\begin{equation}
\frac{\partial}{\partial t} \frac{\partial}{\partial t'} \Delta \frac{\partial}{\partial \Delta} = \rho \Delta
\end{equation}

where the HOLO tensor is given by

\[ \Delta = (\Delta + \lambda) \Delta \]

The non-linear term in the right-hand side is the interaction term of the form

\[ i \phi \Delta \Delta \Delta \rho \Delta = \rho \Delta \phi \]

\[ i \phi \Delta \rho \Delta = \Delta \phi \]

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We need to express the non-linear term in the interaction term of the form

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FIG. 1: Conformal diagram for the case when $\alpha$ is bounded on $\mathbb{R}$, $v$ lies in the range $(-\infty, +\infty)$. Singularities are indicated with a double line.

FIG. 2: Conformal diagram for the case when $\alpha$ is bounded below on $\mathbb{R}$, but there exists $v_F \in \mathbb{R}$ such that $\alpha \to +\infty$ as $v \downarrow v_F$, $v$ lies in the range $(v_F, +\infty)$.

FIG. 3: Conformal diagram for the case when $\alpha$ is bounded above on $\mathbb{R}$, but there exists $v_F \in \mathbb{R}$ such that $\alpha \to -\infty$ as $v \uparrow v_F$, $v$ lies in the range $(-\infty, v_F)$.

and the energy condition now reads

$$\beta(U) \geq 0. \quad (6)$$

The relevant conformal diagrams are obtained by inverting those of Figures 1-4.

In order to make the connection with cosmic censorship, i.e. to study collapse for an initially regular configuration, we replace the interior of a past light cone of a point on $r = 0$ with a portion of Minkowski space. The resulting space-time are asymptotically flat and evolve from regular initial data. They may be considered the cylindrical version of Vaidya space-time. Writing the line element of Minkowski space-time in the cylindrical form

$$ds^2 = -d\tilde{t}^2 + d\tilde{r}^2 + \frac{1}{4}(\tilde{\varphi} - \tilde{\varphi}_0)^2 d\tilde{\varphi}^2,$$

we obtain a smooth matching across an ingoing null hypersurface by taking $\tilde{r}^2 = r^2$, matching across $\tilde{t} = 0$ (a translation may be required to guarantee that $0$ lies in the domain of $\alpha$ and that $v_F < 0 < v_F$ if such exist) and using the boundary condition $\alpha(0) = 0$. We paste the portion $v < 0$ of Minkowski space-time to the portion $v \geq 0$ of the space-time with line element (1).

Two possibilities arise and are shown in Figures 5 and 6 respective. Both contain locally naked singularities. Figure 5 corresponds to space-times modelling the gravitational collapse of cylindrical null dust which are asymptotically flat at fixed $z$ and which arise from regular initial data. The singularity in this case is globally naked.

FIG. 4: Conformal diagram for the case when there exists $v_F, v_F \in \mathbb{R}$ such that $\alpha \to +\infty$ as $v \downarrow v_F$ and $\alpha \to -\infty$ as $v \uparrow v_F$, $v$ lies in the range $(v_F, v_F)$.

FIG. 5: Conformal diagram in the $u - v$ plane for the case when $\alpha$ is bounded below on $\mathbb{R}$. The dashed line is $v = 0$; the portion to the past of $v = 0$ is flat.

A slightly different model has been studied in [4], [5] and [6]. Here, the exterior region is taken to be filled with outgoing null dust, and is matched across a time-like shell of (ordinary) dust to a flat interior. One can
interpret this as a collapsing cylindrical shell emitting both gravitational radiation (the solution is Petrov type \( N \)) and massless particles. It was shown in [5] and [6] that this shell can be constructed in such a manner that the collapse proceeds to \( r = 0 \) without any trapped or marginally trapped surfaces appearing either on the shell or in the exterior geometry. This leads one to suspect that the resulting singularity may be naked; the present results confirm this. The shell is constructed as follows. The metric inside the shell is taken to be flat:

\[
ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\phi^2.
\]

The exterior geometry is given by (5). Define \( T \) by \( U = T - R \), \( V = T + R \), and take the shell to be at \( R = R_0(T) = r_0(t) \). Demanding continuity of the metric yields a relation for \( t \) in terms of \( T \). The surface energy-momentum tensor of the shell general takes the form of an imperfect fluid; following [4], we impose the condition that there are stresses only in the azimuthal direction. This choice of matter admits the interpretation of a thin shell of dust particles in which counter-rotation of one half of the particles results in zero net angular momentum, and is referred to as a “shell of counter-rotating dust”[4, 6, 7]. The field equations for the shell yield the following differential equation for \( R_0(T) \) which includes the terms \( \beta(U_\parallel), \beta(U_\perp) \), where \( U_\parallel = T - R_0(T) \):

\[
R_0^2 = (1 - R_0^2) \left\{ \frac{\Delta^{1/2}}{R_0} + \frac{R_0 - \Delta^{1/2} \beta}{\Delta^{1/2} - 1} \left( 1 - R_0^2 \right) \right\}, \quad \text{(7)}
\]

where

\[
\Delta = \frac{R_0^2}{1 - R_0^2} + e^{a \beta \tau}(1 - R_0^2).
\]

(An erroneous version of this equation was given in [4] and was corrected in [6].)

