Central Charge of the Parallelogram Lattice

Strong Coupling Schwinger Model

Ken Yee

Physics Theory Group, Brookhaven National Laboratory, Upton, NY 11973

We put forth a Fierz'd hopping expansion for strong coupling Wilson fermions. As an application, we show that the strong coupling Schwinger model on parallelogram lattices with nonbacktracking Wilson fermions span, as a function of the lattice skewness angle, the $\Delta = -1$ critical line of 6-vertex models. This Fierz'd formulation also applies to backtracking Wilson fermions, which as we describe apparently correspond to richer systems. However, we have not been able to identify them with exactly solved models.

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I. INTRODUCTION AND RESULTS

In recent years there has been remarkable progress in the classification of two dimensional spin systems and three dimensional topological Yang Mills theories in relation to conformal field theories. [1] Yet, despite considerable activity in lattice gauge theories with fermions [2] and two dimensional lattice toy models, [3] the Wilson lattice transcription of the Schwinger model [1] remains unsolved. The only exactly solved model with Wilson fermions and local gauge invariance is the square lattice strong coupling Schwinger model, whose partition function $Z_{\square}$ at infinite hopping constant equals $Z_{6V}[^1_2, ^1_2, 1]$, a 6-vertex model partition function. [3] By known 6-vertex model features, this mapping reveals that $Z_{\square}$ is critical and its continuum limit is a conformal field theory with central charge $c = 1$.

Since the 6-vertex model has a nontrivial phase structure, it is natural to seek the $Z_{\square}$ extensions which map to other regions of the $Z_{6V}[A, B, C]$ parameter space. To this end, we put forth a Fierzed strong coupling hopping expansion for actions of the form

$$S_F = \sum_{x \in \Lambda} \left( - M \bar{\psi}_x \psi_x + \sum_{\mu = 0}^{1} \bar{\psi}_{x+\bar{\mu}} T^{(+,\mu)} U_{x,\mu}^{+} \psi_x + \bar{\psi}_x T^{(-,\bar{\mu})} U_{x,\bar{\mu}} \psi_{x+\bar{\mu}} \right),$$

(1a)

$$\Lambda \equiv \{ \sum_{\mu = 0}^{1} x^\mu \xi_\mu | x^\mu \in Z \}, \quad M \equiv m + \sum_{\mu = 0}^{1} (T^{(+,\mu)} + T^{(-,\bar{\mu})}).$$

(1b)

$S_F$ is the Wilson fermion action on a square lattice action if $\{\hat{\xi}_1, \hat{\xi}_2\}$ are orthonormal and $T^{(\pm,\mu)} = \frac{1}{2}(r \pm \gamma^\mu)$. More generally, on a parallellogram lattice defined by

$$\hat{\xi}_\mu \equiv \begin{pmatrix} \xi^{(0)}_\mu \\ \xi^{(1)}_\mu \end{pmatrix}, \quad \hat{\xi}_0 \equiv \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\xi}_1 \equiv \lambda' \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

(2a)

$$g \equiv \sum_{a,b=0}^{1} \hat{\xi}^{(a)} \hat{\xi}^{(b)} \delta_{ab} = \begin{pmatrix} \lambda^2 & \lambda \lambda' \cos \theta \\ \lambda \lambda' \cos \theta & \lambda'^2 \end{pmatrix},$$

(2b)

the strong coupling partition function is

$$Z_{SC} \equiv \int_F \int \prod_{x,\mu \in \Lambda} dU_{x,\mu} \ e^{-S_F},$$

(3a)
\[ \int_{F_x} = \int dx_1^1 dx_2^1 dx_2^2 dx_2^3, \quad \int_F = \prod_{x \in A} \int_{F_x}. \]  

(3b)

Assuming nonbacktracking condition

\[ T^{-\mu} T^{+\mu} = 0 = T^{+\mu} T^{-\mu}, \]  

