Quantum Communication Complexity of Symmetric Predicates

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Abstract

We completely (that is, up to a logarithmic factor) characterize the bounded-error quantum communication complexity of every predicate \( f(x, y) \) depending only on \(|x \cap y| (x, y \subseteq [n]) \). Namely, for a predicate \( D \) on \{0, 1, \ldots, n\} let \( \ell_0(D) \stackrel{\text{def}}{=} \max \{ \ell \mid 1 \leq \ell \leq n/2 \land D(\ell) \neq D(\ell - 1) \} \) and \( \ell_1(D) \stackrel{\text{def}}{=} \max \{ n - \ell \mid n/2 \leq \ell < n \land D(\ell) \neq D(\ell + 1) \} \). Then the bounded-error quantum communication complexity of \( f_D(x, y) = D(|x \cap y|) \) is equal (again, up to a logarithmic factor) to \( \sqrt{n \ell_0(D) + \ell_1(D)} \). In particular, the complexity of the set disjointness predicate is \( \Omega(\sqrt{n}) \). This result holds both in the model with prior entanglement and without it.

1. Introduction

The model of communication complexity, originally introduced by Yao [Yao79] has since evolved into a very intriguing and important branch of computational complexity that in particular links and unifies many different things. In this model, Alice holds an input \( x \in X \), Bob holds \( y \in Y \), and they exchange messages to evaluate a Boolean predicate \( f : X \times Y \longrightarrow \{0, 1\} \). The complexity is measured by the number of bits exchanged, and, like in many other areas of computational complexity, one distinguishes between deterministic and probabilistic modes.

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Just like the circuit complexity is quite often concerned with symmetric Boolean functions one class of problems that attracted a considerable interest in communication complexity is made by symmetric predicates which we define as those for which $x, y$ are finite sets and $f_D(x, y) = D(|x \cap y|)$ for some predicate $D$ on integers. Two most prominent members of this class are the disjointness predicate $DISJ_n$ ($D(s) \equiv (s = 0)$) and the inner product function $IP_n$ ($D(s) \equiv s \mod 2$). The rank lower bound by Mehlhorn and Schmidt [MS82] immediately implies a tight $\Omega(n)$ lower bound on the deterministic communication complexity of both $DISJ_n$ and $IP_n$.

For the randomized algorithms, [Vaz87, CG88, BFS86] proved an $\Omega(n)$ lower bound on the complexity of the inner product $IP_n$, and [BFS86] also contained an $\Omega(\sqrt{n})$ lower bound for $DISJ_n$. The latter bound was improved to the optimal $\Omega(n)$ in [KS92], and their proof was further simplified in [Raz92].

The model of quantum communication complexity was also introduced by Yao [Yao93]. Suppose that Alice and Bob can employ the laws of quantum mechanics and are allowed to exchange qubits instead of classical bits. Can it help them to reduce the amount of communication?

Buhrman, Cleve and Wigderson [BCW98] observed that the rank lower bound for deterministic protocols extends to the quantum case (so, after all the answer for such protocols can be “NO”). In particular, both $DISJ_n$ and $IP_n$ require $\Omega(n)$ bits to be exchanged by quantum deterministic (= zero-error) protocols. The rank lower bound was extended in [BW01] to the stronger model with prior entanglement previously introduced in [CB97] (in that model, Alice and Bob share an unlimited number of entangled EPR-pairs before the communication even begins).

The question about the complexity of protocols that allow a small error is by far more interesting. As far as lower bounds are concerned, Kremer [Kre95], based upon some ideas from the seminal paper [Yao93], proved an $\Omega(n)$ lower bound for $IP_n$. This result was extended to the model with prior entanglement in [CDNT98]. Klauck [Kla01] looked at the threshold predicates ($D(s) \equiv (s \geq \ell)$) and exact-$\ell$ predicates ($D(s) \equiv (s = \ell)$) and proved an $\Omega(\ell/\log \ell)$ bound in both cases (without entanglement). The only general lower bound for $DISJ_n$ (that corresponds to $\ell = 0$) prior to this work was $\Omega(\log n)$ [AST+98, BW01]; we can also mention some partial results in this direction such as bounds for constant-round protocols [KNTZ01], protocols with exponentially small error [BW01] and some highly structured
protocols [HW02].

On the upper bounds frontier, the elegant paper [BCW98] established a strong connection between quantum search and quantum communication by showing how to convert every quantum search algorithm for any Boolean function \( f_g(x, y) = g(x \cap y) \) with only a logarithmic delay. Plugging into this procedure Grover's search algorithm [Gro96] immediately gave an \( O(\sqrt{n \log n}) \) upper bound on the bounded-error quantum communication complexity of disjointness (that was later slightly improved in [HW02] to \( O(\sqrt{n \exp(\log^* n)}) \)). [BBC+98] proved that the quantum query complexity of every symmetric Boolean function \( g \) is equal, up to a constant factor, to its approximate degree \( \tilde{\deg}(g) \) (defined as the minimal degree of a real polynomial approximating \( g \) on \( \{0, 1\}^n \) in the \( \ell_\infty \)-norm within accuracy 1/3). Combined with the BCW-reduction, this implies an \( O(\tilde{\deg}(g) \log n) \) upper bound on the bounded error quantum communication complexity of \( f_g(x, y) \).

