On Quantum Radiation in Curved Spacetime

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Abstract: The issue of quantum field theories of elementary particles in curved space-time.

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to reveal the quantum tunneling nature of the Hawking radiation and its back reaction. The analysis of the Hawking radiation in terms of quantum tunneling effect and pair-creations is essential for explicitly revealing the quantum origin of the Hawking effect, and very important for understanding microscopic origin of entropy in black hole thermodynamics.

In a path-integral framework of quantum field theories, we compute the effective action of transition amplitude from vacuum to vacuum in curved spacetime. In the Schwarzschild geometry, we obtain the imaginary part of effective action, which gives rise to the probability of vacuum decay via quantum tunneling process. As results, we obtain the rate and spectrum of quantum emission of particles and antiparticles from vacuum in curved spacetime, which give rise to the rate and spectrum of the Hawking radiation near to black hole's horizon. We show that this quantum emission is dynamically attributed to (i) quantum-field fluctuations of positive- and negative-energy virtual particles in vacuum are polarized by external gravitational field and (ii) effective mass-gap separating positive-energy particles from negative-energy particles is small. The spectrum of particle creations is determined by quantum emissions(absorptions) from(into) vacuum, in unit of quanta of the discrete spectrum of vacuum, which is due to the presence of gravitational field.

**General formulation.** In order to clearly illustrate physics content, we respectively treat a fermion field $\psi$ or a boson field $\Phi$ as a complex scalar field $\phi$ with the Fermi or Bose statistics. The simplest coordinate-invariant action is given by ($\hbar = c = G = k = 1$)

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ g^{\mu\nu} \dot{\phi}_\mu \dot{\phi}_\nu + (m^2 + \xi R) \phi \phi^* \right],$$

where $m$ is particle mass and $R$ the Riemann scalar. The effective action $S_{\text{eff}}$ is defined as

$$S_{\text{eff}} = -i \ln \langle 0|0 \rangle, \quad \langle 0|0 \rangle = \int [\mathcal{D}\phi \mathcal{D}\phi^*] \exp(iS),$$

where $\langle 0|0 \rangle$ is the transition amplitude from vacuum to vacuum in curved spacetime with geometry $g_{\mu\nu}(x)$.

The quantum scalar field $\phi$ can be in principle expressed in terms of a complete and orthogonal basis of quantum-field states $u_k(x)$:

$$\phi(x) = \sum_k \left( a_k u_k(x) + a_k^\dagger u_k^*(x) \right), \quad \left[ a_k, a_k^\dagger \right]_\pm = \delta_{k,k'}$$

where $a_k^\dagger$ and $a_k$ are creation and annihilation operators of the $k$-th quantum-field state. The quantum field states $\{u_k(x)\}$ obey the following equation,

$$(\Delta_x + m^2 + \xi R)u_k(x) = 0,$$
where $\Delta_x$ is the Laplacian operator in curved spacetime. Based on this basis $\{u_k(x)\}$, we can compute the transition amplitude from vacuum to vacuum as
\begin{equation}
\langle 0 | 0 \rangle = \det \pm 1 (\mathcal{M}),
\end{equation}
\[ \mathcal{M}_{k,k'} = \int d^4x \sqrt{-g} u_k(x)(\Delta_x + m^2 + \xi \mathcal{R}) u_{k'}(x), \]
Diagonizing the hermitian matrix $\mathcal{M}$, we obtain the effective action (2):
\begin{equation}
iS_{\text{eff}} = \pm \int \frac{d^4x d^4k}{(2\pi)^4} \ln(\lambda_k^2),
\end{equation}
where $\lambda_k^2$ denotes the $k$-th eigen-value of the matrix $\mathcal{M}$. The eigen-value $\lambda_k^2$ in the logarithmic function is multiplied by the factor $\sqrt{-g}$, which is not explicitly written. The number of quantum-field states, $\int d^4x d^4k/(2\pi)^4$, is invariant in arbitrary coordinate systems, which is the Liouville theorem for the phase-space invariance. In Eqs.(3,5,6) and henceforth, $(\pm)$ indicates for fermions and bosons.

**Spectrum of vacuum.** As a preliminary study, we assume the geometry of the spacetime outside of a massive star $M$ ($r > 2M$) is stationary and spherical, e.g., the Schwarzschild geometry,
\begin{equation}
ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 + r^2d\Omega^2, \quad g(r) \equiv (1 - \frac{2M}{r})
\end{equation}
where $\Omega$ is the spherical solid angle and $r, \theta, \phi, t$ are the Schwarzschild coordinates. The Riemann scalar $\mathcal{R} = 0$.

