(In)finiteness of Spherically Symmetric Static Perfect Fluids
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Abstract

This work is concerned with the finiteness problem for static, spherically symmetric perfect fluids in both Newtonian Gravity and General Relativity. We derive criteria on the barotropic equation of state guaranteeing that the corresponding perfect fluid solutions possess finite/infinite extent. In the Newtonian case, for the large class of monotonic equations of state, and in General Relativity we improve earlier results. Moreover, we are able to treat the two cases in a completely parallel manner, which is accomplished by using a relativistic version of Pohozaev’s identity in the proof of the relativistic criterion. This identity and further generalizations are presented in detail.


Introduction

In this paper we consider non-rotating stellar models, i.e., we consider static, self-gravitating perfect fluids with a barotropic equation of state \( \rho(p) \) relating density and pressure. The basic equations are the Euler-Poisson equations in Newtonian theory and the Euler-Einstein equations in General Relativity. We are concerned with globally regular solutions on \( \mathbb{R}^3 \) consisting of a perfect fluid region and possibly a vacuum region. We focus on spherically symmetric solutions in particular.

In Newtonian theory spherical symmetry is no restriction; static stellar models are necessarily spherically symmetric (see Lindblom:1993 for an overview). In General Relativity, although conjecturable, a general theorem establishing spherical symmetry of solutions does not exist so far, although some progress has been made Beig/Simon:1991, Beig/Simon:1992 Lindblom/Masood-ul-Alam:1994. Some more remarks follow below.


Mathematical issues apart, perfect fluid solutions are of interest in astrophysics. Primarily, perfect fluid solutions represent stellar models (we refer to the classic Chandrasekhar:1957; a recent review addressing some issues is Beig/Schmidt:2000). But also other astrophysical objects as globular clusters possess descriptions in terms of perfect fluid solutions Spitzer:1987.

The main question we pose in this work is the (in)finiteness question. Under which conditions on the equation of state does the corresponding perfect fluid solution possess finite or infinite extent? We briefly recall some criteria existing in the literature: The polytropic equations of state \( p = K \rho^{(n+1)/n} \) \((K > 0, n > 0 \) constants) have been studied extensively, both analytically and numerically. In Newtonian theory finiteness is guaranteed for \( n < 5 \), \( K \) arbitrary (for a general discussion see Horedt:1987), in General Relativity the case is considerably more complex Nilsson/Ugglap:2000. Criteria for more general classes of equations of state are, e.g., the following: In Newtonian theory and in General Relativity, spherical symmetry presupposed, the perfect fluid solution is finite, if \( \rho|_{p=0} > 0 \). This result can be subsumed under the criterion guaranteeing finiteness of the fluid configuration if \( \int_0^\rho dp' \rho'(p')^{-2} < \infty \) holds. In the case \( \rho|_{p=0} = 0 \) monotonicity of the equation of state must be assumed here. As a counterpart to this criterion, under the same assumptions, there exists the following theorem: If \( \int_0^\rho dp' \rho^{-3}(p') < \infty \) (Newtonian theory) or \( \int_0^\rho dp'(p\rho' + c^{-2}p^2)^{-1} < \infty \) (General Relativity), then the fluid solution must necessarily extend to infinity (see Rendall/Schmidt:1991 for a good presentation of these criteria). Note incidentally that the last two quantities will play an important role for our considerations as well (see below). For equations of state of the type \( p = K \rho^{(n+1)/n}(1 + O(\rho^{1/n})) \) \((as \rho \to 0)\) with \( 1 < n < 3 \) finiteness of the fluid solution has been proven in Makino:1998 (some detail follow...
Note that these criteria involve the behavior of the equation of state for small $p$ only. More general criteria, however, must be based on the behavior of the equation of state $\rho(p)$ for all $p$. To show that consider the polytropic equation of state for $n = 5$ in Newtonian theory as an example. Perturbing $\rho(p)$ in a small neighborhood of some finite value of $p$ suffices to produce either finite or infinite fluid configurations. Criteria which are capable of dealing with such phenomena have been derived in Simon:1993 and Simon:2001. Some of those criteria will be reproduced in this paper (see, e.g., theorems J0theorem and relJ0theorem); the other criteria we present here can be regarded as generalizations or modifications in a certain sense. In Simon:2001 the assumptions on the equation of state and the solutions are kept rather general, several theorems are formulated for Sobolev functions. Occasionally we give cross references to Simon:2001.

The paper is divided into three main parts. Part one (Newton): In sections newton:basics–newton:pohozaevandcriteria we treat the (in)finiteness question for perfect fluid solutions in Newtonian theory. Section newton:basics is concerned with the Euler-Poisson equations; in section criteria some quantities are studied which are necessary to formulate the (in)finiteness criteria in section newton:pohozaevandcriteria (theorem J0theorem and theorem J-1theorem). The crucial tool for the proofs of the theorems is Pohozaev’s identity Pohozaev:1965. Part two (General Relativity): Sections euler-einstein–einstein:pohozaevandcriterion deal with general relativistic perfect fluid solutions. The presentation parallels the Newtonian case in order to facilitate comparison; section euler-einstein: the Euler-Einstein equations, section einstein:j0: definitions. The (in)finiteness theorems are formulated in section einstein:pohozaevandcriterion; the main ingredient for the proofs is a relativistic generalization of Pohozaev’s identity. To conclude the first two parts of the paper, in section examples we discuss applications of the criteria we have derived both for the Newtonian and the relativistic case. Part three (Pohozaev-like identities): In section methodofconstruction we present a powerful method of deriving “Pohozaev-like” identities. In particular we treat the relativistic Pohozaev identity which has been used for the proof of theorem relJ0theorem in section einstein:pohozaevandcriterion.

Newton: Basics newton:basics

Newtonian static perfect fluids are regular solutions to the Euler-Poisson equations (on $(\mathbb{R}^3, \delta_{ij})$), given a barotropic equation of state $\rho(p)$ relating the pressure and the density. Subequationspoissonalignpoisson $\Delta u(x) = 4\pi \rho(x)$

In section criteria we discuss which classes of equations of state we consider in this paper (see definition Gammadef ff.). In all cases under consideration the potential $u$ can be viewed as a function of $p$; integrating (euler) we obtain equationup-us $u(p) - u_S = -\int_0^p dp' \rho^{-1}(p') =: -\Gamma(p)$ where $u_S := u|_{p=0}$. 

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