In this paper we prove that for every symmetric predicate \( f_D(x, y) \) this communication algorithm is essentially optimal provided we take care of one "degenerate" case. More specifically, let

\[
\ell_0(D) \overset{\text{def}}{=} \max \{ \ell \mid 1 \leq \ell \leq n/2 \land D(\ell) \neq D(\ell - 1) \} \tag{1}
\]

and

\[
\ell_1(D) \overset{\text{def}}{=} \max \{ n - \ell \mid n/2 \leq \ell < n \land D(\ell) \neq D(\ell + 1) \}. \tag{2}
\]

If we let \( g_D(x_1, \ldots, x_n) = D(|x|) \), then the classical result by Paturi [Pat92] says that \( \tilde{\deg}(g_D) = \Theta(\sqrt{n(\ell_0(D) + \ell_1(D))}) \) which implies, via [BCW98], an upper bound of \( O((\sqrt{n\ell_0(D)} + \sqrt{n\ell_1(D)}) \log n) \) on the quantum bounded-error communication complexity of \( f_D \). This can be easily improved to \( O((\sqrt{n\ell_0(D)} + \ell_1(D)) \log n) \) (large values of \(|x \cap y|\) are taken care of by the trivial algorithm in which Alice sends to Bob her entire input). We prove the lower bound \( \Omega(\sqrt{n\ell_0(D)} + \ell_1(D)) \) matching this upper bound up to a logarithmic factor (Theorem 2.1). Our lower bound works also in the model with prior entanglement.

For the proof of our result we use a multi-dimensional version of the ordinary discrepancy method (Section 5.2). That is, we measure the communication matrix against several probability distributions at the same time.
This allows us to reduce our problem to a classical problem in the discrete polynomial approximation that, quite fortunately, was solved in the above-mentioned paper [Pat92] (Section 5.3). Another specific feature of our approach is that we tend to apply spectral methods (as opposed to combinatorial ones) more systematically than it was done in the previous papers on the subject (this becomes especially critical for handling prior entanglement). In particular, we show a general lower bound on the quantum communication complexity of a function in terms of the approximate trace norm of its communication matrix (Section 5.1).

In the rest of the paper we formulate and prove our main result. Whenever possible, we try to present in reasonable generality those intermediate steps in our proof that might be of independent interest.

2. Quantum communication model and the main result

There are several equivalent definitions of the quantum communication model; in our description we follow [BW01] as this variant seems to be the most convenient to work with.

Let $X, Y$ be finite sets and $f : X \times Y \to \{0, 1\}$ be a Boolean predicate. Let $\mathcal{H}_A, \mathcal{C}, \mathcal{H}_B$ be finite-dimensional Hilbert spaces representing Alice’s part, the channel and Bob’s part, respectively. Like in [BW01] we require that $\mathcal{C}$ consists of a single qubit (that is, $\dim(\mathcal{C}) = 2$, and $|0\rangle, |1\rangle$ is its orthonormal basis).

The models with or without prior entanglement differ only in the unitary vector $\text{Input}(x, y) \in \mathcal{H}_A \otimes \mathcal{C} \otimes \mathcal{H}_B$ prepared at the beginning of the communication. We postpone its definition and describe first how the communication proceeds. A $c$-qubit communication protocol is completely determined by unitary operators $U_1, U_2, \ldots, U_c$, where $U_i$ acts on $\mathcal{H}_A \otimes \mathcal{C}$ if $i$ is odd, and on $\mathcal{C} \otimes \mathcal{H}_B$ if it is even. The output (unitary) vector is then

$$\text{Output}(x, y) \overset{\text{def}}{=} \ldots (U_3 \otimes I_B)(I_A \otimes U_2)(U_1 \otimes I_B)\text{Input}(x, y), \quad (3)$$

where $I_A, I_B$ are identity operators on $\mathcal{H}_A, \mathcal{H}_B$, respectively. The acceptance probability of this protocol on $x, y$ is the result of the measurement of $\text{Output}(x, y)$ with respect to $\mathcal{C}$, i.e., the squared $\ell_2$-norm of its orthogonal projection onto $\mathcal{H}_A \otimes |1\rangle \otimes \mathcal{H}_B$. 

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We are still left to describe $\text{Input}(x, y)$. In the model without prior entanglement, $H_A$ has the orthonormal basis $\{|a, x\rangle \mid a \in W_A, x \in X\}$, where $W_A$ is a finite set with a distinguished element 0 (representing Alice’s internal computations). Likewise, $H_B$ has the orthonormal basis $\{|y, b\rangle \mid b \in W_B, y \in Y\}$ and $\text{Input}(x, y) \overset{\text{def}}{=} |0, x\rangle|0\rangle|y, 0\rangle$.

In the model with prior entanglement, $H_A$ has the basis $\{|a, x, e\rangle \mid a \in W_A, x \in X, e \in E\}$ and $H_B$ has the basis $\{|e, y, b\rangle \mid b \in W_B, y \in Y, e \in E\}$, where $E$ is a new finite set (corresponding to all possible pure states of entangled EPR-pairs). The beginning state in this case is

$$\text{Input}(x, y) = \frac{1}{|E|^{1/2}} \sum_{e \in E} |0, x, e\rangle|0\rangle|e, y, 0\rangle. \tag{4}$$

It is important that in this model we do not have any control of $|E|$ whatsoever, and it must not appear in our bounds.

A quantum protocol computes $f(x, y)$ with error $\epsilon$ if its acceptance probability on every $(x, y)$ is at most $\epsilon$ whenever $f(x, y) = 0$ and at least $1 - \epsilon$ whenever $f(x, y) = 1$. Let $Q_{\epsilon}(f)$ [$Q_{\epsilon}^*(f)$] be the smallest $c$ for which there exists a $c$-qubit communication protocol without [respectively, with] prior entanglement that computes $f$ with error $\epsilon$. Let $Q(f) \overset{\text{def}}{=} Q_{1/3}(f)$ and $Q^*(f) \overset{\text{def}}{=} Q_{1/3}^*(f)$.

Fix an integer $n$, and let $D : \{0, 1, \ldots, n\} \to \{0, 1\}$ be any Boolean predicate. Let $f_{n,D}(x, y) \overset{\text{def}}{=} D(|x \cap y|)$, where $x, y \subseteq [n] \overset{\text{def}}{=} \{1, 2, \ldots, n\}$. Let $\ell_0(D)$ and $\ell_1(D)$ be given by (1), (2) (if no such $\ell$ exists, we naturally let $\ell_{\epsilon}(D) \overset{\text{def}}{=} 0$). The main result of this paper is the following

**Theorem 2.1** For every Boolean predicate $D : \{0, 1, \ldots, n\} \to \{0, 1\}$,

$$\Omega(\sqrt{n\ell_0(D)} + \ell_1(D)) \leq Q^*(f_{n,D}) \leq Q(f_{n,D}) \leq O((\sqrt{n\ell_0(D)} + \ell_1(D)) \log n).$$

Let $\text{DISJ}_n(x, y) \overset{\text{def}}{=} x \cap y = \emptyset$.

