown, so again the unknowns are $A(q)$, $R(q)$ and $f(q)$.

Seeking solutions to Eqs. (19)–(21) as series in $q$, we use again the expansions (32), but, under the present necessary conditions, we put there $A_0 = A_1 = 0$ and suppose nonzero $A_2$, $R_0$, $f_1$. With these expansions substituted, the equations again lead to a chain of recurrent relations for the coefficients, slightly more involved than in Theorem 3.

In the orders $O(1)$ and $O(q)$, Eq. (19) leads to

$$U(\phi_0) = 0, \quad (\overline{A} + D)A_2 + 4U_{f0} = 0$$

(where $U_{f0} = dU/df\big|_{f=0}$), hence $U_{f0} > 0$, in agreement with the necessary conditions. Eq. (20) [$O(1)$] expresses $f_2$ in terms of $f_1$ and $d\phi/df(f = 0)$. Eq. (21) [$O(1)$] gives $A_2 = -2 \overline{(A - 1)}/R_0^2$. Furthermore, Eqs. (19) [$O(q^2)$] and (21) [$O(q^2)$] yield two equations determining $R_1$ and $A_3$:

$$(3\overline{A} + 2)A_3/A_2 + 2\overline{A}(\overline{A} + 1)R_1/R_0 = \ldots ,
3A_3/A_2 + 2(\overline{A} - 1)R_1/R_0 = \ldots , \quad \text{(37)}$$

where the dots on the right replace combinations of previously known constants. The determinant of this set of linear equations with respect to $A_3/A_2$ and $R_1/R_0$ is $-4(2\overline{A} + 1) < 0$ for $\overline{A} > 0$. We thus know the coefficients up to $R_1$, $f_2$, $A_3$. Further coefficients are determined recursively from further orders of magnitude in all three equations:

$$(n + D)\overline{A}a_n + n(n - 1)(\overline{A} + 1)b_n + (n - 1)(n + D)c_n/D = \ldots ,
(n + 2)\overline{A}b_n + c_n = \ldots ,
(n + 1)a_n + n(n^2 - 3\overline{A} - \overline{A} + 9)b_n + (n - 1)c_n = \ldots , \quad \text{(38)}$$

where $n \geq 3$ and

$$a_n = A_{n+1}/A_2, \quad b_n = R_{n-1}/R_0, \quad c_n = f_n/f_1.$$ The determinant of Eqs. (38) with respect to $a_n$, $b_n$, $c_n$ is a multiple (with a nonzero coefficient) of

$$n[\overline{A}(2n^2 - 2n - 7) + n^2 - 9]. \quad \text{(39)}$$

This quantity is nonzero for all values of $\overline{A}$ and $n$ of interest: it is negative for $n = 3$ and positive for $n > 3$. Therefore Eqs. (38) can be consecutively solved for all $n$, leading to a $C^\infty$ solution to the field equations (19)–(21). The proof is completed. \(\square\)

Thus the only kind of STT configurations admitting CC-II is a $D \geq 4$ Kantowski-Sachs cosmology consisting of two T-regions (in fact, epochs, since $\rho$ is a temporal coordinate), separated by a second-order horizon. The qualitative behaviour of the metric function $A(q)$ is shown by the curve 6b in Fig. 1, and the Carter-Penrose diagram is 6b in Fig. 2. We shall see in the next section
5. Global properties of conformally continued solutions

5.1. Horizon dispositions

A solution to the STT equations may a priori undergo a number of CCs, so that each region of \( M_J \) between adjacent surfaces \( S_{\text{trans}} \) is conformally equivalent to some \( M_E \). However, the global properties of \( M_J \) with CCs turn out to be not so diverse as one might expect. The main restriction is that Theorems 2 and 2a on horizon configurations in STT is exhausted by the list presented in Sec. 3 for systems with a cosmological constant in \( D \)- dimensional GR.

There certainly can be a single simple horizon, as in the Schwarzschild or de Sitter space-times, or no horizons at all. Theorem 5 means that, precisely as in GR, the list of possible causal structures of scalar-vacuum configurations in STT is exhausted by the list presented in Sec. 3 for systems with a cosmological constant in \( D \)-dimensional GR.

The situation is still simpler for \( D = 3 \): in this case a CC (more precisely, CC-I) is only possible under the condition \( B = const \neq 0 \), which excludes horizons. A single horizon can exist, but there is then no CC.