In the asymptotically flat space, $2M/r \to 0$ and $g(r) \to 1$, $u_k(x) \sim h_l(k_{r,\infty} r) Y_{lm}(\theta, \phi) e^{i\omega_{\infty} t}$, where $Y_{lm}(\theta, \phi)$ and $h_l(k_{r,\infty} r)$ are standard spherical harmonic and Hankel functions. $\omega_{\infty}$ is the energy-spectrum and the radial momentum $k_{r,\infty}$ is the eigen-value of the operator given by
\begin{equation}
\frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) - \omega^2, \quad \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} - \omega^2.
\end{equation}
On the basis of Hankel functions $h_l(k_{r,\infty} r)$, the eigen-value $\lambda_k^2$ in Eq.(6) is approximately given by,
\begin{equation}
\lambda_k^2 = \omega^2 - \left(k_{r,\infty}^2 + \frac{l(l+1)}{r^2} + m^2\right).
\end{equation}

While, for $2M/r \neq 0$, $u_k(x) \sim R_{\omega}(r) Y_{lm}(\theta, \phi) e^{i\omega t}$. $\omega$ is the energy-spectrum, which is gravitationally red-shifted: $\omega = g^{\frac{1}{2}}(r)\omega_{\infty}$. The radial function $R_{\omega}(r)$ obeys the following differential equation,
\begin{equation}
\left( \omega^2 - \frac{2M}{r^2} \frac{\partial}{\partial r} - V_{\omega}(r) \right) R_{\omega}(r) = 0,
\end{equation}
where \( \hat{k}_r = g(r) \hat{k}_r \big|_\infty \), \( \hat{k}_r^2 = g^2(r) \hat{k}_r^2 \big|_\infty \) and

\[
V_l(r) = g(r) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2 \right].
\]

Eq.(10) is exactly equivalent to the Regge and Wheeler equation. \( R_{\omega l}(r) \) and \( h_l(k_r \big|_\infty r) \) obey Eq.(10) respectively for \( 2M/r \neq 0 \) and \( 2M/r \rightarrow 0 \). Both \( R_{\omega l}(r) \) and \( h_l(k_r \big|_\infty r) \) satisfy the same boundary condition at \( r \rightarrow \infty \). Since \( h_l(k_r \big|_\infty r) \) is a complete set of orthogonal functions, we can expand \( R_{\omega l}(r) \) in terms of \( h_l(k_r \big|_\infty r) \),

\[
R_{\omega l}(r) = \int \frac{dk_r}{2\pi} \left[ b_{\omega l}(k_r \big|_\infty) h_l(k_r \big|_\infty r) + \text{h.c.} \right].
\]

Substituting Eq.(12) into Eq.(10), for \( b_{\omega l}(k_r \big|_\infty) \neq 0 \) we have,

\[
\omega^2 = k_r^2 - \frac{2M}{r^2} k_r + V_l(r),
\]

where \( k_r \simeq g(r) k_r \big|_\infty \) and \( k_r^2 \simeq g^2(r) k_r^2 \big|_\infty \), for simplicity we neglect angular parts and will be back to this point. The eigen-value \( \lambda^2_k \) in Eq.(6) is then approximately given by,

\[
\lambda^2_k = \omega^2 - \left( k_r^2 - \frac{2M}{r^2} k_r + V_l(r) \right),
\]

where \( k_r > 0 \) is for outgoing states of positive-energy particles \( \omega > 0 \) and \( k_r < 0 \) for incoming states of negative energy particles \( \omega < 0 \). This is in accordance with Feynman’s \( \omega \rightarrow \omega \pm i\epsilon \) (\( \epsilon = 2M|\hat{k}_r|/r^2 \)) prescription for particles(+) and antiparticles(-), latter are outgoing states \( (\omega > 0, k_r > 0) \) traveling backward in time. In this prescription, we have \( \omega > 0 \) and \( k_r = |k_r| \).