**Corollary 2.2** $Q^*(\text{DISJ}_n) \geq \Omega(\sqrt{n})$.

Our lower bound proof essentially uses high symmetry of the predicate $f_{n,D}$ and, in particular, we need $x, y$ to be of the same fixed cardinality
k. We formulate the corresponding intermediate result in this section since, although somewhat technical, it still might be of independent interest.

Let \( k \leq n \) and \( D : \{0,1,\ldots,k\} \rightarrow \{0,1\} \). Let \( X = Y \overset{\text{def}}{=} [n]^k \) be the set of all \( k \)-element subsets of \([n]\) and \( f_{n,k,D} : X \times Y \rightarrow \{0,1\} \); \( f_{n,k,D}(x,y) \overset{\text{def}}{=} D(|x \cap y|) \) (thus, \( f_{n,D} = f_{n,n,D} \)).

**Theorem 2.3** Let \( k \leq n/4, \ell \leq k/4 \) and \( D : \{0,1,\ldots,k\} \rightarrow \{0,1\} \) be any predicate such that \( D(\ell) \neq D(\ell - 1) \). Then \( Q^*(f_{n,k,D}) \geq \Omega(\sqrt{kl}) \).

**Remark 1** Nayak and Shi have observed (personal communication) that our lower bound extends to a more general model in which the entanglement need not necessarily be given in the form of shared EPR-pairs. More specifically, in this model (considered e.g. in [NS02]) the input vector \( \text{Input}(x,y) \) has the form

\[
\text{Input}(x,y) \overset{\text{def}}{=} \sum_{e \in E} \lambda_e |0,x,e\rangle |0,e,y,0\rangle,
\]

where \( \{\lambda_e | e \in E\} \) is an arbitrary unitary vector (the case (4) of EPR-pairs corresponds to \( \lambda_e = \frac{1}{|E|^{1/2}} \)). With their kind permission, we include in Section 5.1 the adjustments to our basic proof needed for this generalization (Remark 4).

### 3. Preliminaries

In this section we compile together some definitions and previously known results needed for our proof.

#### 3.1. Quantum search vs. quantum communication

For a precise definition of a quantum decision tree see e.g. [BW00]. Given a Boolean function \( g(x_1, \ldots, x_n) \) we will denote by \( Q_{\text{DT}}(g) \) the minimal number of queries needed to compute \( g \) by a quantum decision tree with error at most 1/3 at any input \( x \in \{0,1\}^n \).

Let us denote by \( f_g : \mathcal{P}([n]) \times \mathcal{P}([n]) \rightarrow \{0,1\} \) the predicate \( f_g(x,y) \overset{\text{def}}{=} g(x \cap y) \), where \( x \cap y \) is identified with its characteristic function. The following is probably the deepest general fact known about quantum communication:
Proposition 3.1 ([BCW98]) For any Boolean function \( g(x_1, \ldots, x_n) \), \( Q(f_g) \leq O(Q_{DT}(g) \log n) \).

3.2. Matrix norms

All material in this section is classical and can be found e.g. in the excellent textbook [Bha97].

After we give up Dirac’s notation (in Section 5.2), all vectors will be represented as columns. For a complex vector \( \xi \) [complex matrix \( A \)], let \( x^* \overset{\text{def}}{=} (\bar{x})^\top \) [\( A^* \overset{\text{def}}{=} (\bar{A})^\top \), respectively] be its conjugate transpose. Let \( ||\xi|| \overset{\text{def}}{=} (\xi^* \xi)^{1/2} \) denote the \( \ell_2 \)-norm of \( \xi \).

For a complex matrix \( A \), we will denote by \( ||A|| \) its operator norm defined as \( ||A|| \overset{\text{def}}{=} \max \{||A\xi|| : ||\xi|| \leq 1\} \). Alternatively, \( ||A|| \overset{\text{def}}{=} \max \{|\eta^\top A\xi| : ||\eta||, ||\xi|| \leq 1\} \).

For two complex matrices \( A, B \) of the same size \( m \times n \) we denote by \( \langle A, B \rangle \) their entrywise scalar product, that is, \( \langle A, B \rangle \overset{\text{def}}{=} \text{Tr}(A^* B) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij} \). \( ||A||_F \) denotes the Frobenius norm corresponding to this scalar product, that is, \( ||A||_F \overset{\text{def}}{=} \left( \sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \). We will also need the following (somewhat more exotic) trace norm \( ||A||_{tr} \) defined as

\[
||A||_{tr} \overset{\text{def}}{=} \max_B \{||\langle A, B \rangle|| : ||B|| \leq 1\},
\]

where \( B \) runs over all (complex) matrices of the same size as \( A \).

The following proposition summarizes some properties of these norms.

Proposition 3.2

1. Let \( ||\cdot|| \) be any one of the three norms \( ||\cdot||, ||\cdot||_F \) or \( ||\cdot||_{tr} \), and \( A \) be a complex \( m \times n \) matrix. Then:

   (a) \( ||A^*|| = ||A^\perp|| = ||A|| \);
   (b) if \( B \) is a submatrix of \( A \) then \( ||B|| \leq ||A|| \);
   (c) \( ||\cdot|| \) is invariant under left and right unitary transformations, that is, for every \( m \times m \) unitary matrix \( U \) and every \( n \times n \) unitary matrix \( V \), \( ||UAV|| = ||A|| \).