5.2. Multiple CCs, singularities and wormholes

A full classification of STT solution behaviours is beyond the scope of this paper; we will instead outline some new features appearing in STT formulated in \( M_J \) as compared with GR or with the Einstein-frame formulation of STT, in particular, in connection with conformal continuations. For simplicity, we will use the STT parametrization such that \( h = 1 \).

A singularity in \( M_J \) can emerge due to the behaviour of the conformal factor \( F = |f|^{-2/d} \) at points where \( M_E \) is regular. The most evident case is that \( f \to \infty \) at some value of \( \phi \), then \( F \to 0 \), and we obtain both \( A = g_{tt} \to 0 \) and \( R^2 = g_{th} \to 0 \).

Two kinds of singularities can appear in an “antigravitational” region where \( f < 0 \), that is, beyond a CC surface. The first one occurs if \( \phi \) blows up while \( \psi \) is finite. Using the relation (6) between \( \phi \) and \( \psi \), one easily finds that this is only possible when, at large \( |\phi| \),

\[
 f(\phi) \approx -\frac{D-2}{4(D-1)} \phi^2; \tag{43}
\]

i.e., when the \( \phi \) field is asymptotically conformally coupled to the curvature. The singularity is then again connected with \( F \to 0 \) that leads to zero \( A(\phi) \) and \( R(\phi) \).

Another kind of singularity is more generic and occurs where

\[
 l(\phi) = f + \frac{D-1}{D-2} \left( \frac{df}{d\phi} \right)^2 \to 0. \tag{44}
\]
Recall that we adhere to the assumption \( l > 0 \) in the whole paper. In the case (44), the conformal factor \( F \) is finite provided \( f \neq 0 \) but its derivatives generically blow up:
\[
\frac{dF}{du} = -\frac{2}{d\sqrt{l(\phi)}} \frac{df}{d\phi} \frac{d\psi}{du}
\]
\((u\) is any admissible coordinate in \( M_E \)), which can only be finite if \( df/d\phi \to 0 \) at the same \( u \). So, a value of \( \phi \) where \( l = 0 \) is generically a singular sphere of finite radius. Recall also Eq. (28) showing that, again generically, the scalar curvature \( R \) is infinite where \( l = 0 \). Special solutions where such a sphere is regular are not completely excluded but are not considered here.

Under the assumption \( l > 0 \), there cannot be more than two values of \( \phi \) where CCs are possible, i.e., those where \( f = 0 \) and \( df/d\phi \neq 0 \); if they do exist, one has \( f > 0 \) between them. Indeed, if the function \( f(\phi) < 0 \) between two zeros, it has to pass a minimum where \( df/d\phi = 0 \), hence \( l < 0 \), contrary to our assumption.

This does not mean, however, that an STT solution cannot contain more than two CCs. The point is that \( \phi \) as a function of the radial coordinate is not necessarily monotonic, so there can be two or more CCs corresponding to the same value of \( \phi \); see the second example in Sec. 6. But there can be no more than one CC-II due to Theorem 5: other CCs in the same solution, if any, belong to type CC-I and occur in a T-region.

Though, multiple CCs can appear in rather special (if not artificial) situations since the very existence of a CC imposes restrictions on the solution parameters, such as (30) for CC-I. A transition surface \( S_{\text{trans}} \in M_J \) corresponds to a singularity \( r = 0 \) in \( M_E \), therefore an Einstein-frame manifold \( M_E \), describing a region between two transitions, should contain "two centres", more precisely, two values of the radial coordinate (say, \( \rho \)) at which \( r(\rho) = 0 \). This property, resembling that of a closed cosmological model, is quite generic due to \( r'' \leq 0 \) in Eq. (14), but a special feature is that the conditions (30) should hold at both centres.

Another generic (and more usual) behaviour of \( M_E \) is that \( r \) varies from zero to infinity. Let there be a family of such static solutions and an STT with \( f(\phi) \) having a simple zero. Then, by Theorem 3, there is a subfamily of solutions admitting CC-I. A particular solution from this subfamily can come beyond \( S_{\text{trans}} \) either to one of the above-mentioned types of singularities, or, if "everything is quiet", to another spatial asymptotic and will then describe a static, traversable wormhole. Each of the asymptotics can be either flat (if \( U(\phi) \to 0 \), or anti-de Sitter (if \( U(\phi) \to \text{const} < 0 \)). Thus wormholes are among generic structures that emerge due to conformal continuations.