(4)

which prevents hopping expansion quarks from backtracking, we will show that

\[ Z_{SC} = \int_F e^{-S_{SCF}}, \quad S_{SCF} = \sum_{x \in A} \left( -M \overline{\psi}_x \psi_x + \frac{1}{\mu} \Theta_x^{(\mu)} + \sum_{\mu=0} \Theta_x^{(\mu)} \Theta_{x+\hat{\mu}}^{(-\mu)} \right). \]  

(5)

Commuting operators \( \Theta_x^{(\mu)} \) are characterized by \( \Theta \)

\[ \int_{F_x} \left( \overline{\psi}_x \psi_x \right)^q (\Theta_x^{(\mu)})^p = 2\delta_{q,2} \delta_{p,0} \quad (\forall q, p \in \{0, 1, 2, \ldots\}), \]  

(6a)

\[ -\int_{F_x} \Theta_x^{(\mu)} \Theta_{x+\hat{\mu}}^{(-\mu)} = r^{(\mu)} r^{(\mu)} \left( \delta_{\mu \nu} \delta_{\nu \nu} + \delta_{\mu \theta} \delta_{\nu \theta} S^2_{\nu \nu} \right), \]  

(6b)

where \( r^{(\mu)} \) and \( S_{\nu \nu} \) are functions (given below) of \( \lambda, \lambda' \) and \( \theta \). Since \( \Theta_x^{(\mu)} \Theta_{x+\hat{\mu}}^{(-\mu)} \) completely saturates \( \int_{F_x} \), \( \Theta_x^{(\mu)} \) hops along selfavoiding paths. Monomers \( (M \overline{\psi}_x \psi_x)^2 \) fill in unhopped sites. We call this the Fierz hopping expansion because the \( \Theta_x^{(\mu)} \Theta_{x+\hat{\mu}}^{(-\mu)} \) form in (6) is achieved using Fierz identities. As described in Section \( III \), the Fierz hopping expansion also applies to backtracking models.

Following \( II \), \( \Theta_x^{(\mu)} \) hopping amplitudes are identified in Section \( III \) with Boltzmann weights of two-state vertex models. Nonperturbative solution of these vertex models (by Bethe ansatz or whatever) is tantamount to resummation of the hopping expansion. In this way we identify \( Z_{SC} \) with \( Z_{SV} \), the 8-vertex model partition function with Boltzmann weights

\[ \omega_1 = M^2, \quad \omega_2 = 0, \quad \omega_3 = (\lambda' \sin \theta)^{-2}, \quad \omega_4 = (\lambda \sin \theta)^{-2}, \]  

(7a)

\[ \omega_5 = \omega_6 = \frac{1}{4\lambda \lambda'} \left( \frac{1}{\cos \theta} \right)^2, \quad \omega_7 = \omega_8 = \frac{1}{4\lambda \lambda'} \left( \frac{1}{\sin \theta} \right)^2. \]  

(7b)

The \( \{\omega_i\} \) are defined, for example, p. 347 of Lieb and Wu \( III \): \( \omega_1 \) corresponds to the empty vertex; \( \omega_3 \) and \( \omega_4 \) to vertical and horizontal lines; \( \omega_5 \) and \( \omega_6 \) to lower-right and upper-left
corners; and $\omega_7$ and $\omega_8$ to lower-left and upper-right corners. Setting $\theta = \pi/2$ and $r^{(\mu)} = 1$ recovers the square lattice result of Ref. [3].

The 8-vertex model is not solved in general for the Boltzmann weights given in (24)-(27). A solvable case is $M = 0$ and $\lambda' = \lambda$. In this subspace, define

$$D \equiv \omega_1 = \omega_2 = 0, \quad C \equiv \omega_3 = \omega_4, \quad A \equiv \omega_5 = \omega_6, \quad B \equiv \omega_7 = \omega_8.$$  \hspace{1cm} (8)

Then 8-vertex model symmetries [3] imply

$$Z_{SC} = Z_{8V}[D, C, A, B] = Z_{8V}[A, B, C, D] = Z_{8V}[A, B, C].$$  \hspace{1cm} (9)

In the thermodynamic limit, these 6-vertex models fall on the critical line characterized by Lieb parameter [3]

$$\Delta \equiv \frac{A^2 + B^2 - C^2}{2AB} = -1.$$  \hspace{1cm} (10)

Varying skewness angle $\theta$ from 0 to $\pi$ spans the critical line between the antiferroelectric and disorder phases of the 6-vertex model.

