A UNIVERSAL PROBABILITY DISTRIBUTION FUNCTION FOR WEAK-LENSING AMPLIFICATION

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Abstract

We present an approximate form for the weak lensing magnification distribution of standard candles, valid for all cosmological models, with arbitrary matter distributions, over all redshifts. Our results are based on a universal probability distribution function (UPDF), \( P(\eta) \), for the reduced convergence, \( \eta \). For a given cosmological model, the magnification probability distribution, \( P(\mu) \), at redshift \( z \) is related to the UPDF by \( P(\mu) = P(\eta)/2|\kappa_{\text{min}}| \), where \( \eta = 1 + (\mu - 1)/2|\kappa_{\text{min}}| \), and \( \kappa_{\text{min}} \) (the minimum convergence) can be directly computed from the cosmological parameters (\( \Omega_m \) and \( \Omega_\Lambda \)). We show that the UPDF can be well approximated by a three-parameter stretched Gaussian distribution, where the values of the three parameters depend only on \( \xi_\eta \), the variance of \( \eta \). In short, all possible weak lensing probability distributions can be well approximated by a one-parameter family. We establish this family, normalizing to the numerical ray-shooting results for a \( \Lambda \)CDM model by Wambsganss et al. (1997). Each alternative cosmological model is then described by a single function \( \xi_\eta(z) \). We find that this method gives \( P(\mu) \) in excellent agreement with numerical ray-shooting calculations, and provide numerical fits for three representative models (SCDM, \( \Lambda \)CDM, and OCDM). Our results provide an easy, accurate, and efficient method to calculate the weak lensing magnification distribution of standard candles, and should be useful in the analysis of future high-redshift supernova data.

Subject headings: cosmology: observations—cosmology: theory—gravitational lensing

1. INTRODUCTION

The luminosity distance-redshift relations of cosmological standard candles provide a powerful probe of the cosmological parameters \( H_0, \Omega_m \), and \( \Omega_\Lambda \) (Garnavich et al. 1998a; Perlmutter et al. 1999; Wang 2000b; Branch et al. 2001), as well as of the nature of the dark energy (Garnavich et al. 1998b; White 1998; Podariu & Ratra 2000; Waga & Frieman 2000; Maor et al. 2001; Podariu, Nugent, & Ratra 2001; Wang & Garnavich 2001; Wang & Lovelace 2001; Kujat et al. 2002). At present, type Ia supernovae (SNe Ia) are our best candidates for cosmological standard candles (Phillips 1993; Riess, Press, & Kirshner 1995). The main systematic uncertainties of SNe Ia as cosmological standard candles are weak gravitational lensing (Kantowski et al. 1995; Frieman 1997; Wang et al. 1997; Holz 1998; Holz & Wald 1998; Wang 1999; Valageas 2000a,b; Munshi & Jain 2000; Barber et al. 2000; Premadi et al. 2001), and luminosity evolution (Drell, Loredo, & Wasserman 2000; Riess et al. 1999; Wang 2000b). Future SN surveys (Wang 2000a, SNAP\(^1\)) could yield thousands of SNe Ia out to redshifts of a few. Since the effect of weak lensing increases with redshift, the appropriate modeling of the weak lensing of high-redshift SNe Ia will be important in the correct interpretation of future data. In addition, with high statistics it may be possible to directly measure the lensing distributions, and thereby infer properties of the dark matter (Metcalf & Silk 1999; Seljak & Holz 1999).

In general, determining the magnification distributions of standard candles due to weak lensing is a laborious and time-consuming process, involving such techniques as ray-tracing through N-body codes or Monte-Carlo approximations to inhomogeneous universes. Here we present an easy, accurate, and efficient method to calculate the weak lensing magnification distribution of standard candles, \( P(\mu) \). Our method avails itself of a universal probability distribution function (UPDF), \( P(\eta) \), which we fit to a simple analytic form (normalized by the cosmological N-body simulations of Wambsganss et al. (1997)). All magnification probability distributions, for all cosmological models over all redshifts, can then be approximated by a one-parameter family of solutions. The underlying fundamental parameter is \( \xi_\eta \), the variance of the reduced convergence, \( \eta \). To determine the magnification PDF for a given model it is thus sufficient to determine \( \xi_\eta(z) \) for that model. We demonstrate this method with a number of examples, and provide fitting formulae for three fiducial cosmologies (see Table 1).