6. Examples

We will present two explicit examples of configurations with CC-I. The first example is well known and is given here to illustrate the generic character of wormholes appearing due to CC. The second one is a 3-dimensional periodic structure with an infinite sequence of CCs.

6.1. Conformal scalar field in GR

Conformal scalar field in GR can be viewed as a special case of STT, such that, in Eq. (2), \( D = 4 \) and
\[
f(\phi) = 1 - \phi^2/6, \quad h(\phi) = 1, \quad U(\phi) = 0. \quad (45)
\]
After the transformation (5) \( g_{\mu\nu} = F(\psi) \tilde{g}_{\mu\nu} \) with
\[
\phi = \sqrt{6} \tanh(\psi + \psi_0)/\sqrt{6}, \quad \psi_0 = \text{const},
\]
\[
F(\psi) = \cosh^2[(\psi + \psi_0)/\sqrt{6}], \quad (46)
\]
we obtain the action (1) with \( D = 4 \) and \( V \equiv 0 \), describing a minimally coupled massless scalar field in GR. The corresponding static, spherically symmetric solution is well known: it is the Fisher solution [23]. In terms of the harmonic radial coordinate \( u \in \mathbb{R}_+ \), specified by the condition \( g_{uu} = -g_{u}(\theta\phi)^2 \), the solution is [5]
\[
ds_E^2 = e^{-2m_0} dt^2 - k^2 e^{2m_0} \frac{k^2 du^2}{\sinh^2(ku)} + d\Omega^2,
\]
\[
\psi = C u, \quad (47)
\]
where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \), \( m \) (the mass), \( C \) (the scalar charge), \( k > 0 \) and \( u_0 \) are integration constants, and \( k \) is expressed in terms of \( m \) and \( C \):
\[
k^2 = m^2 + C^2/2. \quad (48)
\]
In case \( C = 0 \), \( k = m \) we recover the Schwarzschild solution, as is easily verified using the coordinate \( \rho = 2k/(1 - e^{-2ku}) \). The metric (47) takes the form (11) with \( A(\rho) = (1 - 2k/\rho)^{m/k} \).

Another convenient form of the solution is obtained in isotropic coordinates: with \( y = \tanh(ku/2) \), Eqs. (47) are converted to
\[
ds_E^2 = A(y) dy^2 - \frac{k^2(1 - y)^2}{y^4 A(y)} (dy^2 + y^2 d\Omega^2),
\]
\[
A(y) = \left| \frac{1 - y}{1 + y} \right|^{2m/k}; \quad \psi = \frac{C}{k} \ln \left| \frac{1 + y}{1 - y} \right|. \quad (49)
\]
The solution is asymptotically flat at \( u \to 0 \) (\( y \to 0 \)), has no horizon when \( C \neq 0 \) (as should be the case according to the no-hair theorem) and is singular at the centre (\( u \to \infty, \ y \to 1 - 0, \ \psi \to \infty \)).

A feature of importance is the invariance of (49) under the inversion \( y \to 1/y \), noticed probably for the first time by Mitskevich [24]. Due to this invariance, the solution (49) considered in the range \( y > 1 \) describe quite a similar configuration, but now \( y \to \infty \) is a flat asymptotic and \( y \to 1 + 0 \) is a singular centre. An attempt to unify the two ranges of \( y \) is meaningless due to the singularity at \( y = 1 \). We shall see that such a unification, leading to a wormhole, is achieved when the singularity...
is smoothed out in \( M_J \) in case \( C = \sqrt{6}m \) due to the conformal factor.

The Jordan-frame solution for (45) is described by the metric 
\[ ds^2 = F(\psi)ds^2_E, \]
and the \( \phi \) field according to (46). It is the conformal scalar field solution [4, 25], its properties are more diverse and can be described as follows (putting, for definiteness, \( m > 0 \) and \( C > 0 \)):

1. \( C < \sqrt{6}m \). The metric behaves qualitatively as in the Fisher solution: it is flat at \( y \to 0 \) \((u \to 0)\), and both \( y_0u \) and \( r^2 = |g_{00}| \) vanish at \( y \to 1 \) \((u \to \infty)\) — a singular attracting centre. A difference is that here the scalar field is finite: \( \phi \to \sqrt{6} \).

