Localization of Gravity on Brane Embedded in $AdS_5$ and $dS_5$

Iver Brevik ¹
Division of Applied Mechanics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway

Kazuo Ghoroku ²
²Fukuoka Institute of Technology, Wajiro, Higashi-ku
Fukuoka 811-0295, Japan

Sergei D. Odintsov ³
³Tomsk State Pedagogical University, 634041 Tomsk, Russia

Masanobu Yahiro ⁴
⁴Department of Physics and Earth Sciences, University of the Ryukyus, Nishihara-chou, Okinawa 903-0213, Japan

Abstract

We address the localization of gravity on the Friedmann-Robertson-Walker type brane embedded in either $AdS_5$ or $dS_5$ bulk space, and derive two definite limits between which the value of the bulk cosmological constant has to lie in order to localize the graviton on the brane. The lower limit implies that the brane should be either $dS_4$ or $4d$ Minkowski in the $AdS_5$ bulk. The positive upper limit indicates that the gravity can be trapped also on curved brane in the $dS_5$ bulk space. Some implications to recent cosmological scenarios are also discussed.

¹iver.h.brevik@mtf.ntnu.no
²gouroku@dontaku.fit.ac.jp
³odintsov@mail.tomsknet.ru
⁴yahiro@sci.u-ryukyu.ac.jp
1 Introduction

It is quite expectable that four dimensional world is formed in the process of compactification from the ten-dimensional superstring theory. The D-brane approach is getting very useful for the study of such theory. In particular, there is some interest in the geometry obtained from the D3-brane of type IIB theory. Near the horizon of the stacked D3-branes, the configuration $AdS_5 \times S^5$ is realized and the string theory on this background describes the four-dimensional SUSY Yang-Mills theory which lives on the corresponding boundary [1, 2, 3, 4].

On the other hand, a thin three-brane (Randall-Sundrum brane) embedded in $AdS_5$ space has been proposed [5, 6] as a model of our world. The position of the brane is possible at any value of the transverse coordinate, which is considered as the energy scale of the dual conformal field theory (CFT) living on the boundary. Another interesting point is that this idea gives an alternative to the standard Kaluza-Klein (KK) compactification via the localization of the zero mode of the graviton [6]. Brane approach opened also a new way to the construction of the hierarchy between four-dimensional Planck mass and the electro-weak scale, and also for realization of the small observable cosmological constant with lesser fine-tuning [7, 8].

The gravity theory under consideration is five-dimensional. However, its zero mode is trapped on the brane, and as a result the usual 4-dim Newton law is realized on the brane. Non-trapped, massive KK modes correspond to a correction to Newton’s law, and they are also understood from the idea of the AdS/CFT correspondence. Localization of the various fields, especially of gravity, is thus essential for the brane-world to be realistic. The study of localization, however, was limited to the case of $AdS_5$ bulk space and not made for $dS_5$ bulk space.

The purpose of this paper is to study the localization of the graviton on our brane when it is taken to be time dependent and embedded in $AdS_5$ or in $dS_5$. In Section 2, we give various brane solutions obtained on the basis of a simple ansatz imposed on the bulk metric. For these solutions, we make some brief comments from a cosmological viewpoint. In Section 3, the localization of the graviton on those brane solutions are examined, and a restriction for the parameters of the theory is given in order to realize the localization (cf. Eq. (61) below). In Section 4, the relation of our solutions to cosmology, especially to the recently observed mini-inflation, is discussed. Also, the sub-class of viscous cosmology is briefly dealt with. Concluding remarks are given in the final section.
2 Cosmological solutions of brane-universe

We start from the five-dimensional gravitational action. It is given in the Einstein frame as

\[ S_5 = \frac{1}{2\kappa^2} \left\{ \int d^5X \sqrt{-G} (R - \Lambda + \cdots) + 2 \int d^4x \sqrt{-g} K \right\}, \]  

(1)

where the dots denote the contribution from matter, \( K \) being the extrinsic curvature on the boundary. This term is formal here and it plays no role until one considers the AdS/CFT correspondence. The fields represented by the dots are not needed to construct the background of the brane. The other ingredient is the brane action,

\[ S_b = -\tau \int d^4x \sqrt{-g}, \]  

(2)

which is added to \( S_5 \), and the Einstein equation is written as

\[ R_{MN} - \frac{1}{2} g_{MN} R = \kappa^2 T_{MN} \]  

(3)

where \( \kappa^2 T_{MN} = -(\Lambda + \frac{1}{b}\delta(y)\kappa^2 \tau \delta^M \delta^N) g_{MN} \) and \( b = \sqrt{-g}/\sqrt{-G} \). Here we solve the Einstein equation (3) with the following metric,

\[ ds^2 = -n^2(t, y) dt^2 + a^2(t, y) \gamma_{ij}(x^i) dx^i dx^j + dy^2, \]  

(4)

where the coordinates parallel to the brane are denoted by \( x^\mu = (t, x^i) \), \( y \) being the coordinate transverse to the brane. The position of the brane is taken at \( y = 0 \).