2. Let now \( B \) be another complex \( n \times k \) matrix, and \( AB \) stand for the ordinary matrix multiplication. Then:

   (a) \( ||AB|| \leq ||A|| \cdot ||B|| \);
(b) $||AB||_{tr} \leq ||A||_F \cdot ||B||_F$. (Hölder inequality, see e.g. [Bha97, Corollary IV.2.6])

3. $||A|| \leq ||A||_F \leq (\min\{m,n\})^{1/2} \cdot ||A||$.

4. For a square $n \times n$ matrix $A$, $||A||_{tr} \geq \sum_{i=1}^{n} |a_{ii}|$.

**Remark 2**
If $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_p(A)$, $p = \min\{m,n\}$ are the singular values of $A$ then $||A|| = \sigma_1(A)$, $||A||_F = (\sum_{i=1}^{p} \sigma_i^2(A))^{1/2}$ and $||A||_{tr} = \sum_{i=1}^{p} \sigma_i(A)$ (which, along with Proposition 3.2.1c) and the SVD-theorem almost immediately implies all non-trivial parts of that proposition). We, however, will not need this singular value characterization in our proof.

**Remark 3**
The same proposition 3.2.1c) implies that we can unambiguously talk of the operator, Frobenius or trace norm of an operator from one (finite-dimensional) Hilbert space to another.

Two more matrix norms we will be using are the $\ell_1$-norm and $\ell_\infty$-norm defined entrywise:

$$\ell_1(A) \overset{\text{def}}{=} \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |a_{ij}|;$$

$$\ell_\infty(A) \overset{\text{def}}{=} \max \left\{ \max_{1 \leq i \leq m} \max_{1 \leq j \leq n} |a_{ij}| \right\}.$$

Of course, these norms are not invariant under unitary transformations. However, they are at least somewhat related to unitary invariant norms via the following (obvious) observation:

$$||\langle A,B \rangle|| \leq \ell_1(A) \cdot \ell_\infty(B).$$

### 3.3. Decomposition of quantum communication protocols

**Proposition 3.3 ([Yao93, Kre95])** Let $P$ be a $c$-qubit communication protocol, and let $U_p$ be the unitary operator in the right-hand side of (3). Then there exist linear operators $A_u$ on $\mathcal{H}_A$ and $B_u$ on $\mathcal{H}_B$ ($u \in \{0,1\}^c$) such that for every vector $a \in \mathcal{H}_A$ and every vector $b \in \mathcal{H}_B$,

$$U_p(|a\rangle|0\rangle|b\rangle) = \sum_{u \in \{0,1\}^c} |A_u(a)\rangle|u_c\rangle|B_u(b)\rangle.$$

Moreover, $||A||, ||B|| \leq 1$ for every $u \in \{0,1\}^c$. 

Proof. It is only the last observation (about the operator norms) that is (apparently) new. This, however, immediately follows from Proposition 3.2.2a) and the fact that every operator $A_u, B_u$ is composed from unitary operators and orthogonal projections onto the subspaces $\mathcal{H}_A \otimes |\epsilon\rangle \otimes \mathcal{H}_B$, $\epsilon \in \{0, 1\}$. \[\square\]

3.4. Symmetric functions and predicates

For a Boolean predicate $D : \{0, 1, \ldots, n\} \rightarrow \{0, 1\}$, denote by $\tilde{\deg}(D)$ the approximate degree of this predicate defined as the minimal degree of a univariate real polynomial $f(x)$ such that $|f(s) - D(s)| \leq 1/3$ for every $s \in \{0, 1, \ldots, n\}$. Let $g_D(x_1, \ldots, x_n)$ be the symmetric Boolean function defined as $g_D(x) \overset{\text{def}}{=} D(\sum_{i=1}^{n} x_i)$ (note that $f_{n,D}(x, y) = f_{g_D}(x, y)$). [NS94] observed that $\deg(D) = \tilde{\deg}(g_D)$, where $\tilde{\deg}(g)$ is the minimal degree of a multi-variate polynomial approximating $g$ on $\{0, 1\}^n$ within error $1/3$ in the $\ell_\infty$-norm.

Proposition 3.4 ([Pat92]) $\tilde{\deg}(D) = \theta(\sqrt{n(\ell_0(D) + \ell_1(D)))}$.

It was proved in [BBC+98] that $\Omega(\tilde{\deg}(g))$ is a general lower bound on $Q_{DT}(g)$. In the opposite direction, they show that for symmetric functions this bound is tight:

Proposition 3.5 ([BBC+98]) $Q_{DT}(g_D) \leq O(\tilde{\deg}(D))$.

Assume now that $X = Y \overset{\text{def}}{=} [n]^k$. For $0 \leq s \leq k$, denote by $J_{n,k,s}$ the $0-1$ $n(k) \times n(k)$ matrix whose rows and columns are indexed by $[n]^k$ and $(J_{n,k,s})_{xy} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } |x \cap y| = s \\ 0 & \text{otherwise} \end{cases}$. The spectrum of these matrices is described by the so-called Hahn polynomials (see e.g. [Del78]). The latter, being classical objects, were re-discovered many times in different contexts; the expression that is the most convenient for our purposes was proposed by Knuth [Knu91]; remarkably, it is based on a direct computation of the eigenvalues.
Proposition 3.6 ([Knu91]) Let \( k \leq n/2 \). The matrices \( J_{n,k,s}(0 \leq s \leq k) \) share the same eigenspaces \( E_0, E_1, \ldots, E_k \). The eigenvalue of \( J_{n,k,s} \) corresponding to the eigenspace \( E_t \) is given by

\[
\min\{s,t\} \sum_{i=\max\{0,s+t-k\}} (-1)^{t-i}t^{ik - i(s - in - k - t + i) - s - t + i}.
\]

4. Upper bound

In this section we show that the upper bound \( Q(f_{n,D}) \leq O((\sqrt{n\ell_0(D)} + \ell_1(D)) \log n) \) in Theorem 2.1 is almost immediately implied by the previously known results from Section 3.