2. \( C > \sqrt{6}m \). Instead of a singular centre, at \( y \to 1 \) \((u \to \infty)\) one has a repulsive singularity of infinite radius: \( y_0u \to \infty \) and \( r^2 \to \infty \). Again \( \phi \to \sqrt{6} \).

3. \( C = \sqrt{6}m, \ k = 2m \). Now the metric and \( \phi \) are regular at \( y = 1 \); this is \( S_{\text{trans}} \), and the coordinate \( y \) provides a continuation. The solution acquires the form
\[
 ds^2 = \frac{(1+yy_0)^2}{1-y_0^2} \left[ \frac{dt^2}{1+y^2} - \frac{m^2(1+y)^2}{y^2}(dy^2 + y^2d\Omega^2) \right], \\
 \phi = \sqrt{6} \left( \frac{y + y_0}{1 + yy_0} \right), 
\]
where \( y_0 = \tanh(\psi_0/\sqrt{6}) \). The range \( u \in \mathbb{R}_+ \), describing the whole manifold \( M_E \) in the Fisher solution, corresponds to the range \( 0 < y < 1 \), describing only a region \( M_J' \) of the manifold \( M_J \) of the solution (50). The properties of the latter depend on the sign of \( y_0 \) [5]. In all cases, \( y = 0 \) corresponds to a flat asymptotic, where \( \phi \to \sqrt{6}y_0 \), \( |y_0| < 1 \).

3a: \( y_0 < 0 \). The solution is defined in the range \( 0 < y < 1/|y_0| \). At \( y = 1/|y_0| \), there is a naked attracting central singularity: \( y_0u \to 0 \), \( r^2 \to 0 \), \( \phi \to \infty \). Such singularities have been mentioned in Sec. 5.2 as a characteristic feature of solutions for conformally and asymptotically conformally coupled scalar fields, see Eq. (43).

3b: \( y_0 > 0 \). The solution is defined in the range \( y \in \mathbb{R}_+ \). At \( y \to \infty \), we find another flat spatial infinity, where \( \phi \to \sqrt{6}/y_0 \), \( r^2 \to \infty \) and \( y_0u \) tends to a finite limit. This is a wormhole solution, found for the first time in Ref. [5] and recently discussed by Barcelo and Visser [9].

3c: \( y_0 = 0 \), \( \phi = \sqrt{6}y \), \( y \in \mathbb{R}_+ \). In this case it is helpful to pass to the conventional coordinate \( r \), substituting \( y = m/(r - m) \). The solution
\[
 ds^2 = \frac{(1-m/r)^2}{1-m/r} dr^2 - \frac{r^2}{(1-m/r)^2} d\Omega^2, \\
 \phi = \sqrt{6}m/(r - m) 
\]
is the well-known BH with a conformal scalar field [4, 25]. The infinite value of \( \phi \) at the horizon \( r = m \) does not make the metric singular since, as is easily verified, the energy-momentum tensor remains finite there.

This solution turns out to be unstable under radial perturbations [26].

Case 3 belongs to variant III in the classification of Sec. 3.2, and the whole manifold \( M_J \) can be represented as the union
\[
 M_J = M_J' \cup S_{\text{trans}} \cup M_J'' 
\]
where \( M_J' \) is the region \( y < 1 \), which is, according to (46), in one-to-one correspondence with the manifold \( M_E \) of the Fisher solution (47). The "antigravitational" \( (f(\phi) < 0) \) region \( M_J'' (y > 1) \) is in similar correspondence with another "copy" of the Fisher solution, where, instead of (46),
\[
 \phi = \sqrt{6} \coth(\psi/\sqrt{6}), \quad F(\psi) = \sinh^2(\psi/\sqrt{6}). 
\]
Static wormhole solutions have also been found [9] for more general nonminimally coupled massless scalar fields in GR, represented as STT where
\[
 f(\phi) = 1 - \xi \phi^2, \quad h(\phi) = 1, \quad U(\phi) = 0
\]
with \( \xi = \text{const} > 0 \). In full agreement with the observations of Sec. 5.2, there appear wormholes similar to (50), but in case \( \xi < 1/6 \) some of the conformally continued solutions possess singularities connected with \( l(\phi) = 1 - \xi(1 - 6\xi)\phi^2 \to 0 \). In case \( \xi > 1/6 \) all continued solutions describe wormholes.