The solutions studied in [4], [5] and [6] are similar to the following simple example. For our present purposes, it suffices to show that there exists a function \( \beta(U_\parallel) \) and a solution \( R_0(T) \) of (7) with the following properties: \( R_0(T) \) decreases monotonically from a finite positive value to \( R_0 = 0 \) in finite proper time; \( |R_0'| < 1; \beta(U_\parallel) \geq 0; \Delta > 0 \) along the solution; a suitable energy condition is satisfied by the shell. In order to do this, we make the choice \( R_0(T) = a - bT \) where \( a, b > 0 \) and \( b < 1 \). Then (7) may be written as the following equation for \( y(T) := \beta(T - R_0(T)) \) where \( \tau = -\ln|a - bT| \):

\[
\frac{dy}{d\tau} = \frac{2 \Delta - \Delta^{1/2}}{b - b + \Delta^{1/2}}.
\]

The right-hand side here is a \( C^1 \) function of \( y \), and so this equation admits a unique solution through any point \((\tau, y)\). If we choose data with \( y < 0 \), we see that \( y \) increases towards \( y = 0 \) asymptotically as \( \tau \to \infty \). Such a solution cannot reach \( y = 0 \) in a finite time since \( y \equiv 0 \) is the unique solution through any point \((\tau, y) = (\tau, 0)\). In other words, there exist solutions of (7) for which \( R_0 \) decreases to zero in finite time \( T = a/b \), with \( \beta \) satisfying \( \beta_0 \leq \beta \leq 0 \) on \( T \in [0, a/b] \). The shell has density \( \rho \) and azimuthal stress \( p_{\phi} \) satisfying

\[
\rho = p_{\phi} = e^{\beta/2} \left( 1 - R_0^2 \right)^{-1/2} R_0^{-1} \left( \Delta^{1/2} - 1 \right),
\]

which is clearly positive for the solution constructed, and the other conditions mentioned above are clearly satisfied. We note that Gleiher’s general analysis shows that there are numerous possibilities which lead to the outcome exemplified by this case [6].

To see that the resulting singularity is globally naked, we note that since \( \beta(U_\parallel) \leq 0 \) during the collapse, \( \beta \) cannot diverge to \( +\infty \) before the shell undergoes complete collapse and hence the conformal diagram for the matched space-time must be of the form shown in either Figure 7 or Figure 8. When there is no future singularity \( U_F \), i.e. when Figure 7 applies, the space-times give more examples of cylindrical collapse resulting in a globally naked singularity. The space-time is asymptotically flat at \( J^+ (r \to \infty \) at fixed \( z) \) for fixed \( z \).
FIG. 8: Conformal diagram in the $U-V$ plane for collapse of a shell of counter-rotating dust particles. The dashed curve represent the shell; space-time is flat in the interior of the shell. The exterior corresponds to the time-reversal of Figures 2 or 4 with $U_F = -v_F$. The possible past singularity is ignored.

Element (1) can be matched smoothly across any surface $z = \text{constant}$ with the spherically symmetric space-time with line element

$$ds^2 = -\exp(-\alpha(v)) du dv + r^2 (u, v) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (9)$$

provided the matching is done across the equator $\theta = \pi/2$ of the spherically symmetric space-time. Doing this at two different values of $z$ and choosing normals in the appropriate directions corresponds [10] to replacing the infinite cylinder by a finite cylinder bounded by hemispherical caps [8]. The Ricci tensor of (9) has non-vanishing components

$$R_{uv} = -\frac{\alpha'}{4r}, \quad R_{\theta\theta} = \cos^2 \theta R_{\phi\phi} = 1 - e^\alpha.$$

The models of Figures 5 and 6 used the boundary condition $\alpha(0) = 0$, and the energy conditions for the cylinder gave $\alpha'(v) \leq 0$. Using these, it is straightforward to show that all energy conditions (weak, strong, dominant) are satisfied in the hemispheres. A similar conclusion holds for the off-shell regions of the models of Figures 7 and 8 using the condition $\beta \leq 0$ found above. The collapsing shell of matter now has a finite cylindrical body and hemispherical caps. We take its equation of motion to be given by the same equation $r = r_0(T/\tau)$ in both the cylindrical and spherical regions (this is in line with our ‘recycling’ of the cylindrical metric functions in the spherical regions). It is straightforward to show that the surface energy-momentum tensor of the shell in the cylindrical region is given by [4, 5, 6]

$$S_{ab} = \rho \delta^2 \delta_a \delta_b + p_v \delta^2 \delta_a \delta_b + \rho r^2 \sin^2 \theta \delta_a \delta_b.$$  

Then it transpires that the surface energy-momentum tensor in the spherical region is

$$S_{ab} = 2\rho \delta^2 \delta_a \delta_b + p_v \delta^2 \delta_a \delta_b + (p_v + p) r^2 \sin^2 \theta \delta_a \delta_b.$$  

Imposing the ‘counter-rotating dust’ condition $p_v = 0$ shows that the collapsing shell is composed of dust with no tangential stresses in the hemispheres. In particular, the energy conditions in the hemispheres are inherited from the energy conditions in the cylindrical region. Asymptotic flatness in the hemispherical regions is guaranteed by asymptotic flatness at fixed $z$ in the cylindrical region. The properties of the singularities which we have examined depend only on the geometry of the Lorentzian $\mathbb{R}^2$-space $z = \text{constant}$, $\theta = \text{constant}$ of (1).

Since this 2-space is identical to the 2-space $\theta = \text{constant}$, $\phi = \text{constant}$ of (9), the singularity structure in the cylindrical region applies throughout the entire space-time.

Finally, we note the unsuitability of these models for studying the hoop conjecture. Since the hemispheres may be attached at any pair of values of $z$, the resulting object may have an arbitrary degree of prolateness, from zero (spherical) to infinity (infinite cylinder). Thus the lack of occurrence of an apparent horizon (i.e. trapped surfaces) relates only to the particular choice of metric functions and not to the asphericity of the collapsing object.

I am grateful to S. Gonçalves and S. Jhingan for allowing me to read their work [5] prior to publication.