Let \( D : \{0,1,\ldots,n\} \longrightarrow \{0,1\} \) be any predicate. \( D \) is constant on the interval \( [\ell_0(D), n - \ell_1(D)] \). Negating it if necessary, we can assume that \( D \) takes on value 0 in this interval. Then \( D = D_0 \lor D_1 \), where \( D_0^{-1}(1) \subseteq [0, \ell_0(D) - 1] \) and \( D_1^{-1}(1) \subseteq [n - \ell_1(D) + 1, n] \). Also, \( f_D = f_{D_0} \lor f_{D_1} \), and Alice and Bob compute \( f_{D_0} \) and \( f_{D_1} \) separately.

For computing \( f_{D_0} \), they apply the BCW-reduction (Proposition 3.1) and Propositions 3.5, 3.4:

\[
Q(f_{D_0}) \leq O(Q_{DT}(g_{D_0}) \log n) \leq O(\deg(D_0) \log n) \leq O(\sqrt{\ell_0(D)} \log n).
\]

For computing \( f_{D_1} \), Alice and Bob use the following trivial (classical) protocol. Alice first checks whether her input \( x \) has \( \leq n - \ell_1(D) \) ones or not. In the first case \( f_{D_1}(x, y) = 0 \) and she declares the result. Otherwise she sends her entire input to Bob. This will take at most \( \log_2 \left( \sum_{k=n-\ell_1(D)+1}^n \binom{n}{k} \right) \) bits which is \( O(\ell_1(D) \log n) \) since \( \ell_1(D) \leq n/2 \). Then Bob computes \( f_{D_1}(x, y) \).

5. Lower bounds

In this section we prove the lower bound in Theorem 2.1 and Theorem 2.3. First we show that the latter implies the first, and this is done by a straightforward reduction.

Definition 5.1 For a Boolean predicate \( D \) on \( \{0,1,\ldots,n\} \) and \( 0 \leq r \leq n \), let \( D - r : \{0,1,\ldots,n-r\} \longrightarrow \{0,1\} \) be given by \( (D - r)(s) \overset{\text{def}}{=} D(r + s) \). Let also \( D|_k \) be the restriction of \( D \) onto \( \{0,1,\ldots,k\} \), \( k \leq n \).
Lemma 5.2 For every predicate $D$ on $\{0,1,\ldots,n\}$ and every integers $k,r$ satisfying $0 \leq r \leq n$, $k \leq n - r$, we have $Q^*(f_{n,D}) \geq Q^*(f_{n-r,k,(D-r)|k})$.

Proof. Alice and Bob use the optimal protocol for $f_{n,D} : \mathcal{P}([n]) \times \mathcal{P}([n]) \rightarrow \{0,1\}$ to compute $f_{n-r,k,(D-r)|k} : [n-r]^k \times [n-r]^k \rightarrow \{0,1\}$. For this they simply map their inputs $x,y \in [n-r]^k$ to the inputs $\phi(x), \phi(y) \in \mathcal{P}([n])$ using the mapping $\phi(x) \overset{\text{def}}{=} x \cup \{n-r+1,\ldots,n\}$, and feed $\phi(x), \phi(y)$ into the protocol for $f_{n,D}$. ■

Proof of lower bound in Theorem 2.1 from Theorem 2.3. We need to establish two separate bounds, $Q^*(f_{n,D}) \geq \Omega(\sqrt{n\ell_0(D)})$ and $Q^*(f_{n,D}) \geq \Omega(\ell_1(D))$, and both are proved via a reduction from Lemma 5.2 (with different values $r, k$ of course). In choosing $r, k$ we must satisfy the two conditions

$$k \leq (n-r)/4, \ (\ell-r) \leq k/4$$

(6) arising from the statement of Theorem 2.3), where $\ell \overset{\text{def}}{=} \ell_0(D)$ for the first bound and $\ell \overset{\text{def}}{=} n - \ell_1(D)$ for the second. As long as they are satisfied, Theorem 2.3 gives $Q^*(f_{n,D}) \geq \Omega(\sqrt{k(\ell-r)})$.

If $\ell \leq n/16$ (and, in particular, $\ell = \ell_0(D)$), we simply let $r \overset{\text{def}}{=} 0$ and $k \overset{\text{def}}{=} n/4$. Then the bound of Theorem 2.3 becomes $\Omega(\sqrt{n\ell})$; i.e., exactly what we are proving.

If $\ell \geq n/16$, we satisfy the conditions (6) with equality for which we set $r \overset{\text{def}}{=} \frac{16\ell-n}{15}$ and $k \overset{\text{def}}{=} \frac{4}{15}(n-\ell)$. Then $\ell-r \geq \Omega(n-\ell)$, and Theorem 2.3 still gives us the required bound $Q^*(f_{n,D}) \geq \Omega(n-\ell)$. ■

In the rest of the paper we prove Theorem 2.3. The proof splits into three fairly independent blocks.

5.1. Approximate trace norm lower bound

Definition 5.3 For a real matrix $M$, let $||M||_\epsilon \overset{\text{def}}{=} \min\{||P||_\epsilon : \ell_\infty(M-P) \leq \epsilon\}$ be its $\epsilon$-approximate trace norm ($P$ runs over all real matrices of the same size as $M$).

Definition 5.4 For a predicate $f : X \times Y \rightarrow \{0,1\}$, $M_f$ denotes its communication 0-1 matrix $(M_f)_{xy} \overset{\text{def}}{=} f(x,y)$.
Theorem 5.5 For any predicate \( f : X \times Y \rightarrow \{0, 1\} \), where \(|X| = |Y| = N\), and any \( \epsilon > 0 \), \( Q^*_s(f) \geq \Omega(\log(||M_f||_{tr}/N)) \).

Proof. Fix a \( c \)-qubit communication protocol with prior entanglement computing \( f \) with probability \( \epsilon \). Let \( p_{xy} \) be the acceptance probabilities of this protocol on the input \((x, y)\); arrange them into an \((N \times N)\) matrix \( P \).