Although the form (4) is simple, it could describe various geometries of both the bulk and the brane. The solutions are controlled by two parameters, \( \Lambda \) and \( \tau \). The configuration of the bulk is determined by \( \Lambda \), i.e., anti-de Sitter (AdS) space for \( \Lambda < 0 \) and de Sitter (dS) space for \( \Lambda > 0 \). The geometry of the four-dimensional brane is controlled by the effective four-dimensional cosmological constant, \( \lambda \), which follows from Eq. (15) below and is given explicitly by

\[ \lambda = \kappa^4 \tau^2 / 36 + \Lambda / 6. \]  

(5)

Thus \( \lambda \) is determined by both \( \Lambda \) and the intrinsic four-dimensional cosmological constant, \( \tau \). Here we notice that there are two values of \( \tau = \pm |\tau| \) for the same \( \lambda \), but positive \( \tau \) should be chosen for the localization of the gravity since one needs attractive \( \delta \)-function force (see Eqs. (43) and (57)). We restrict our interest here to the case of a Friedmann-Robertson-Walker type (FRW) universe. In this case, the three-dimensional metric \( \gamma_{ij} \) is described in Cartesian coordinates as

\[ \gamma_{ij} = (1 + k \delta_{mn} x^m x^n / 4)^{-2} \delta_{ij}, \]  

(6)

\(^5\)Here we take the following definition, \( R^\mu_{\nu \lambda \sigma} = \partial_{(\lambda} \Gamma^\mu_{\nu \sigma)} - \cdots \), \( R_{\nu \sigma} = R^\rho_{\nu \sigma} \) and \( \eta_{AB} = \text{diag}(-1,1,1,1,1) \). Five dimensional suffices are denoted by capital Latin and the four-dimensional one by the Greek ones.
where the parameter values \( k = 0, 1, -1 \) correspond to a flat, closed, or open universe respectively.

We consider the solution of the Einstein equation (3) with the ansatz (4). This leads to the following equations:

\[
3\left\{ \frac{\dot{a}}{a} \right\}^2 - n^2 \left( \frac{a''}{a} + \left( \frac{a'}{a} \right)^2 + \frac{k}{a^2} \right) = \kappa^2 T_{tt},
\]

\[
a^2 \gamma_{ij} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + 2 \frac{n'}{n} \right) + \frac{n''}{n} \right\} + \frac{\dot{a}^2}{a^2} \gamma_{ij} \left\{ \frac{\dot{a}^2}{a^2} \left( - \frac{\dot{a}}{a} + 2 \frac{\dot{n}}{n} \right) - 2 \frac{\ddot{a}}{a} \right\} = \kappa^2 T_{ij},
\]

\[
3 \left( \frac{n'}{n} \frac{\dot{a}}{a} - \frac{\dot{a}'}{a} \right) = \kappa^2 T_{ty},
\]

\[
3 \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{1}{n^2} \left( \frac{\dot{a}}{a} \left( \frac{a'}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} - \frac{k}{a^2} \right) \right\} = \kappa^2 T_{yy}.
\]

Integrating once the \((t, t)\) and \((y, y)\) components of the Einstein equation with respect to \( y \) \[9\], one gets

\[
\left( \frac{\dot{a}}{na} \right)^2 = \frac{\Lambda}{6} + \left( \frac{a'}{a} \right)^2 - \frac{k}{a^2} + \frac{C}{a^4},
\]

where \( C \) is a constant of integration. Let us consider this constant more closely. The solution of Eq. (11) can be given as \[10\],

\[
a(t, y) = \left\{ \frac{1}{2} \left[ 1 + \frac{k^4 \tau^2}{6 \Lambda} \right] a_0^2 + \frac{3C}{\Lambda a_0^2} + \frac{1}{2} \left( 1 - \frac{k^4 \tau^2}{6 \Lambda} \right) a_0^2 - \frac{3C}{\Lambda a_0^2} \right\} \cosh(2\mu y)
\]

\[
- \frac{\kappa^2 \tau}{\sqrt{-6 \Lambda}} a_0^2 \sinh(2\mu |y|) \}^{1/2},
\]

for negative \( \Lambda \) where \( \mu = \sqrt{-\Lambda/6} \). For positive \( \Lambda \), the solution is given as

\[
a(t, y) = \left\{ \frac{1}{2} \left[ 1 + \frac{k^4 \tau^2}{6 \Lambda} \right] a_0^2 + \frac{3C}{\Lambda a_0^2} + \frac{1}{2} \left( 1 - \frac{k^4 \tau^2}{6 \Lambda} \right) a_0^2 - \frac{3C}{\Lambda a_0^2} \right\} \cos(2\mu_d y)
\]

\[
- \frac{\kappa^2 \tau}{\sqrt{6 \Lambda}} a_0^2 \sin(2\mu_d |y|) \}^{1/2},
\]

where \( \mu_d = \sqrt{\Lambda/6} \). In both cases, \( a_0(t) = a(t, y = 0) \) and \( n(t, y) = \dot{a}(t, y)/a_0(t) \). As for \( a_0(t) \), its governing equation can be obtained by considering Eq. (11) at \( y = 0 \) with the boundary condition at \( y = 0 \),

\[
\frac{a'(t, 0+) - a'(t, 0-)}{a_0(t)} = - \frac{\kappa^2 \tau}{3},
\]
the latter following from integrating Eq. (7) across the surface $y = 0$. We get

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \lambda - \frac{k}{a_0^2} + \frac{C}{a_0^4},$$

(15)

(cf. Eq. (5)). The solution of this equation is obtained with a new constant of integration, $c_0$:

$$a_0(t) = \frac{\lambda^{-3/4}}{2f(t)} \left(\sqrt{\lambda} f^4(t) - 4C\right) + 2k\sqrt{\lambda} f^2(t) + k^2)^{1/2},$$

(16)

$$f(t) = e^{\sqrt{\lambda}(t-c_0)}.$$  

(17)

This solution shows inflationary behavior for positive $\lambda$. It then describes the inflation at the early universe and the mini-inflation at the present universe. Further remarks on this point are given in Section 4.