**Effective action.** We define a quantum shell comprising the spacetime shell \( dtdr \) and the energy-momentum shell \( d\omega dk_r \) at the spacetime coordinates \((t, r)\) and energy-momentum coordinates \((\omega, |k_r|)\). In unit of the number of quantum-field states in this quantum shell, the effective action is:

\[
S_{\text{eff}} = \frac{i}{16\pi} \int_0^\infty \frac{ds}{s} e^{i(s\lambda^2 + i\epsilon)},
\]

where logarithmic function in Eq.(6) is represented by an \( s \)-integration and infrared convergence at \( s \rightarrow 0 \) is insured by \( i\epsilon \) prescription \( (\epsilon \rightarrow 0) \). In order to sum over quantum numbers \( "l, m" \) of angular momenta in Eq.(15), we introduce continuous transverse momenta \( \tilde{k}_\perp \) and \( k_\perp^2 = l(l+1)/r^2 \) so that,

\[
\sum_{lm} e^{-i\beta \frac{l(l+1)}{r^2}} \sim 4\pi r^2 \int \frac{d^2 k_\perp}{(2\pi)^2} e^{-i\beta k_\perp^2} = \frac{r^2}{i\beta}, \]

(16)
for \( \text{Im}(\beta) < 0 \). As a result, we obtain the effective action

\[
S_{\text{eff}} = (\pm) \frac{r^2}{g(r)} \int_0^\infty \frac{ds}{s^4} e^{i\lambda_k^2 s + i\epsilon},
\]

(17)

where and henceforth, \( \lambda_k^2 \) is given by Eq.(14) without the angular term \( l(l+1)/r^2 \).

In order to compute the integration over “s” in Eq.(17), we introduce a complex variable 
\( z = -1 + \delta (|\delta| \to 0) \), and use the following integral representation of the \( \Gamma(z) \)-function by an analytical continuation for \( \text{Im}(\alpha) > 0 \):

\[
\int_0^\infty e^{ia\tau} \tau^{2z-1} d\tau = (\pm i\alpha)^{-z} \Gamma(z),
\]

where \( \alpha \) is dimensionless. This analytical continuation is equivalent to analytical continuation of dimensionality of the momentum-space \( \int \frac{d^3q}{(2\pi)^3} \) in Eq.(16). In the neighborhood of singularity, where \( |\delta| \to 0 \) and \( z \to -1 \) in Eq.(18), we have

\[
\Gamma(z) = -\frac{1}{\delta} - C_\gamma + O(\delta), \quad \alpha^{-z} = \alpha e^{-i\ln \alpha},
\]

\[
(\pm i)^{-z} = -i e^{i\left(\frac{\pi}{2} + 2\pi n\right)}, \quad n' = 0, 1, 2, \cdots
\]

(19)

where \( C_\gamma = 1 - \gamma, \gamma \approx 0.577 \) is the Euler constant and \( 1/\delta \) term corresponds to the ultra-violet divergence of summing over \( (l, m) \) in Eq.(15). Analogously to the treatment of dimensional and \( \zeta \)-functional regularization schemes, using Eqs.(18,19) to calculate integration over “s” in Eq.(17) and keeping up to the order \( O(\delta^*) \), we cast the effective action Eq.(17) to be:

\[
S_{\text{eff}} = (\pm) \frac{r^2 \lambda_k^2}{g(r)} \left[ \frac{\pi}{2} - 2\pi n' + i \left( C_\gamma - \ln \frac{\lambda_k^2}{\mu^2} \right) \right]
\]

(20)

where \( \mu^2 = \sqrt{-g} = 4\pi r^2 \). In the normal prescription of renormalization of quantum field theories, we consistently add an appropriate counterterm to cancel the ultra-violet divergence \( 1/\delta \) at the energy-scale \( \mu \), to obtain a finite effective action (20).

Given \( \lambda_k^2 (14) \), we have

\[
\text{Im} \left( \ln \frac{\lambda_k^2}{\mu^2} \right) = 2\pi n' + \theta, \quad \theta \equiv \tan^{-1} \left( \frac{\text{Im}(\lambda_k^2)}{\text{Re}(\lambda_k^2)} \right),
\]

(21)

\( n' = 0, \pm 1, \pm 2, \cdots \). As a result, per unit of the number of quantum-field states of virtual particles and antiparticles in the quantum shell, the imaginary part of the effective action is given by,

\[
\text{Im}(S_{\text{eff}}) = (\pm) \left[ \frac{2M|k_r|}{g(r)} \left( \frac{\pi}{2} + \theta(|k_r|) - 2\pi n \right) + r^2 \frac{\text{Re}(\lambda_k^2)}{g(r)} \left( C_\gamma - \ln \frac{|\lambda_k^2|}{\mu^2} \right) \right],
\]

(22)
where and henceforth integer \( n = 1, 2, 3, \ldots \).