\(^1\) see http://snap.lbl.gov

Table 1
Three fiducial models
2. WEAK LENSING OF POINT SOURCES

Due to the deflection of light by density fluctuations along the line of sight, a source (at redshift $z_s$) will be magnified by a factor $\mu = 1 + 2\kappa$, where the convergence $\kappa$ is given by (Bernardeau, Van Waerbeke, & Mellier 1997; Kaiser 1998)

$$\kappa \equiv 3 \Omega_m \int_0^{X_s} \frac{d\chi}{\sqrt{\Omega_\Lambda + \Omega_k(1+z)^2 + \Omega_m(1+z)^3}} w(\chi, \chi_s) \delta(\chi),$$

with

$$\frac{d\chi}{\sqrt{\Omega_k}} \equiv \frac{c H_0^{-1}}{\sqrt{\Omega_k}} dz,$$

$$w(\chi, \chi_s) = \frac{H_0^2}{c^2} \frac{D(\chi)}{D(\chi_s)} (1+z),$$

$$D(\chi) = \frac{c H_0^{-1}}{\sqrt{\Omega_k}} \sinh \left( \sqrt{\Omega_k} |\chi| \right),$$

and where $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$, and “sinn” is defined as sinh if $\Omega_k > 0$, and sin if $\Omega_k < 0$. If $\Omega_k = 0$, the sinn and $\Omega_\Lambda$’s disappear. The density contrast $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$. Since $\rho \geq 0$, there exists a minimum value of the convergence:

$$\kappa_{\text{min}} = -\frac{3}{2} \Omega_m \int_0^{X_s} \frac{d\chi}{\sqrt{\Omega_k}} w(\chi, \chi_s).$$

The minimum magnification is thus given by $\mu_{\text{min}} = 1 + 2\kappa_{\text{min}}$.

Now we define (Valageas 2000a)

$$\eta \equiv \frac{\mu - \mu_{\text{min}}}{1 - \mu_{\text{min}}} = 1 + \frac{\kappa}{|\kappa_{\text{min}}|} = \frac{\int_0^{X_s} \frac{d\chi}{\sqrt{\Omega_k}} w(\chi, \chi_s) (\rho/\bar{\rho})}{\int_0^{X_s} \frac{d\chi}{\sqrt{\Omega_k}} w(\chi, \chi_s)},$$

Note that $\eta$ is the average matter density relative to the global mean, weighted by the gravitational lensing cross section of a unit mass lens along the line of sight to the source. This is the same as the direction-dependent smoothness parameter introduced by Wang (1999) in the weak lensing limit (Wang, in preparation).

The variance of $\eta$ is given by

$$\xi_\eta = \int_0^{X_s} \frac{d\chi}{F_s} \left( \frac{w}{F_s} \right)^2 I_\mu(\chi),$$

with

$$F_s = \int_0^{X_s} \frac{d\chi}{\sqrt{\Omega_k}} w(\chi, \chi_s),$$

$$I_\mu(z) = \pi \int_0^{\infty} \frac{dk}{k} \frac{\Delta^2(k, z)}{k},$$

and where $\Delta^2(k, z) = 4\pi k^3 P(k, z)$. Using the hierarchical ansatz to model non-linear gravitational clustering (Balian & Schaeffer 1989), Valageas (2000a, b) showed that

$$P(\eta) = \int_{-\infty}^{\infty} \frac{dy}{2\pi i \xi_\eta} e^{i[y - \phi_\eta(y)]/\xi_\eta},$$

where $\phi_\eta(y) \simeq \int_{-\infty}^{\infty} dx \left( 1 - e^{-x^2} \right) h(x)$. The scaling function $h(x)$ can be obtained from numerical simulations of large scale structure. We found that the scaling function given by Valageas (2000a) leads to large errors in $P(\eta)$ for small $\xi_\eta$, making it less useful for calculating $P(\mu)$ at higher redshifts. Although $P(\mu)$ becomes increasingly broad as source redshift increases, $P(\eta)$ becomes increasingly narrow, since the universe becomes more smoothly distributed at high $z$ (Wang 1999).

3. THE UNIVERSAL PROBABILITY DISTRIBUTION FUNCTION

Munshi & Jain (2000) showed that $P(\eta)$ is independent of the background geometry of the universe, as can be seen from equation (5): since weak lensing contributions are dominated by a narrow range of the matter power spectrum, the scaling function $h(x)$ is independent of cosmological parameters, and hence $P(\eta)$ has no explicit dependence on cosmology (Munshi & Jain 2000). Thus the dependence of $P(\eta)$ on cosmology enters entirely through the variance, $\xi_\eta$. We can determine the functional form of $P(\eta|\xi_\eta)$ by fitting it to accurate calculations of $P(\mu)$ for any cosmological model. The amplification distribution, $P(\mu)$, for arbitrary alternative cosmological models can then be found by computing the appropriate $\kappa_{\text{min}}$ [equation (2)] and $\xi_\eta$. Utilizing $\mu = 1 + 2\kappa_{\text{min}}|\eta - 1|$ we find

$$P(\mu) = \frac{P(\eta|\xi_\eta)}{2\kappa_{\text{min}}}. \quad (6)$$

We call $P(\eta)$ the universal probability distribution function (UPDF), as this one-parameter family of solutions underlies all magnification PDFs for all cosmologies, at all redshifts.