In terms of these general solutions, it is possible to discuss more closely the meaning of the parameter $C$. For example, $C$ can be related to the CFT radiation field energy in a cosmological context, from the viewpoint of the AdS/CFT correspondence [11, 12]. In this sense, the above solution with non-zero $C$ may be important to see the cosmological dS/CFT correspondence (for the introduction, see[13]).

In this general case, it will however be difficult to proceed with the analysis of the localization problem of the graviton. We intend to discuss this general case elsewhere. Instead, we impose here the following ansatz on the metric (4):

$$a(t, y) = a_0(t)A(y), \quad n(t, y) = A(y).$$

(18)

This turns out to be convenient for our purpose of examining the localization problem. This form is suitable to see the effects of the bulk geometry $A(y)$ on the cosmological evolution expressed by the scale factor $a_0(t)$. It is easy to see that the solution of this type can be obtained by taking $C = 0$ in Eq. (13). Then the solutions obtained in the form of (18) are restricted to the one where the radiation energy content of CFT is neglected. We will see, in Section 4, that this simplification is justified in a realistic cosmological scenario. Hereafter, we discuss the solutions of the form (4) with the ansatz (18) for FRW brane-universes, with $k = 0, \pm 1$.

When considering Eq. (18), we obtain from Eq. (11) the following equations when $C = 0$:

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 + \frac{k}{a_0^2} = A^2 + \frac{\Lambda}{6}A^2 = D,$$

(19)

where $D$ is a constant being independent of $t$ and $y$. In view of the boundary condition at the brane position,

$$A'(0+) - A'(0-) = -\frac{k^2\tau}{3} A(0),$$

(20)

one gets

$$D = \lambda$$

(21)

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if $A(0) = 1$. The normalization condition $A(0) = 1$ does not affect the generality of our discussion. In the following, we discuss solutions for the cases where $\lambda$ takes zero, positive, or negative values.

2.1 Solutions for $\lambda = 0$

When $\lambda = 0$, solutions are available only for $k = 0, -1$, because of Eq. (19). The solution for the flat three space ($k = 0$) is known as the Randall-Sundrum (RS) brane [5, 6], and is

$$ds^2 = e^{-2|y|/L} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$$  \hfill (22)

where $\tau = 6/(L\kappa^2)$, $L = \sqrt{-6/\Lambda}$ being the AdS radius. Here $\lambda = 0$ is achieved by the fine-tuning of $\Lambda$ and $\tau$ as given above. This is assured from Eq. (5). The solution (22) represents a static and flat brane situated at $y = 0$, and the configuration is taken to be $Z_2$ symmetric for $y \rightarrow -y$.

Another solution is obtained for $k = -1$, open three space, as

$$ds^2 = e^{-2|y|/L} (-dt^2 + a_0^2(t) \gamma_{ij} dx^i dx^j) + dy^2$$  \hfill (23)

where $\gamma_{ij}$ is taken from Eq. (6) with $k = -1$ and $a_0(t) = \pm t + c_0$ with a constant $c_0$. This solution leads to the curvature dominated universe whose open three space size expands or shrinks linearly with time. But this universe would not correspond to our present universe, since recent analyses of the cosmic microwave background indicates that our universe is almost flat.

In the evolution of our universe, the cosmological constant $\lambda$ may be not zero but positive. It is considered to be large in the early inflation epoch and tiny in the present epoch. Recent observations of Type Ia supernovae and the cosmic microwave background indicate that our universe is dominated by a positive $\lambda$ [14, 15, 16]. Next, we consider this case.

2.2 Solutions for $\lambda > 0$

When $\lambda > 0$, we obtain time-dependent solutions for each value of $k = 0, 1, -1$. When $k = 0$, the typical, inflationary brane is obtained, and it has the following form [17, 18]

$$a_0(t) = e^{H_0 t}, \quad A(y) = \frac{\sqrt{\lambda}}{\mu} \sinh[\mu(y_H - |y|)]$$  \hfill (24)

where $\mu = \sqrt{-\Lambda/6}$, and the Hubble constant is represented as $H_0 = \sqrt{\lambda}$. This solution is obtained for $\Lambda < 0$, $y_H$ representing the position of the horizon, and the five dimensional space-time is expressed as

$$ds^2 = A(y)^2 (-dt^2 + e^{H_0 t} \delta_{ij} dx^i dx^j) + dy^2.$$  \hfill (25)

This solution represents a brane at $y = 0$. The configuration is taken to be $Z_2$ symmetric as in Eq. (22). It should be noticed that the solution approaches the RS solution
(22) in the limit $H_0 \to 0$. In this sense, the solution can be understood as an extension of the RS brane to a time-dependent brane due to the non-zero $\lambda$, which in turn can be considered as a result of the failure of the fine-tuning to obtain zero $\lambda$.

For $k = \pm 1$, the solutions are given by the same $A(y)$ but with a different $a_0(t)$;

\[ a_0(t) = \frac{1}{H_0} \cosh(H_0 t + \alpha_1), \tag{26} \]

for $k = 1$ and

\[ a_0(t) = \frac{1}{H_0} \sinh(H_0 t + \alpha_2), \tag{27} \]

for $k = -1$, where $\alpha_i$ are constants. These solutions also represent inflation with curved three space. For any value of $k = 0, \pm 1$, we obtain the same solution for $A(y)$ as given above. Thus $a_0(t)$ has nothing to do with the problem of localization since it depends only on the form of $A(y)$, as we will see below.