To obtain \( \text{Im}(S_{\text{eff}}) \) per unit of time and unit of the number of quantum-field states in the phase space \((r, \omega)\), we further assume that energy \( \omega \) and radial momentum \( |k_r| \) of virtual particles and antiparticles in the quantum shell are approximately in “the mass-shell condition” \( \text{Re}(\lambda_k^2) \simeq 0 \) of real particles and antiparticles. The reason will be discussed. Thus, \( \theta \simeq -\frac{\pi}{2} \) and we approximately make integration \( \int \frac{dk_r}{(2\pi)} \) in Eq.(6) in the phase space, with a proper measure \( 2\pi\delta(\text{Re}(\lambda_k^2)) \) in the energy-momentum space \((\omega, |k_r|)\). As a result, \( \text{Im}(S_{\text{eff}}) \) is approximately given by,

\[
\text{Im}(S_{\text{eff}}) \simeq (\pm) \left( -4\pi M |k_r| n \right) |k_r| \simeq \sqrt{\omega^2 - V(r)}, \tag{23}
\]

where the effective mass-gap \( V(r) \) is Eq.(11) without angular term \( l(l+1)/r^2 \).

**Vacuum decay and Radiation.** The imaginary part of \( S_{\text{eff}} \) indicates vacuum decay caused by quantum tunneling of virtual particles, leading to pair-creations of real particles and antiparticles. The probability is given by

\[
|\langle 0 \rangle|^2 = e^{-2\text{Im}S_{\text{eff}}}, \tag{24}
\]

per unit of time, and the number of quantum-field states in the phase space \((\omega, r)\).

The integer \( n \) in Eq.(22) describes all possible quantum bound states of vacuum in the presence of gravitational potential,

\[
\int dr' |k_r| = \int dr' \sqrt{\omega^2 - V(r')} \simeq \left( n + \frac{1}{2} \right) |k_r| \simeq \frac{n}{r}. \tag{25}
\]

We sum over all these possible quantum states “\( n \)” in Eq.(22) with respect to either the Fermi-distribution or Bose-distribution in the occupation of these quantum states of vacuum. The rate of vacuum decay is given as,

\[
|\langle 0 \rangle|^2 \simeq \frac{1}{\exp\left( 8M\pi |k_r| \right) \pm 1}, \tag{26}
\]

per unit of the number of quantum-field states of real particles and antiparticles created in the phase-space \((\omega, r)\).

Regarding these outgoing states as an outward radiation flux, Eq.(26) gives the rate and spectrum of such a radiation from the spatial shell \( r \to r + dr \), with respect to a static observer located at \( r \). While, with respect to an infinity observer, the rate and spectrum of
such a radiation are given by Eq.(26) with replacing $\omega \rightarrow \omega_\infty$ and $V(r) \rightarrow m^2 + 2M/r^3$, for the gravitational red-shift.

In the case that black hole’s horizon is formed, the effective mass-gap vanishes $V(r) \sim 0$ for $r \sim r_+ \equiv 2M$ and $\omega \simeq |k_\rho| < 1/r_+$ for massless particles $m = 0$. Eq.(26) gives the emission rate and black-body spectrum of the Hawking radiation with temperature $T = 1/(8\pi M)$. In the proper frame of a free-falling observer passing the point $r$ at time $t$, the rate of radiation is Eq.(26) multiplied by $1/\sqrt{g(r)}$ and the energy of radiation is $\omega_\infty$ with respect to his proper time $\tau$. The characteristic energy scale (temperature) of radiation is $1/(8\pi M \sqrt{g(r)})$, relating to the acceleration of the observer $a = M/(r^2 \sqrt{g(r)})$. This is reminiscent of the Unruh effect[5].