Figure 1 shows $\xi_\eta$ and $\kappa_{\text{min}}$, for the three cosmological models from Table 1. For illustration, we give accurate fitting formulae for the curves in Figure 1:

$$\xi_\eta(\theta_0=1')^{\text{SCDM}} = 0.032 + \frac{0.986}{5z} - \frac{0.452}{(5z)^2} + \frac{0.114}{(5z)^3}$$

$$\xi_\eta(\theta_0=1')^{\text{LCDM}} = 0.021 + \frac{1.384}{5z} - \frac{0.642}{(5z)^2} + \frac{0.147}{(5z)^3} \quad (7)$$

<table>
<thead>
<tr>
<th>Model</th>
<th>$\Omega_m$</th>
<th>$\Omega_\Lambda$</th>
<th>$k$</th>
<th>$\sigma_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCDM</td>
<td>0.3</td>
<td>0.7</td>
<td>0.9</td>
<td>0.85</td>
</tr>
<tr>
<td>LCDM</td>
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<td>0.7</td>
<td>0.9</td>
<td>0.85</td>
</tr>
<tr>
<td>ΩCDM</td>
<td>0.3</td>
<td>0.7</td>
<td>0.9</td>
<td>0.85</td>
</tr>
</tbody>
</table>
solid curves are \( \kappa_{\text{min}}^{\text{CDM}} \) independent of \( \eta \). The parameter \( q \) together with \( P \) of Barber et al. (2000). We extract the UPDF, \( P(\eta) \), from ray-tracing within the large scale structure simulations of Wambgsanss et al. (1997). We then fit the UPDF to the stretched Gaussian (Wang 1999):

\[
P(\eta|\xi_\eta) = C_{\text{norm}} \exp \left[ -\frac{(\eta - \eta_{\text{peak}})^2}{w \eta^q} \right], \tag{8}
\]

where \( C_{\text{norm}} \), \( \eta_{\text{peak}} \), \( w \), and \( q \) depend solely on \( \xi_\eta \) and are independent of \( \eta \). \( C_{\text{norm}}(\xi_\eta) \) is a normalization constant, chosen so that \( \int_0^\infty P(\eta) \, d\eta = 1 \). Figure 2 shows \( \eta_{\text{peak}}, w, \) and \( q \) as functions of \( \xi_\eta \). The points denoted by crosses are extracted from the numerical \( P(\mu|z) \) by Wambgsanss et al. (1997) for a \( \Lambda \)CDM model with \( \Omega_m = 0.4, \Omega_\Lambda = 0.6 \); the solid curves are \( (\chi^2 \text{ minimizing}) \) best fits to the crosses:

\[
\eta_{\text{peak}}(\xi_\eta) = 1.002 - 1.145 \left( \frac{\xi_\eta}{5} \right) - 20.427 \left( \frac{\xi_\eta}{5} \right)^2,
\]

\[
w(\xi_\eta) = 0.028 + 3.952 \left( \frac{\xi_\eta}{5} \right) - 1.262 \left( \frac{\xi_\eta}{5} \right)^2, \tag{9}
\]

\[
q(\xi_\eta) = 0.702 + 0.509 \left( \frac{1}{\delta \xi_\eta} \right) + 0.008 \left( \frac{1}{\delta \xi_\eta} \right)^2.
\]

The parameter \( \eta_{\text{peak}}(\xi_\eta) \) indicates the average smoothness of a universe; it increases with decreasing \( \xi_\eta \) (i.e., increasing \( z \)) and approaches \( \eta_{\text{peak}}(\xi_\eta) = 1 \) for \( \xi_\eta \rightarrow 0 \). The parameter \( w(\xi_\eta) \) indicates the width of the distribution in the smoothness parameter \( \eta \); it decreases with decreasing \( \xi_\eta \) (i.e., increasing \( z \)). The \( \xi_\eta \) dependencies of \( \eta_{\text{peak}}(\xi_\eta) \) and \( w(\xi_\eta) \) are as expected because as we look back to earlier times, lines of sight sample more of the universe, and the universe becomes smoother on average. The parameter \( q(\xi_\eta) \) indicates the deviation of \( P(\eta|\xi_\eta) \) from Gaussianity (which corresponds to \( q = 0 \)).