When $\Lambda$ is positive, the solution for $a_0(t)$ is the same as above, but $A(y)$ becomes different. One has

\[ A(y) = \frac{\sqrt{\Lambda}}{\mu_d} \sin[\mu_d(y_H - |y|)], \tag{28} \]

\[ \mu_d = \sqrt{\Lambda/6}, \quad \sin(\mu_d y_H) = \mu_d/\sqrt{\Lambda}. \tag{29} \]

Here $y_H$ represents the position of the horizon in the bulk $dS_5$, where there is no spatial boundary as in $AdS_5$. This configuration represents a brane with $dS_4$ embedded in the bulk $dS_5$ at $y = 0$. The $Z_2$ symmetry is also imposed.

Related to this solution for positive $\Lambda$, we comment on another form of solution [19] which can be solved by the following ansatz,

\[ ds^2 = A^2(y)a_0^2(T)(-dT^2 + \gamma_{ij} dx^i dx^j) + dy^2, \tag{30} \]

where we take as $a(T, y) = n(T, y) = a_0(T)A(y)$. In this case, the solution is obtained for $k = 1$ as

\[ a_0(T) = \frac{l}{\sqrt{\Lambda} \cos(T)}, \quad A(y) = \frac{\sqrt{\Lambda}}{\mu_d} \sin[\mu_d(y_H - |y|)], \tag{31} \]

where $A(y)$ is the same as in Eq. (28) and $a_0$ has a different form. But we can see that this metric is transformed to (28) by a coordinate transformation from $T$ to $t$ by $dT/dt = a_0(t)$. So the properties of this metric will be the same as of the metric given above. Similar coordinate transformations for the other solutions given above would lead to different form of the solutions, but we will not discuss this further here.
2.3 Solutions for $\lambda < 0$

Another type of time-dependent solution occurs for $\lambda < 0$. In this case, we obtain the $AdS_4$ brane only with $k = -1$ as seen from Eq.(19). The solution is obtained as

$$a_0(t) = \frac{1}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda}t), \quad A(y) = \frac{\sqrt{-\lambda}}{\mu} \cosh(\mu[c - |y|]),$$

where $c$ is a constant. (We cannot get this solution for positive $\Lambda$ since $\lambda$ is then positive, as is seen from Eq. (5).) This solution, however, would not represent our universe since we cannot see the graviton as a massless field in this AdS brane world [21].

In any case, it would be important to observe the localization of various fields, especially the graviton, which is needed in the brane-world in order to get a realistic theory. In the next section, we discuss this problem.

3 Localization of gravity

The case of $\lambda = 0$ is well known and studied widely, and it is known that there is no normalizable zero-mode for $\lambda < 0$. So we discuss here the case of $\lambda > 0$, where we can get solutions for both $\Lambda > 0$ and $\Lambda < 0$. Consider the perturbed metric $h_{ij}$ in the form

$$ds^2 = -n^2(t,y)dt^2 + a^2(t,y)[\gamma_{ij}(x^i) + h_{ij}(x^i)]dx^i dx^j + dy^2. \quad (33)$$

We are interested in the localization of the traceless transverse component, which represents the graviton on the brane, of the perturbation. It is projected out by the conditions, $h_{ii} = 0$ and $\nabla_i h^{ij} = 0$, where $\nabla_i$ denotes the covariant derivative with respect to the three-metric $\gamma_{ij}$ which is used to raise and lower the three-indices $i, j$. The transverse and traceless part is denoted by $h$ hereafter for simplicity.

We first consider the simple case where $\Lambda < 0$ and $\gamma_{ij}(x^i) = \delta_{ij}$. Then the transverse and traceless part $h$ is projected out by $\partial_i h^{ij} = 0$ and $h^i_i = 0$, where $\delta_{ij}$ is used to raise and lower the indices $ij$. One arrives at the following linearized equation of $h$ in terms of the five dimensional covariant derivative $\nabla^2_5 = \nabla_M \nabla^M$:

$$\nabla^2_5 h = 0. \quad (34)$$

This is equivalent to the field equation of a five dimensional free scalar. The general form of this linearized equation for (33) is given in [9], so we abbreviate it here.

First, consider the case of (18), where the metric is written as

$$ds^2 = A(y)^2(-dt^2 + a_0(t)^2 \delta_{ij} dx^i dx^j) + dy^2. \quad (35)$$

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In this case, Eq. (34) is written by expanding $h$ in terms of the four-dimensional continuous mass eigenstates:

$$h = \int dm \phi_m(t, x^i) \Phi(m, y),$$

where the mass $m$ is defined by

$$\ddot{\phi}_m + \frac{3 \dot{a}_0}{a_0} \dot{\phi}_m + \frac{-\partial^2}{\partial y^2} \phi_m = -m^2 \phi_m,$$

and $\dot{\cdot} = d/dt$. For $a_0 = 1$, we get the usual relation, $-k^2 = -\gamma^{\mu\nu} k_\mu k_\nu = m^2$, where $\phi = e^{ik_\mu x^\mu}$, $m$ representing the four-dimensional mass. So we can consider $m$ as the mass of the field on the brane. The explicit form of the solution of Eq. (37) is not shown here since it is not used hereafter. The equation for $\Phi(m, y)$ is obtained as

$$\Phi'' + 4 \frac{A'}{A} \Phi' + \frac{m^2}{A^2} \Phi = 0,$$

where $\Phi = A^{-3/2} u(z)$ and $\partial z/\partial y = \pm A^{-1}$. So the eigenvalue $m^2$ should be non-negative, i.e., no tachyon in four dimension. Then the zero mode $m = 0$ is the lowest state which would be localized on the brane. The localization is seen by solving the one-dimensional Schrödinger-like equation in the $y$-direction with the eigenvalue $m^2$, in the form of Eq.(41). The potential $V(z)$ in Eq. (41) is determined by $A(y)$, and it should contain a $\delta$-function attractive force at the brane position to trap the zero-mode of the bulk graviton. Another condition for the localization is the existence of a normalizable state for the wave function of $m = 0$ eigenvalue. For the solution (24) with $\gamma_{ij} = \delta_{ij}$, the discussions are also given in [20, 21].