Then, clearly, \( \ell^\infty(M_f - P) \leq \epsilon \), and we only have to prove that \( ||P||_{tr} \leq N \cdot \exp(O(c)) \).

Apply the decomposition from Proposition 3.3 to the input string (4). We get:

\[
\text{Output}(x, y) = \frac{1}{|E|^{1/2}} \sum_{e \in E} \sum_{u \in \{0, 1\}^c} A_u |0, x, e \rangle |u_c \rangle B_u |e, y, 0 \rangle
\]

and then

\[
p_{xy} = \frac{1}{|E|} \left| \sum_{e \in E} \sum_{u \in \Pi} A_u |0, x, e \rangle B_u |e, y, 0 \rangle \right|^2
\]

\[
= \frac{1}{|E|} \cdot \sum_{e, f \in E} \sum_{u, v \in \Pi} (\langle f, x, 0 | A_v | A_u | 0, x, e \rangle \cdot \langle f, y, 0 | B_v | B_u | 0, y, e \rangle),
\]

where \( \Pi \overset{\text{def}}{=} \{ u \in \{0, 1\}^c \mid u_c = 1 \} \).

Let us now define \( N \times (|E|^2 \times |\Pi|^2)\)-matrices \( A, B \) by letting \( a_{x,(e,fuv)} \overset{\text{def}}{=} \langle f, x, 0 | A_v | A_u | 0, x, e \rangle \) and \( b_{y,(e,fuv)} \overset{\text{def}}{=} \langle f, y, 0 | B_v | B_u | 0, y, e \rangle \). Then \( P = \frac{1}{|E|} AB^\perp \), and Proposition 3.2.2b) implies

\[
||P||_{tr} \leq \frac{1}{|E|} \cdot ||A||_F \cdot ||B||_F. \tag{7}
\]

For estimating \( ||A||_F, ||B||_F \), we divide these matrices into \( N \cdot |\Pi|^2 \) blocks, and interpret every block as an \((|E| \times |E|)\) matrix. Namely, for any fixed \( x \in X \) and \( u, v \in \Pi \), let \( A^{xuv} \) be the square \((|E| \times |E|)\) matrix given by

\[
a_{e,f}^{xuv} \overset{\text{def}}{=} a_{x,(e,fuv)} = \langle f, x, 0 | A_v | A_u | 0, x, e \rangle.
\]

Then

\[
||A||_F^2 \leq N \cdot |\Pi|^2 \cdot \max_{x,u,v} ||A^{xuv}||_F^2. \tag{8}
\]

For bounding \( ||A^{xuv}||_F \) we first use Proposition 3.2.3:

\[
||A^{xuv}||_F \leq |E|^{1/2} \cdot ||A^{xuv}||. \tag{9}
\]
Finally we claim that

$$||A^{xuv}|| \leq 1.$$  \hfill (10)

Indeed, let $\eta, \xi$ be any vectors of length $|E|$ with $||\eta||, ||\xi|| \leq 1$. Then we have

$$\eta^\top A^{xuv} \xi = \langle \sum_{f \in E} \eta_f f, x, 0 | A_u | 0, x, \sum_{e \in E} \xi_e e \rangle$$

and, since $||A_u||, ||A_v|| \leq 1$,

$$||\eta^\top A^{xuv} \xi|| \leq ||0, x, \sum_{e \in E} \xi_e e|| \cdot ||0, x, \sum_{f \in E} \eta_f f|| = ||\xi|| \cdot ||\eta|| \leq 1.$$

(10) is proved. Along with (9) and (8) this implies $||A||_F \leq N^{1/2} \cdot ||\Pi|| \cdot |E|^{1/2}$, and the same bound holds for $||B||_F$. Substituting them into (7), we get $||P||_{tr} \leq N \cdot ||\Pi||^2 \leq N \cdot \exp(O(c))$ which completes the proof of Theorem 5.5.

**Remark 4 (Nayak, Shi)** Theorem 5.5 (and, hence, all lower bounds following from it) extends to the case of more general entanglement in which the input vector is given by (5). In order to see this, first note the following generalization of the right-hand side in Proposition 3.2.3:

$$||\hat{A}^{xuv}|| \leq ||\hat{A}^{xuv}||_F \cdot ||\hat{B}^{xuv}||_F,$$

(11)

(the original statement corresponds to $L = L_{\min (m,n)}$). If $\hat{a}_{x,(efuv)} \overset{\text{def}}{=} \lambda_e a_{x,(efuv)}$ and $\hat{b}_{y,(efuv)} \overset{\text{def}}{=} \lambda_f b_{y,(efuv)}$ then $\hat{P} = \hat{A}^{xuv} \hat{B}^\top$, where $\hat{P}$ is the matrix of acceptance probabilities relative to the input vector (5), and $||\hat{P}||_{tr} \leq ||\hat{A}^{xuv}|| \cdot ||\hat{B}^{xuv}||$. As before, $||\hat{A}^{xuv}||^2 \leq N \cdot ||\Pi||^2 \cdot \max_{x,u,v} ||\hat{A}^{xuv}||_F^2$. We, however, know that $\hat{A}^{xuv} = L A^{xuv}$, where $L$ is the diagonal matrix with elements $\{\lambda_e | e \in E\}$. Since $\lambda$ is unitary, $||L||_F = 1$ and (11) implies $||\hat{A}^{xuv}||_F \leq ||A^{xuv}|| \leq 1$. The remaining calculations are the same as in the basic proof.