4. Comparison with other published results

The universal probability distribution function, \( P(\eta) \), encapsulated in eqs. (8) and (9), can now be used to determine the magnification probability distribution, \( P(\mu) \), for arbitrary cosmological models at arbitrary redshifts. For each parameter and redshift, the single free parameter \( \xi_\eta \) determines the full probability distribution.

Figure 3 shows the \( P(\mu) \) from ray-tracing simulations by Munshi & Jain (2000) (circles) for smoothing angle \( \theta_0 = 1' \), source redshift \( z_s = 1 \), and three cosmological models from Table 1, together with \( P(\mu) \) computed using our UPDF for the \( \kappa_{\text{min}} \) computed using equation (2), and \( \xi_\eta \) given by Figure 1 of Munshi & Jain (2000).

Figure 4 shows the \( P(\mu) \) from ray-tracing simulations by Barber et al. (2000) (circles) for a \( \Lambda \)CDM model with \( \Omega_m = 0.3, \Omega_\Lambda = 0.7 \) at source redshifts \( z_s = 1, 2, 3.6 \), together with \( P(\mu) \) computed using our UPDF, with \( \kappa_{\text{min}} \) computed using equation (2) and \( \xi_\eta \) inferred from Table 4 of Barber et al. (2000).

Our UPDF gives \( P(\mu) \) in excellent agreement with ray-tracing simulations. To make this more apparent, we have extracted \( P(\eta) \) from the \( P(\mu) \) obtained via ray-tracing by Munshi & Jain (2000) and Barber et al. (2000), and fitted them to the functional form of equation (8). The resultant coefficients are plotted in Figure 2. There is very good agreement in the peak location \( \eta_{\text{peak}}(\xi_\eta) \) and width indicator \( w(\xi_\eta) \), but larger scatter in the non-Gaussianity indicator \( q(\xi_\eta) \) extracted from Munshi & Jain (2000) and Barber et al. (2000). The latter may arise partly due to the fact that in both cases, we poorly resolve the non-Gaussian tails, which are crucial to determining accurate values of \( q \). In addition, the weak lensing condition breaks down for the high \( \mu \) tails which could be significant for small \( \xi_\eta \). Also plotted in Figure 2 are the coefficients extracted from fitting the analytically computed \( P(\eta) \), following Munshi & Jain (2000), for the scaling parameter \( \omega = 0.3 \pm 0.05 \). These \( P(\eta) \) have not been tested for \( z > 1 \) (i.e., for small \( \xi_\eta \), although the deviations are expected to be small, since \( P(\eta) \) peaks close to \( \eta = 1 \) small \( \xi_\eta \) [see equation (8)].
5. Summary and Discussion

We have derived a simple and accurate method to compute the magnification distribution, \( P(\mu) \), for standard candles placed at any redshift in an arbitrary cosmological model. We achieve this by using a universal probability distribution function (UPDF), \( P(\eta|\xi_{\eta}) \), which is independent of the cosmological model; the dependence on cosmology entering only through the variance, \( \xi_{\eta} \), of the reduced convergence, \( \eta \). The UPDF is fit accurately by a 3-parameter stretched Gaussian distribution [eq. (8)]. We give polynomial fitting formulae [eq. (9)] for the three parameters \( \eta_{\text{peak}}(\xi_{\eta}) \) (average smoothness), \( w(\xi_{\eta}) \) (smoothness variation), and \( q(\xi_{\eta}) \) (non-Gaussianity), which we normalize to the N-body simulations of Wambsganss et al. (1997). The magnification PDF, \( P(\mu, z) \), can then be determined from the UPDF using equation (6).

To test the robustness of this method, we have compared our results against three alternate independent methods (see Fig. 2). We find excellent agreement with: (1) the ray-shooting calculations of Munshi & Jain (2000) and Barber et al. (2000), with some scatter in the non-Gaussianity indicator \( q(\xi_{\eta}) \), which is consistent with the limited statistics at low \( z \) (i.e., large \( \xi_{\eta} \)) of these ray-shooting results and the breaking down of the weak lensing condition at high \( \mu \), (2) the analytical calculation following Munshi & Jain (2000), where the latter has been verified by ray tracing experiments, and (3) the ray-shooting of randomly placed SIS mass distributions, following the prescription of Holz & Wald (1998).

We expect these simple, universal forms for the weak lensing distribution to be useful in addressing high redshift data, and in particular, in the analysis of results from future supernova surveys.

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