For the cases of $k = \pm 1$, some of the solutions given above are also considered in [21], especially for $\lambda < 0$ and $k = -1$ (AdS$_4$ brane). An interesting feature is that the graviton on the brane in this case might be massive [21, 22]. We will not discuss this point further here. The procedure to examine the localization for $k = \pm 1$ is parallel to the procedure for $k = 0$. The metric perturbation taken as in Eq. (33) is projected out by the condition $\nabla_i h^{ij} = 0$ applied to the transverse component, and its traceless part is denoted by $h$ as above. It satisfies Eq. (34). By expanding it as in Eq. (36), one obtains the equation for $\phi_m(t, x)$:

$$\ddot{\phi}_m + \frac{2 \dot{a}_0}{a_0} \dot{\phi}_m + \frac{-\nabla^2 + 2k}{a_0^2} \phi_m = -m^2 \phi_m.$$
Similarly to the case of Eq. (37), it is easy to see that \( m \) corresponds to the mass on the brane. And we obtain the same equation (38) as before, for \( \Phi(m, y) \). So in the solution of the above equation the mass has to obey the restriction \( m^2 \geq 0 \).

For any solution, Eq. (38) can be rewritten in terms of \( u(z) \), which is defined by \( \Phi = A^{-3/2}u(z) \) as given above, as follows:

\[
[-\partial_z^2 + V(z)]u(z) = m^2 u(z),
\]

where the potential \( V(z) \) depends on \( A(y) \) as

\[
V(z) = \frac{9}{4} (A')^2 + \frac{3}{2} AA''
\]

Hereafter, we examine the spectrum of \( u(z) \) for various solutions given above.

First, we discuss the solution (28) obtained for \( \Lambda > 0 \). In this case,

\[
z = \text{sgn}(y)(\lambda)^{-1/2} \ln(\cot[\mu_d(y_H - |y|)/2])
\]

where \( V(z) \) is expressed as

\[
V(z) = \frac{15}{4} \lambda \left[-\frac{1}{\cosh^2(\sqrt{\lambda}z)} + \frac{3}{5}\right] - \frac{\kappa^2 \tau}{2} \delta(z - z_0),
\]

\[
z_0 = \frac{1}{\sqrt{\lambda}} \text{arccosh}(\frac{\sqrt{\lambda}}{\mu_d}).
\]

Here \( z_0 \) corresponds to \( y = 0 \), the position of the brane. (Note that \( \sqrt{\lambda}/\mu_d \geq 1 \), because of Eq. (5).) We notice several points with respect to this potential.

1. One sees the presence of an attractive \( \delta \)-function force at the brane for \( \tau > 0 \). Then one would expect one bound state on the brane, and it should be the ground state, i.e., the zero-mode of \( m = 0 \).

2. The potential monotonically increases with \( z \) and approaches

\[
V_{\infty} = \frac{9}{4} \lambda
\]

at \( z = \infty \) or at the horizon \( y = y_H \). (See Fig. 1.) Then one might suppose the existence of discrete Kaluza-Klein (KK) modes in the range \( 0 < m < V_{\infty} \). However, as will be shown below, there is no such discrete mode except from the zero-mode, \( m = 0 \).

3. For \( m > V_{\infty} \), there appear the continuum KK modes. Due to this lower bound of the continuum spectrum of KK modes, the shift from the Newton law on the brane is qualitatively different from the case of the RS brane, where the continuum KK modes are observed with \( m > 0 \).
Fig. 1: The curves $A'(A)$ and $B'(B)$ show the finite part of $V(z)$ given in (43) ((57)) for the parameters $\lambda = 2$ and $\lambda = 1.2$ respectively with $\mu_d = 1(\mu = 1)$. The left hand end-points of each curve represents $V(z_0)$, where $\delta$-function attractive potential appears.

4. The potential takes its minimum,

$$V(z_0) = \frac{5}{8} \lambda \left( \frac{18}{5} - \frac{\Lambda}{\lambda} \right),$$

at $z = z_0$, which is the left-hand end point of curves $A'$ and $B'$ in Fig.1. The value of $V(z_0)$ should be non-negative to confine the zero mode on the brane. This requirement leads to the condition

$$\Lambda \leq \frac{(K^2 \tau)}{2}.$$  \hspace{1cm} (47)

This constraint is necessary for $\Lambda > 0$ since $V(z_0)$ is always positive for $\Lambda < 0$, where volcano type potential is realized although the tail of the mountain does not approach zero but instead $V_\infty = 9\lambda / 4$ (see curves $A$ and $B$ in Fig.1).