While the points on this critical line all have central charge $c = 1$, [4] to our knowledge it has not been demonstrated whether they are the same $c = 1$ conformal field theory or different ones. Renormalization group arguments suggest that the subset of points $A = B = \frac{C}{2}$, the so-called F-model, corresponding to $\theta \in \{ \frac{\pi}{3}, \frac{2\pi}{3} \}$ can be identified with free massless boson field theory. [8]

II. FIERZED STRONG COUPLING HOPPING EXPANSION

In the finite coupling ($\beta > 0$) hopping expansion [3] $\psi$ hops from site-to-site with weight $T^{(\epsilon, \mu)}_{x, y, m}$. At strong coupling such quark motions are suppressed. Define

$$D^{(\mu)}_{\tilde{v}, j, k} \equiv T^{(-\mu)}_{ij} T^{(+\mu)}_{kl}, \quad \Xi(x, y; D) \equiv \overline{\psi}^x_x \psi^y_y D_{\tilde{v}, j, k} \overline{\psi}^y_y \psi^x_x.$$  \hspace{1cm} (11)

Since $$(\Xi)^3 = 0$$ in two dimensions, integrating out $U(1)$ links $\{U_{x, \mu}\}$ using $f dU \exp[aU^\dagger + bU] = \sum_{k=0}^\infty (a b)^k / (k!)^2$ yields
\[
Z_{SC} = \int_F \sum_{x, \mu} e^M \sum_{\psi} \prod_{x, \mu} [1 + \Xi(x, x + \mu; D^{(\mu)}) + \frac{1}{4} \Xi^2(x, x + \mu; D^{(\mu)})] \\
= \int_F \sum_{x, \mu} [M \bar{\psi}_x \psi_x + \sum_{\mu} \Xi(x, x + \mu; D^{(\mu)}) - \frac{2}{4} \Xi^2(x, x + \mu; D^{(\mu)})].
\]  
(12a) (12b)

Let \( n, m \in \{ \pm 0, \pm 1 \} \), \( T^{(\alpha, -\alpha)} = T^{(-\alpha, m)} \), and \( k_{\alpha, -\alpha} \equiv k_{\alpha - n, n} \). Expanding \( \exp(M \bar{\psi}_x \psi_x) = \sum_{s_x = 0}^{2} (M \bar{\psi}_x \psi_x)^{s_x} / s_x! \) reveals that \( \psi \otimes \bar{\psi} \) hops with amplitude \( D^{(m)} \) such that \( s_x + \sum_{m=-1}^{1} k_{x, m} = 2 \) for each \( x \in \Lambda \). Thus only three events are possible at each site: \( (\text{3a}) s_x = 2, k_{x, m} = 0; (\text{3b}) s_x = 1, k_{x, m} = 1; \) or \( (\text{3c}) s_x = 0, k_{x, -n} = k_{x, m} = 1 \). The associated amplitudes are

\[
\int_{F_x} \frac{1}{2} (M \bar{\psi}_x \psi_x)^2 = M^2 ,
\]

(13a)

\[
\int_{F_x} M \bar{\psi}_x \psi_x \Xi(x, x + \hat{m}; D^{(m)}) = M \text{tr}(T^{(+ \mu)} T^{(- \mu)} \bar{\psi}_{x + \hat{m}} \psi_{x + \hat{m}}),
\]

(13b)

\[
\int_{F_x} \Xi(x, -\hat{n}, x; D^{(n)}) \Xi(x, x + \hat{m}; D^{(m)}) = \Xi(x, -\hat{n}, x + \hat{m}; \tilde{D}^{(n, m)}),
\]