All the wormhole solutions mentioned in this section prove to be unstable under radial perturbations [27], which seems to be a common feature of transitions to regions with \( f < 0 \), where the effective gravitational constant becomes negative. This problem deserves further study. A similar instability was pointed out by Starobinsky [28] for cosmological models with conformally coupled scalar fields.

### 6.2. A solution with multiple CCs

Trying to obtain a simple analytical solution, we choose \( D = 3 \) and the functions \( f(\phi) \) and \( h(\phi) \) in the action (2) corresponding to conformal coupling:
\[
 f(\phi) = 1 - \phi^2/8, \quad h(\phi) = 1. 
\]
The \( \phi - \psi \) connection according to (6) (with a proper choice of the integration constant) and the conformal factor \( F(\psi) \) in (5) may be written as
\[
 \phi = \sqrt{8}\tanh(\psi/\sqrt{8}), \\
 F(\psi) = \cosh^2(\psi/\sqrt{8}) \quad \text{for } \phi^2 < 8, \\
 \phi = \sqrt{8} \coth(\psi/\sqrt{8}), \\
 F(\psi) = \sinh^2(\psi/\sqrt{8}) \quad \text{for } \phi^2 > 8. 
\]
A solution in \( M_J \), including regions with \( |\phi| \) larger and smaller than \( \sqrt{8} \), is built from solutions in different \( M_E \) with CC through the surfaces \( S_{\text{trans}} \) on which \( \phi^2 = 8 \). Let us first construct a solution in the Einstein frame, to be put into correspondence any of the regions
As follows from the aforesaid, to provide a CC we must choose a solution to (15) in the form
\[ A(\rho) = c_A r^2(\rho), \quad c_A = \text{const}, \] (58)
where \( c_A > 0 \) and \( c_A < 0 \) corresponds to static and cosmological solutions, respectively.

By (31), near \( S_{\text{trans}} \) \((\rho = \rho_c)\) the function \( r(\rho) \) behaves as \((\rho - \rho_c)^{1/2}\). Accordingly, let us choose this function as follows:
\[ r(\rho) = r_0(1 - x^4)^{1/3}, \quad x = \rho/\rho_0, \]
\[ \rho_0 = \text{const} > 0. \] (59)
Thus a CC-I can occur at \( x = \pm 1 \).

Now, Eq.(14) makes it possible to find \( \psi(\rho) \), or equivalently \( \psi(x) \):
\[ \pm 3 \frac{d\psi}{dx} = 2x\sqrt{9 - x^4} \quad \text{or} \quad \mp 3 \frac{d\psi}{dx} = 2x\sqrt{9 - x^4}. \] (60)
Choosing the plus sign and integrating, we find
\[ \psi(x) = -\frac{\sqrt{2}}{3} \ln(1 - x^2) + \psi_1(x) + \psi_0 \] (61)
where \( \psi_0 \) is an integration constant while the function \( \psi_1(x) \) is analytic and finite for all \( |x| \leq 1 \):
\[ \psi_1(x) = \arcsin(x^2/3) + \sqrt{2} \ln \left( \frac{1 + x^2}{9 + x^2 + \sqrt{8}(9 - x^4)} \right) \] (62)
The potential \( V(\psi) \) as a function of \( x \) is found from (13):
\[ V(\psi) = 2 \frac{cA^2}{\rho_0^3} \frac{x^2}{(1 - x^4)^{1/3}}. \] (63)
This completes the solution in \( \mathcal{M}_E \), specified in the range \(-1 < x < 1\). All the constituent functions are even. The potential \( V(\psi) \) is well-defined due to monotonicity of \( \psi(x) \) in each half-range \( 0 < |x| < 1 \). At possible CCs, \( x = \pm 1, \psi \to \infty \), and the minimum value of \( \psi \) is \( \psi(0) = \psi_0 \); assuming \( \psi_0 > 0 \), we make sure that \( \psi > 0 \) everywhere.

The corresponding solutions in the Jordan frame are different for \( \phi < \sqrt{8} \) and \( \phi > \sqrt{8} \), according to (56) and (57) where we put for certainty \( \phi > 0 \).