5. When one considers the above constraints (47) and (29), the distance between the horizon and the brane is restricted to be

$$y_H \leq \frac{1}{\mu_d} \sin^{-1} \left( \sqrt{\frac{3}{5}} \right).$$  \hspace{1cm} (48)

6. If $\lambda$ is large, implying that also $\sqrt{\lambda/\mu_d} \gg 1$, then the position coordinate of the brane approaches $z_0 = \lambda^{-1/2} \ln(2\sqrt{\lambda/\mu_d})$, which is a small quantity.
The first and second points above are explicitly examined through the solution of Eq. (41), which is given as

\[ u(z) = c_1 X^{-id} {}_2F_1(a, b; c; X) + c_2 X^{id} {}_2F_1(a', b'; c'; X), \]  

(49)

where \(c_{1,2}\) are constants of integration and

\[ X = \frac{1}{\cosh^2(\sqrt{\lambda}z)}, \quad d = \frac{\sqrt{-9 + 4m^2/\lambda}}{4}, \]  

(50)

\[ a = -\frac{3}{4} - id, \quad b = \frac{5}{4} - id, \quad c = 1 - 2id, \]  

(51)

\[ a' = -\frac{3}{4} + id, \quad b' = \frac{5}{4} + id, \quad c' = 1 + 2id. \]  

(52)

Here \( {}_2F_1(a, b; c; X) \) denotes the Gauss’s hypergeometric function. It follows from this solution that \(u(z)\) oscillates with \(z\) when \(m > 3\sqrt{\lambda}/2\), where the continuum KK modes appear. While for \(m < 3\sqrt{\lambda}/2\), \(u(z)\) should decrease rapidly for large \(z\) since the mode in this region should be a bound state. Then one must take \(c_2 = 0\), and this solution must satisfy the boundary condition at \(z = z_0\),

\[ u'(z_0) = -\frac{\kappa^2 r}{4} u(z_0), \]  

(53)

because of the \(\delta\)-function in the potential. However, we find such a state, which satisfies the above condition, only at \(m = 0\) as shown in Fig. 2. Then there is no other bound state than the zero-mode, which is confined on the brane, and the remaining mass eigen-modes, \(m > V_\infty\), are the continuum KK modes with the lower mass-bound.

The next point to be shown for the localization is the normalizability of this zero-mode in the sense

\[ \int_0^{y_0} dy A^2(y) \Phi^2(0, y) < \infty, \]  

(54)

for the zero-mode solution \(\Phi(0, y)\) which is obtained as

\[ \Phi(0, y) = \tilde{c}_1 \frac{\cos(\mu_d |y_H - |y||)}{3 \sin^3(\mu_d |y_H - |y||)} (2 \sin^2(\mu_d |y_H - |y||) + 1) + \tilde{c}_2, \]  

(55)

where \(\tilde{c}_1\) and \(\tilde{c}_2\) are integral constants. This solution must satisfy the boundary condition, \(\Phi'(0, y = 0) = 0\), so \(\tilde{c}_1 = 0\) and \(\Phi(0, y) \equiv \tilde{c}_2\). Then the condition (54) is satisfied. This problem can also be studied in terms of the propagators in the bulk space [23, 24], but it is difficult to present such analysis here.

Next is the case of \(\Lambda < 0\) and \(\lambda > 0\). In this case, \(A(y)\) is given as follows,

\[ A(z) = \frac{\sqrt{\lambda}}{\mu \sinh(\sqrt{\lambda}|z|)}, \]  

(56)
Fig. 2: The curves A and B show $u(z_0)$ and $-4u'(z_0)/(\kappa^2 \tau)$ respectively for $\lambda = 5/3$ and $\mu_d = 1$. Mass $m$ is parameterized as $x = m/(\sqrt{3} \lambda)$. The two curves coincide only at $x = 0$, i.e., at $m = 0$.

where $z = \text{sgn}(y)(\lambda)^{-1/2} \ln(\coth[\mu(y_H - |y|)/2])$ and $V(z)$ is expressed as

$$V(z) = \frac{15}{4} \lambda \left[ \frac{1}{\sinh^2(\sqrt{\lambda} z)} + \frac{3}{5} \right] - \frac{\kappa^2 \tau}{2} \delta(z - z_0).$$  \hspace{1cm} (57)

The position of the brane $z_0$ ($y = 0$) is given by

$$z_0 = \frac{1}{\sqrt{\lambda}} \text{arcsinh}(\frac{\sqrt{\lambda}}{\mu}).$$  \hspace{1cm} (58)

In contrast to the case of $\Lambda > 0$, the potential (57) has a volcano form, although this mountain ends at $z = \infty$ with $V_\infty$ as mentioned above. (See Fig. 1., curves A and B.) However, the discussions are parallel and the conclusions for the localization are the same except for the constraint on $\Lambda$. Namely, (i) Only the zero mode ($m = 0$) is bounded and there is no other bound state in the expected region, $0 < m < 3\sqrt{\lambda}/2$. (ii) Only the continuum KK mode for $m > 3\sqrt{\lambda}/2$ occurs. Also, it is to be noted from Eq. (58) that in the limiting case when $\lambda \to 0$, $z_0$ approaches $z_0 = 1/\mu = \sqrt{-6/\Lambda}$, which is independent of $\tau$.

As in the previous case, these results can be assured by the explicit solution for $u(z)$ with the above potential (57). It is obtained as

$$u(z) = \bar{c}_1 Y^{-id}_2 F_1(a, b; c; -Y) + \bar{c}_2 Y^{id}_2 F_1(a', b'; c'; -Y),$$  \hspace{1cm} (59)

where $\bar{c}_{1,2}$ are integration constants and

$$Y = \frac{1}{\sinh^2(\sqrt{\lambda} z)}. \hspace{1cm} (60)$$
Here \((a, b, c), (a', b', c')\) and \(d\) are the same as those given in Eqs. (50) \(\sim\) (52). The function \(u(z)\) oscillates with \(z\) for \(m > 3\sqrt{\lambda}/2\), where the continuum KK modes appear. For \(m < 3\sqrt{\lambda}/2\), \(u(z)\) decreases rapidly for large \(z\) as before. Then one must take \(\bar{c}_2 = 0\), and this solution must satisfy the boundary condition (53) at \(z = z_0\). Then we find the above condition only at \(m = 0\). This is shown in Fig.3.