### 5.2. Multi-dimensional discrepancy bound

This section is central to our argument, so we begin with a brief overview of the ordinary discrepancy bound.
Suppose that we want to get a lower bound on the approximate trace norm (or any other approximate norm) of a matrix $M$. That is, we need to rule out the existence of a decomposition $M = P + \Delta$, where $\|P\|_{\text{tr}}$ is small and $\ell_\infty(\Delta)$ is small. The ordinary discrepancy method [Yao93, Kre95] proceeds as follows. Assume that $M$ is a $\pm 1$-matrix, take any probability distribution $\mu$ on its entries and form the Hadamard product $M \odot \mu$ ($((M \odot \mu)_{ij})_{ij} \overset{\text{def}}{=} M_{ij} \mu_{ij}$). Then $\langle M, M \odot \mu \rangle = 1$ and $|\langle \Delta, M \odot \mu \rangle| \leq \ell_1(M \odot \mu) \cdot \ell_\infty(\Delta) = \ell_\infty(\Delta)$. Therefore, if $|\langle P, M \odot \mu \rangle|$ is small for every matrix $P$ with small trace norm (in other words, $M \odot \mu$ has a low discrepancy with such matrices), we are done.

The next logical step was taken by Klauck in [Kla01, Theorem 4] who observed that the “test matrix” need not be of the particular form $M \odot \mu$. As long as $\mu$ is any matrix with $\ell_1(\mu) = 1$ and of low discrepancy, we are still in a good shape for all matrices $M$ for which $|\langle M, \mu \rangle|$ is at least somewhat large.

It is well known, however, that even in this form the discrepancy method does not work for (say) the disjointness predicate. In this paper we take it one step further and instead of considering the linear functional $X \mapsto \langle X, \mu \rangle$ for a single “test matrix” $\mu$, we consider the multi-dimensional “trace operator” $X \mapsto (\langle X, \mu_1 \rangle, \ldots, \langle X, \mu_r \rangle)$ for a family of matrices $\mu_1, \ldots, \mu_r$ with $\ell_1(\mu_s) \leq 1$. In order to be able to apply spectral methods, we will assume that $\mu_1, \ldots, \mu_r$ are real symmetric commuting matrices (although it would be sufficient to assume that they allow singular value decompositions $U \mu_1 V, \ldots, U \mu_r V$ with common unitary matrices $U, V$).

**Definition 5.6** An $r$-dimensional discrepancy test consists of real symmetric matrices $\mu_1, \ldots, \mu_r$ with $\ell_1(\mu_s) \leq 1$ ($1 \leq s \leq r$) that have the same size $N \times N$ and commute with each other, along with an orthogonal decomposition

$$\mathbb{R}^N = E_1 \oplus E_2 \oplus \ldots \oplus E_k$$

of $\mathbb{R}^N$ into their shared eigenspaces $E_1, E_2, \ldots, E_k$.

Note that the commutativity alone implies the existence of at least one decomposition (12). For our application we, however, need $k \ll N$ (that is, eigenvalues substantially repeat themselves), and for this reason we prefer to fix the decomposition explicitly in the definition.

Given a discrepancy test $(\mu_1, \ldots, \mu_r, E_1, \ldots, E_k)$, denote by $\lambda_{st}$ the eigenvalue of $\mu_s$ corresponding to $E_t$. Let the trace of $E_t$ be the $r$-dimensional vec-
tor $\lambda'$ naturally defined as $(\lambda')_s \overset{\text{def}}{=} \lambda_{st}$, and let $\text{Trace}(\bar{\mu}, \bar{E}) \overset{\text{def}}{=} \{ \lambda' \in \mathbb{R}^r \mid 1 \leq t \leq k \}$ be the set of all these vectors.

**Definition 5.7** Given a set of vectors $T \subseteq \mathbb{R}^r$ and $C > 0$, let $\text{Conv}_C(T) \overset{\text{def}}{=} \{ \sum_{\lambda \in T} a_\lambda \lambda : \sum_{\lambda \in T} |a_\lambda| \leq C \}$ be the convex hull of the segments $\{ [-C, C] \lambda \mid \lambda \in T \}$. Given another vector $\xi \in \mathbb{R}^r$ and $\epsilon > 0$, let $\phi^\epsilon(\xi, T) \overset{\text{def}}{=} \min \{ C \mid \rho_\infty(\xi, \text{Conv}_C(T)) \leq \epsilon \}$, where $\rho_\infty$ is the distance in the $\ell_\infty$-norm.

**Theorem 5.8** Let $M$ be a real squared matrix, and $(\mu_1, \ldots, \mu_r, E_1, \ldots, E_k)$ be an arbitrary $r$-dimensional test of the same size. Let $\xi_M \in \mathbb{R}^r$ be defined as $(\xi_M)_s \overset{\text{def}}{=} \langle M, \mu_s \rangle$. Then

$$\|M\|_{\text{tr}} \geq \phi^\epsilon(\xi_M, \text{Trace}(\bar{\mu}, \bar{E})).$$

**Proof.** Let $\|M\|_{\text{tr}} = C$ and $M = P + \Delta$, where $\|P\|_{\text{tr}} = C$ and $\ell_\infty(\Delta) \leq \epsilon$. Then $\xi_M = \xi_P + \xi_\Delta$ and, moreover, $|\langle \xi_\Delta \rangle_s| = |\langle \Delta, \mu_s \rangle| \leq \ell_1(\mu_s) \cdot \ell_\infty(\Delta) \leq \epsilon$ for every $s \in [r]$ which implies $\ell_\infty(\xi_\Delta) \leq \epsilon$. Thus, we only need to prove that $\xi_P \in \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E}))$.