(13c)

\[
\tilde{D}^{(n, m)}_{i, j \ell k} \equiv (T^{(- \alpha)} T^{(- \alpha)})_{ij}(T^{(+ \mu)} T^{(+ \mu)})_{k\ell} - (T^{(- \alpha)} T^{(+ \mu)})_{i\ell}(T^{(+ \mu)} T^{(- \mu)})_{k j}.
\]

(13d)

A backtracking \( \psi \otimes \bar{\psi} \) pair, \( k_{x, m} = 2 \) or \( n = -m \) in \( (\text{3c}) \), saturates \( \int_F \) at the two sites it occupies and, hence, makes a dimer. Since 2-state vertex models cannot model dimer-loop mixtures, \( Z_{SC} \) is not a 2-state vertex model unless backtracking is forbidden. In this Section we adopt \( (\text{3}), \) which sets \( \tilde{D}^{(n, -n)} = 0 \). Then \( \psi \otimes \bar{\psi} \) worldlines comprise a selfavoiding loop gas with Boltzmann weight \( M^2 \) for unoccupied sites and weights \( \tilde{D}^{(n, m)} = T^{(- n)} T^{(- m)} \otimes T^{(+ \mu)} T^{(- \mu)} \) along loops.

We now recast the problem so that the Boltzmann weights are more succinctly related to hopping amplitudes. Define \( \gamma_a \equiv i \gamma_0 \gamma_1, \)

\[
\gamma_0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^\mu = \sum_{x, \mu} g^{\mu \nu} c_\nu^{(x)} \gamma_a,
\]

(14a)

\[
T^{(\pm \mu)} \equiv \frac{1}{2} (r^{(\mu)} \pm \Gamma^\mu), \quad r^{(0)} = (\lambda \sin \theta)^{-1}, \quad r^{(1)} = (\lambda' \sin \theta)^{-1}.
\]

(14b)
Wilson regulators \( \{ r^{(\mu)} \} \) are chosen so that, in addition to (14),
\[
T_{ij}^{(\epsilon,\mu)} T_{kl}^{(-\epsilon,\mu)} = (T^{(\epsilon,\mu)\gamma_5})_i^j (T^{(-\epsilon,\mu)\gamma_5})_k^l \quad (\mu \text{ fixed}),
\]
(14c)
\[
T^{(\epsilon,\mu)} T^{(\epsilon',\mu)} = \delta^{\epsilon\epsilon'} r^{(\mu)} T^{(\epsilon,\mu)}, \quad r^{(1)} T^{(\epsilon,0)} = r^{(0)} S^\dagger T^{(\epsilon,1)} S,
\]
(14d)
\[
S = \begin{pmatrix}
\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \\
-\cos \frac{\theta}{2} & \sin \frac{\theta}{2}
\end{pmatrix} \equiv \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}.
\]
(14e)
Fierz identity (14c) implies
\[
\Xi(x, x + \hat{m}, D^{(m)}) = -\Theta_x^{(m,5)} \Theta_x^{(+m,5)} = -\bar{\psi}_x T^{(\epsilon,\mu)\gamma_5} \psi_x.
\]
and transforms (12) to (3). Following (3), \( \Theta_x^{(\epsilon,\mu)} \) hops from direction \( n \equiv \text{sign}(n) \nu \) to direction \( m \equiv \text{sign}(m) \mu \) with amplitude (43) where \( \epsilon' = \text{sign}(n) \) and \( \epsilon = -\text{sign}(m) \). Boltzmann weights \( \{ \omega_k \} \) can be read off from (43). Straight \( n = m \) vertical or horizontal hops, corresponding to \( \omega_3 \) and \( \omega_4 \), have weight \( r^{(n)} \). If \( x \) is approached from \( n = 0 \) and exited in the \( m = 1 \) direction, \( \omega_5 = r^{(0)} r^{(1)} S^{2}_{++} \). Similarly, \( n = -1 \) and \( m = 0 \) yields \( \omega_7 = r^{(0)} r^{(1)} S^{2}_{--} \).