![Fig. 3: The curves A and B show \(u(z_0)\) and \(-4u'(z_0)/(\kappa^2\tau)\) respectively for \(\lambda = 10/3\) and \(\mu = 1\). And mass \(m\) is parameterized as \(x = m/(\frac{2}{3}\sqrt{\lambda})\). The two curves coincide only at \(x = 0\), i.e., at \(m = 0\).](image)

As a result, the localization of zero-mode occurs both for positive and negative \(\Lambda\), but the value of \(\Lambda\) is restricted as

\[
-\frac{1}{6} \leq \frac{\Lambda}{\kappa^4\tau^2} < 0, \tag{61}
\]

for \(AdS_5\) bulk space, and

\[
0 < \frac{\Lambda}{\kappa^4\tau^2} \leq \frac{1}{4}, \tag{62}
\]

for \(dS_5\) bulk. The constraint (61) comes from the positivity of \(\lambda\), and (62) represents Eq. (47), which is required in the case of positive \(\Lambda\). So we conclude that the brane with a tiny positive cosmological constant should be embedded in the 5d bulk space with \(\Lambda\) in the range given above in order to localize gravity.
4 Cosmological Implications

4.1 On the conformal temperature

The evolution of our universe consists of four epochs; (1) the early inflationary epoch, (2) the radiation dominated epoch, (3) the matter dominated epoch and (4) the present mini-inflationary epoch. Epochs (1)-(3) are remarkably successful [25]. Epoch (4) is supported by recent observations of Type Ia supernovae and the cosmic microwave background [14, 15, 16], since they indicate that our universe is accelerating. The present universe is then in epoch (4). The energy density of our universe is dominated by the effective cosmological constant \( \lambda \) in epochs (1) and (4), while it is dominated by radiation, \( \rho_r \equiv C_r/a_0(t)^4 \) in (2), and by matter, \( \rho_m \equiv C_m/a_0(t)^3 \) in (3), where \( C_r \) and \( C_m \) are known constants. The present analyses in sections 1-3 are then applicable for (1) and (4). The effective cosmological constant is large (of order \( M_{\text{pl}}^2 \)) in (1) and tiny (of order \( 10^{-122} M_{\text{pl}}^2 \)) in (4), where \( M_{\text{pl}} \) is the Planck mass.

The energy density \( \rho_{DR} \equiv C/a_0(t)^4 \) of the dark radiation in Eq. (11) is constrained by Big-Bang Nucleosynthesis (BBN) [26]. The result is \( \rho_{DR} \lesssim 0.05 \rho_r \). This means that the dark radiation is negligible in epochs (1) and (4). The product ansatz (18) is then true, since it can be obtained by taking \( C = 0 \) from (13).

The dark radiation can be related to the CFT radiation from the viewpoint of the AdS/CFT correspondence [12]. The radiation energy density at the BBN era is obtained as \( \rho_r = g(T_{BBN})T_{BBN}^4 \pi^2/30 \), where \( g(T_{BBN}) \) is the effective number of relativistic degrees of freedom at temperature \( T_{BBN} \) in the BBN era. In the standard model, \( g(T_{BBN}) = 10.75 \). In the four-dimensional conformal symmetric Yang-Mills theory with \( \mathcal{N} = 4 \), the corresponding effective number \( g_{\text{CFT}} \) is \( 15 N^2 \) for SU(N) gauge. The CFT radiation (dark radiation) energy density is then \( \rho_{\text{CFT}} = g_{\text{CFT}} T_{\text{CFT}}^4 \pi^2/30 \). The temperature \( T_{\text{CFT}} \) of CFT differs from real temperature \( T \), since CFT has no coupling with ordinary fields except graviton. Hence, we obtain

\[
\delta \equiv \frac{g_{\text{CFT}} T_{\text{CFT}}^4}{g(T_{BBN})T_{BBN}^4} \lesssim 0.05,
\]

and then

\[
\frac{T_{\text{CFT}}}{T_{BBN}} \lesssim \frac{1}{\sqrt{5N}}.
\]

This ratio is estimated in the BBN era, but it is almost time independent since real and CFT temperatures, \( T \) and \( T_{\text{CFT}} \), are almost proportional to \( 1/a_0(t) \). The ratio has a maximum \( 1/\sqrt{5} \) at \( N = 1 \). Thus, \( T_{\text{CFT}} \) is slightly smaller than \( T \) for \( N = 1 \). From a theoretical point of view, \( N \) should be large. For such large \( N \), \( T_{\text{CFT}} \) is much smaller than \( T \).

It is likely that both \( \rho_{\text{CFT}} \) and \( \rho_r \) are generated in the reheating era just after the early inflation. The constraint (63) for \( \delta \) is somewhat modified in the era, since the effective number \( g(T_*) \) at the era is 106.75 in the standard model and different from \( g(T_{BBN}) \). A precise estimate is possible on the basis of entropy conservation.
\[ a_0(t_{BBN})^3 g(T_{BBN}) T_{BBN}^3 = a_0(t_*)^3 g(T_*) T_3^3 \] and \[ a_0(t_{BBN})^3 g_{CFT} T_{CFT}^3 = a_0(t_*)^3 g_{CFT} T_{CFT}^3, \]
where * stands for the reheating era. For example, \( T_{CFT} \) is the CFT temperature in the era. The ratio \( \delta(t_*) \) at the reheating era is
\[
\delta(t_*) \equiv \frac{g_{CFT} T_{CFT}^4}{g(T_*) T_3^4} \lesssim 0.05 \left( \frac{g(T_*)}{g(T_{BBN})} \right)^{1/3} \sim 0.11.
\] (65)

This ratio can be regarded as a ratio of the coupling of inflaton with CFT fields to that with ordinary fields. The former coupling is thus at least an order of magnitude smaller than the latter.