Let $U$ be the orthogonal matrix corresponding to the decomposition (12), so that all $(U^T \mu_s U)$ are diagonal. Consider the matrix $(U^T PU)$, for every $t \in [k]$ let $(U^T PU)_t$ be its principal submatrix corresponding to the eigenspace $E_t$, and let $a_t \overset{\text{def}}{=} \text{Tr}((U^T PU)_t)$. Then $\xi_P \in \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E}))$ is implied by the following two facts:

$$\xi_P = \sum_{t=1}^k a_t \lambda^t$$

and

$$\sum_{t=1}^k |a_t| \leq C.$$

Both are proved by easy matrix manipulations (with heavy use of Proposition 3.2):

$$(\xi_P)_s = \langle P, \mu_s \rangle = \langle (U^T PU), (U^T \mu_s U) \rangle = \sum_{t=1}^k \text{Tr}((U^T PU)_t) \cdot \lambda_{st} = \sum_{t=1}^k a_t \lambda_{st}$$

and

$$\sum_{t=1}^k |a_t| \leq \sum_{i=1}^N |(U^T PU)_i| \leq \| (U^T PU) \|_{\text{tr}} = \| P \|_{\text{tr}} = C.$$
5.3. Putting things together

Now we are ready to finish the proof of Theorem 2.3. Fix integers $n$ and $k \leq n/4$. Set $N \overset{\text{def}}{=} n(k)$. Let $D : \{0,1,\ldots,k\} \rightarrow \{0,1\}$ be any predicate such that $D(\ell) \neq D(\ell - 1)$ for some $\ell \leq k/4$. Applying Theorem 5.5 (and observing that the error probability can be always reduced from $1/3$ to $1/4$ with an increase in complexity by at most a constant multiplicative factor), we get

$$Q^*(f_{n,k,D}) \geq \Omega(\log(||M_{f_{n,k,D}}||_{\text{tr}}^{1/4}/N)). \quad (13)$$

Let now $\mu_s \overset{\text{def}}{=} N^{-1}k(s^{-1}n - k)k - s^{-1}J_{n,k,s}$, and let $E_0, \ldots, E_k$ be the shared eigenspaces of these matrices from Proposition 3.6. Note that $\ell(\mu_s) = 1$ and $\langle M_{f_{n,k,D}}, \mu_s \rangle = D(s)$. Applying Theorem 5.8 with the $(k/2 + 1)$-dimensional test $(\mu_0, \mu_1, \ldots, \mu_k, E_0, E_1, \ldots, E_k)$, we get

$$||M_{f_{n,k,D}}||_{\text{tr}}^{1/4} \geq \varphi^{1/4}(D|_{k/2}, \text{Trace}(\bar{\mu}, \bar{E})). \quad (14)$$

Claim 5.9 Let $\lambda_{st}$ be the eigenvalue of the matrix $\mu_s$ corresponding to the eigenspace $E_t$. Then:

1. $\lambda_{st} = F_t(s)$, where $F_t$ is a polynomial of degree $t$ (known, up to a normalizing factor, as Hahn polynomial);

2. whenever $k \leq n/4$ and $s \leq k/2$, $|\lambda_{st}| \leq N^{-1} \cdot \exp(-\Omega(t))$.

Proof. By Proposition 3.6,

$$\lambda_{st} = N^{-1}k(s^{-1}n - k)k - s^{-1} \sum_{i=\max\{0,s+t-k\}}^{\min\{s,t\}} (-1)^{t-i} (ik - i(s - in - k - t + i(k - s - t + i)}$$

$$= N^{-1} \sum_{i=\max\{0,s+t-k\}}^{\min\{s,t\}} (-1)^{t-i} (ik - i(s - in - k - t + i(k - s - t + i})$$

$$= N^{-1} \sum_{i=0}^{t} \left((-1)^{t-i} \frac{s(s - 1) \ldots (s - i + 1)}{k(k - 1) \ldots (k - i + 1)} \times \frac{(k - s)(k - s - 1) \ldots (k - s - t + i + 1)}{(n - k)(n - k - 1) \ldots (n - k - t + i + 1)} \right).$$
Part 1 is already obvious from this expression. Part 2 is also easy:

\[ |\lambda_{st}| \leq N^{-1} \sum_{i=0}^{t} t() \cdot \frac{s(s-1) \ldots (s-i+1)}{k(k-1) \ldots (k-i+1)} \times \frac{(k-s)(k-s-1) \ldots (k-s-t+i+1)}{(n-k)(n-k-1) \ldots (n-k-t+i+1)} \times \left( \frac{k-s}{n-k} \right)^{t-i} = N^{-1} \cdot \left( \frac{s}{k} + \frac{k-s}{n-k} \right)^t \leq N^{-1} \cdot \left( \frac{1}{2} + \frac{1}{3} \right)^t. \]

This claim implies that for every \( t_0 \leq k, \{ \lambda^t \mid t \leq t_0 \} \subseteq P(t_0), \) where \( P(t_0) \) is the set of all real-valued functions on \( \{0, 1, \ldots, k/2\} \) representable by (real) polynomials of degree \( \leq t_0 \). Whereas \( \ell_{\infty}(\lambda^t) \leq N^{-1} \exp(-\Omega(t_0)) \)

\[ \forall \xi \in \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E})), \rho_{\infty}(\xi, P(t_0)) \leq N^{-1} \cdot C \cdot \exp(-\Omega(t_0)). \]  

Set now \( t_0 \overset{\text{def}}{=} \deg(D|_{k/2}) - 1 \) and \( C \overset{\text{def}}{=} \phi^{1/4}(D|_{k/2}, \text{Trace}(\bar{\mu}, \bar{E})). \) Note that since \( \ell \leq k/4, \)

\[ t_0 \geq \Omega(\sqrt{k\ell}) \]  

by Proposition 3.4. Also, by definition of the approximate degree, \( \rho_{\infty}(D|_{k/2}, P(t_0)) > 1/3. \) On the other hand, by (15),

\[ \rho_{\infty}(D|_{k/2}, P(t_0)) \leq N^{-1} \cdot C \cdot \exp(-\Omega(t_0)) + \rho_{\infty}(D|_{k/2}, \text{Conv}_C(\text{Trace}(\bar{\mu}, \bar{E}))) \leq N^{-1} \cdot C \cdot \exp(-\Omega(t_0)) + 1/4. \]

Combining these two bounds with (16), we get

\[ \phi^{1/4}(D|_{k/2}, \text{Trace}(\bar{\mu}, \bar{E})) = C \geq N \cdot \exp(\Omega(\sqrt{k\ell})). \]  

Theorem 2.3 now follows from (13), (14) and (17).

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