Eqs. (7a)-(13) list the remaining vertices.

III. BACKTRACKING

To study backtracking effects when (3) is removed, we work on a square lattice with
\[
T^{(\pm,\mu)} \equiv \frac{1}{2} (r \pm \gamma^\mu).
\]
Fixed-\( \mu \) Fierz transformations are
\[
T_{ij}^{(\epsilon,\mu)} T_{kl}^{(-\epsilon,\mu)} = (T^{(\epsilon,\mu)\gamma_5})_i^j (T^{(-\epsilon,\mu)\gamma_5})_k^l + \left( \frac{r^2 - 1}{8} \right) \sum_\sigma (-1)^\sigma \gamma_5^\sigma \gamma_5^\sigma,
\]
(16)
where \( \gamma^\sigma \in \{ \gamma^\sigma \equiv 1, \gamma_5, \gamma_0, \gamma_1 \} \), \( (-1)^\sigma = -1 \) for \( \gamma_5 \), and \( (-1)^\sigma = +1 \) otherwise. Hermitian operators
\[
\Theta_x^\sigma \equiv \begin{cases}
\bar{\psi}_x \gamma^\sigma \psi_x & \text{if } \gamma^\sigma \in \{ \gamma^\sigma, \gamma_5, T^{(\epsilon,\mu)\gamma_5} \}; \\
\bar{i \psi}_x \gamma^\sigma \psi_x & \text{if } \gamma^\sigma \in \{ \gamma_0, \gamma_1 \};
\end{cases}
\]
(17)
are characterized by
\[ \int_{F_x} (\Theta_x^s)^2 = 2, \quad (\Theta_x^s)^3 = 0, \quad (18a) \]

\[
\Theta_x^{s'} \Theta_x^s = (-1)^s \delta_{s's} (\Theta_x^s)^2 \quad (\sigma, \sigma' \in \{s, 5, 0, 1\}), \quad (18b)
\]

\[
\Theta_x^s \Theta_x^{(\varepsilon, \mu\bar{5})} = \left[ \frac{1}{4} \left( \epsilon \epsilon' \delta_{\mu\nu} - r^2 \right) \delta_{\sigma(\varepsilon', \mu\bar{5})} - \frac{r}{2} \delta_{s5} + \frac{\epsilon}{2} \tilde{\varepsilon} \right] (\Theta_x^s)^2. \quad (18c)
\]

Following (121) and (16)

\[
S_{SCF}^s = S_{SCF}^s + \sum_{x, \mu \in \Lambda} \left[ \Theta_x^{(-\mu5)} \Theta_x^{(+\mu\bar{5})} + \left( \frac{1}{8} \right) \sum_{\sigma=5,0,1} \Theta_x^s \Theta_x^{s+\mu} \right], \quad (19a)
\]

\[
S_{SCF}^s \equiv \sum_{x \in \Lambda} \left[ -M \Theta_x^s + \frac{1}{32} \sum_{\mu=0}^{1} ((r^2 - 1) \Theta_x^s \Theta_x^{s+\mu} + 2)^2 \right]. \quad (19b)
\]

Note that \( S_{SCF}^s \) contains dimers. These systems are currently under study. We have not been able to identify them with solved models.

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* ken@jimi.phy.bnl.gov or (after July 1, 1992) kyee@rouge.phys.lsu.edu

** Under the Euclidean analog of hermitian conjugation, $$\bar{\psi}_E^\dagger \equiv \bar{\psi}\gamma_5, ~ \bar{\psi}_E^\dagger = \gamma_5\bar{\psi}, ~ (S_F)_E^\dagger = S_F, ~ (\Theta^{(\epsilon,\mu^5)}_E)^\dagger = \Theta^{(\epsilon,\mu^5)}_E.$$