The solution in \( \mathcal{M}_J \) for \( \phi < \sqrt{8} \), obtained from \( \mathcal{M}_E \) using (56), occupies in \( \mathcal{M}_J \) a certain region \( M_0 \) parametrized by \( x \in (-1, 1) \). The solution can be continued through the surface, say, \( x = 1 \) to \( \phi > \sqrt{8} \); to do that one can, i.e., consider the metric coefficients as functions of \( \phi \) and analytically continue them beyond the value \( \phi = \sqrt{8} \). However, one cannot do that explicitly since the transcendental equation (61) cannot be resolved with respect to \( x \).

Figure 3: The behaviour of \( \phi(q), \, R(q), \, U(\phi(q)) \) in the model of Sec. 6.2 with multiple CCs, in case \( \psi_0 = \sqrt{8} \), on a segment of the infinite sequence of regions \( M_i \). The latter are separated by vertical lines \( q = q_i \) corresponding to the surfaces \( S_{\text{trans}} \). The potential \( U(\phi) \) is shown in an arbitrary scale.

In the new region \( M_1 \), which can again be parametrized by \( x \in (-1, +1) \), another “copy” of the Einstein-frame solution (58)–(63) is valid. To make sure that this is the same solution as the one used in \( M_0 \), let us consider the transition between them and recall the proof of Theorem 3 (sufficiency). Namely, there is a unique solution in \( \mathcal{M}_J \) near \( S_{\text{trans}} \) (in the form of a power series in \( \phi \)) if the functions \( f(\phi) \) and \( U(\phi) \) and the constants \( A_0, \, R_0 \) and \( f_1 \) are specified. In our case \( f(\phi) \) is given, \( U(\phi) \) is not known explicitly but its existence follows from the existence of the analytic continuation in terms of \( \phi \). So it is sufficient to show that the constants \( A_0, \, R_0 \) and \( f_1 \) calculated as the limiting values of \( A = AF, \, R = rvF \) and \( df/dq \), respectively, from the two solutions, are finite and coincide with each other. A direct inspection shows that this is indeed the case if all the parameters of the solutions are the same, including \( \psi_0 \).

In \( M_1 \) the field \( \phi \) reaches its maximum at \( x = 0 \),
\[ \phi_{\text{max}} = \sqrt{8} \coth(\psi_0/\sqrt{8}), \] (64)
and returns to the value \( \sqrt{8} \) on the other end of the region. One more CC leads to one more region \( M_2 \) with \( \phi < \sqrt{8} \) and so on. The same happens starting from \( x = -1 \) of the initial region \( M_0 \). We obtain an infinite sequence of regions \( M_i, \, i \in \mathbb{Z} \), where adjacent regions are connected by CCs. In regions with even and odd numbers \( i \), one has \( \phi < \sqrt{8} \) and \( \phi > \sqrt{8} \), respectively. Each region \( M_i \) in \( \mathcal{M}_J \) corresponds to its own Einstein-
frame manifold \( M_{i+1} \), described by the solution (58)–(63) with singularities at \( x \to \pm 1 \).

The whole manifold \( M_j \) can be parametrized by a unique “radial” coordinate: it can be, e.g., the quasiglobal coordinate \( q \) used in Theorem 3, or the proper length \( \ell = \int dq/\sqrt{A} \); both quantities take finite values on the transition surfaces from \( M_i \) to \( M_{i+1} \) and change monotonically inside \( M_i \).

The manifold \( M_j \) is thus nonsingular and has the topology \( \mathbb{R} \times \mathbb{R} \times S^1 \): an infinitely long static tube with a periodically changing diameter. This is true if \( c_A > 0 \), when we deal with a static model. One can identify any two \( M_i \) with the numbers \( i \) of equal parity, and this leads to the topology \( \mathbb{R} \times S^1 \times S^1 \), in other words, 2-torus times the time axis. If \( c_A < 0 \), then \( q \) is a temporal coordinate, \( \ell \) becomes proper time, and the solution describes a (2+1)-dimensional cosmology with a periodically and isotropically (since \( A \propto R^2 \)) changing scale factor. The spatial section is \( \mathbb{R} \times S^1 \), but any two points on the \( t \) axis (\( t \) is now spacelike) may be identified, and we obtain a periodically “breathing” 2-torus.

The properties of the model are illustrated in Fig. 3.

Acknowledgements

I am grateful to Vitaly Melnikov and Georgy Shikin for helpful discussions. The work was supported in part by the Russian Foundation for Basic Research and the Russian Ministry of Industry, Science and Technologies.

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