### 4.2 On viscous cosmology

In cosmological theory, the cosmic fluid with four-velocity \( U^\mu \) is most often taken to be ideal, *i.e.*, to be nonviscous. From a hydrodynamical viewpoint this idealization is almost surprising, all the time that the viscosity property is so often found to be of great physical significance in ordinary hydrodynamics. As one might expect, the viscosity concept has gradually come into more use in cosmology in recent years, and it may seem to be pertinent to deal briefly with this topic here also, emphasizing in particular the close relationship between viscous theories and nonconformally invariant field theories.

Consider the fluid’s energy-momentum tensor (cf., for instance, Ref. [27]):
\[
T_{\mu\nu} = \rho U_\mu U_\nu + (p - \zeta \theta) h_{\mu\nu} - 2 \eta \sigma_{\mu\nu},
\] (66)
where \( \zeta \) is the bulk viscosity and \( \eta \) the shear viscosity, \( h_{\mu\nu} = (g_{\mu\nu} + U_\mu U_\nu) \) the projection tensor, \( \theta = U^\nu \gamma_\nu \) the scalar expansion, and \( \sigma_{\mu\nu} \) the shear tensor. Here we ought to notice that under normal circumstances in the early universe the value of \( \eta \) is enormously larger than the value of \( \zeta \). As an example, if we consider the instant \( t = 1000 \text{ s} \) after Big Bang, meaning that the universe is in the plasma era and is characterized by ionized H and He in equilibrium with radiation, one can estimate on the basis of kinetic theory [28] that the value of \( \eta \) is about \( 2.8 \times 10^{14} \text{ g cm}^{-1} \text{ s}^{-1} \), whereas the value of \( \zeta \) is only about \( 7.0 \times 10^{-3} \text{ g cm}^{-1} \text{ s}^{-1} \). Even a minute deviation from isotropy, such as we encounter in connection with a Kasner metric, for instance, would thus be sufficient to let the strong shear viscosity come into play. However, let us leave this point aside here, and follow common usage in taking the universe to be perfectly homogeneous and isotropic. Then, \( \eta \) can be ignored and we obtain, for a radiation dominated universe with \( p = \rho/3 \) [27],
\[
\frac{d}{dt} (\rho_r a^4) = \zeta \theta^2 a^4,
\] (67)
where \( a \) is the conventional scale factor. The important point in our context is the following: In the presence of viscosity the content \( \rho_r \) of radiation energy is no longer proportional to \( a^{-4} \). We recall that energy density terms of the form \( C/a^4 \) were found above, both in the five-dimensional theory (Eq. (11)), and on the brane (Eq. (15)).
Physically, the inclusion of even a simple bulk viscosity coefficient means a violation of conformal invariance (like mass terms).

5 Concluding remarks

We have examined the localization of gravity on various cosmological branes with a non-trivial curvature. The cosmological constant \( \Lambda \) in the bulk space is considered for both negative and positive values. The latter case \( (\Lambda > 0) \) has recently attracted interest in connection with the proposed dS/CFT correspondence. In both cases, we find a localized zero mode of the gravity fluctuation for a restricted region of \( \Lambda \).

For negative \( \Lambda \), the bulk is asymptotically \( AdS_5 \) and the solution known as RS brane with flat four-dimensional metric can be obtained by fine-tuning the parameters. On this brane, the localization of gravity has been demonstrated by number of previous works. Different choices for the parameters in our model can lead to positive as well as negative values for the four-dimensional cosmological constant \( (\lambda) \). In the case of negative \( \lambda \), \( AdS_4 \) space is obtained as a solution for the brane-world with \( k = -1 \), and we cannot obtain normalizable zero-mode of gravity fluctuations. This implies that we cannot observe the usual four-dimensional gravity on this brane. There is a normalizable zero-mode in the case of positive \( \lambda \), and we find that this mode is confined on the brane. In contrast to the case of \( \lambda = 0 \), the continuous mass of the KK modes has a lower bound which is proportional to \( \lambda \). We also demonstrated that there is no bound-state on the brane below this lower bound other than the zero-mode, the four-dimensional graviton.

For positive \( \Lambda \), the bulk is asymptotically \( dS_5 \) and the brane is realized only for positive \( \lambda \). Although the graviton could be localized on the brane, the situation in the bulk is different from the case of negative \( \Lambda \). In fact, we find a critical value of \( \Lambda \), below which we can see the localized graviton, which could lead to the usual Newton law, on the brane. Above this critical value, the zero mode expands in the fifth dimension and one cannot see the four-dimensional graviton anymore on the brane. The KK mode has the same lower mass-bound as in the case of negative \( \Lambda \), but the wave function is different due to the part dependent on the fifth coordinate. So, the contribution to the shift from the Newton law will be discriminated from the case of negative \( \Lambda \).

Also from the smallness of the present \( \lambda \), the value of the positive \( \Lambda \) should be bounded, but we cannot reject the possibility of positive \( \Lambda \) in considering our brane-world. Our current conclusion is that the value of \( \Lambda \) can be restricted to the region given in Eqs. (61) and (62). Detailed analysis is necessary in order to restrict the parameters region in a realistic brane-world. Moreover, such an analysis may indicate if the bulk space should be de Sitter or anti-de Sitter one.
References

[25] There are many good references. For example, A. R. Liddle and D. H. Lyth, Cosmological inflation and Large-Scale Structure (Cambridge University Press); E. W. Kolb and M. S. Turner, The Early Universe (Perseus Publishing).