Discussions. We turn to discussions of the dynamical reason for such a vacuum decay and particle creations. Quantum-field fluctuations in vacuum indicate pair creations and annihilations of positive- and negative-energy virtual particles, represented by closed fermion-loops in Feynman diagrams. In the presence of external gravitational field, pair-creation process is energetically favorable for it approximately gains a gravitational energy (per quantum state);

$$
\delta E \simeq \frac{M|\xi|}{r + |\xi|/2} - \frac{M|\xi|}{r - |\xi|/2} \sim \frac{M|\delta \omega|}{2\pi r^2} |\delta x| \simeq \frac{M}{2\pi r^2},
$$

(27)

where $\delta x$ is the separation of positive- and negative-energy virtual particles in spacetime and $\delta \omega$ is the variation of their energy-momenta ($\delta x \delta \omega \simeq 1$). This shows that gravitational field polarizes vacuum by separating positive-energy virtual particles from negative-energy virtual particles, analogously to the phenomenon of electric field polarizing vacuum. This gravitational polarization is characterized by the energy scale $|\delta E|$. As a consequence, vacuum possibly decays via pair-creation process. To have a large probability of vacuum decay, $|\delta E|$ must be larger than effective mass-gap $V(r)$ that is the energy-gap between positive-energy particles and negative-energy particles. In a local proper frame, $\delta E$ is related to the acceleration of observer by the equivalent principle.

Even on black hole’s horizon, $|\delta E|$ is very small, compared with neutrino masses $m$. However, $V(r)$ in Eq.(23) almost vanishes around black hole’s horizon. By quantum-field fluctuations, virtual particles that are in quasi zero-energy states $\omega \sim 0^-$ just bellow the zero-energy level of vacuum, turn to be real particles that are in quasi zero-energy states
\(\omega \sim 0^+\) just above the zero-energy level of vacuum, since it almost costs no energy for such a quantum tunneling process crossing the zero-energy level of vacuum and leading to pair-creation. These particles in quasi zero-energy states have a typical energy-scale \(\delta E \sim 1/(8\pi M)\) \((27)\), which in fact determines the Hawking temperature. The effective mass-gap \(V(r \sim r_+) \sim \delta E\).

In the region away from black hole’s horizon \((r > r_+)\) and massless particles \(m = 0\), the effective mass-gap \(V(r)\) owing to the potential term \(2M/r^3\) is extremely small and comparable with the energy-scale \(\delta E\). Pair-creation rate is given by Eq.\((26)\) in very low-energy regime \((\sim 1/r)\). This could be case for extremely low-energy emission of massless particles. However, all known fermion masses are not zero and \(V(r)\) is much larger than \(\delta E\). As a consequence, the rate \((26)\) of quantum tunneling effect is exponentially suppressed \((\sim e^{-8\pi Mm} \sim 0)\).

Due to the smallness of the effective mass-gap \(V(r) \sim 0\), real particles and antiparticles, which are created, are in the mass-shell condition \(\omega = |k_r| \sim 0^+\), which are quasi zero-energy states \(\omega \sim 0^+\). By the continuation of energy-momentum dispersion relation, virtual particles and antiparticles, which are in quasi zero-energy states \(\omega \sim 0^-\), must be in an approximate mass-shell condition \(\omega \sim -|k_r| \sim 0^-\). This justifies the “mass-shell condition” \(\text{Re}(\lambda_+^2) \sim 0^-\), and \(\text{Re}(\lambda_-^2) < 0\) is due to virtual particles bounded by the effective mass-gap \(V(r)\).

Attributed to the nature of quantum-field fluctuations of virtual particles tunneling through a very small effective mass-gap, vacuum decays, leading to pair-creations of real particles and antiparticles. Accordingly, the nature of quantum-field fluctuations of real particles and anti-particles, clearly implies the inverse process: pairs of real particles and antiparticles annihilate into virtual particles and antiparticles in vacuum. The rate of pair-annihilation process must be the same as the rate \((26)\) of pair-creation process, as the CTP invariance is preserved in such processes. In these emission(creation) and absorption(annihilation) processes, the vacuum acts as a reserve of temperature given by Eq.\((27)\). In the presence of gravitational field, the spectrum of this reserve(vacuum) is quantized and described by the integer \(n\) and quanta \(|k_r|\) (see Eq.\((25)\)). Particle creations(emissions) from vacuum and annihilation(absorptions) into vacuum are permitted, if these processes take place in unit of quanta \(|k_r|\). The detail balance of these emission and absorption processes in unit of quanta \(|k_r|\), leads to the black-body spectrum \((26)\) of particle creations. In general,
such a spectrum is different from black-body one up to a gray factor, since \( \text{Re}(\lambda_k^2) \) is not exactly zero for virtual particles in quasi zero-energy states \( (\omega \sim 0^-) \) and angular parts are neglected in Eq.(13). This is actually because of particle creation and annihilation processes scattered by the potential terms \( \frac{2M}{r^2} \), \( l(l + 1)/r^2 \) and \( g(r) \) in the effective mass-gap \( V(r) \).


