Gravity on noncommutative D-branes

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Abstract

The effective action for the low energy scattering of two gravitons with a D-brane in the presence of a constant antisymmetric $B$ field in bosonic string theory is calculated and the modification to the standard D-brane action to first order in $\alpha'$ is obtained.

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1 Introduction

The discovery in the early days of string theory that string amplitudes in low energy regimes maybe reproduced by Yang-Milles field theory for open strings and gravitational field theory for closed strings, was the beginning of a long, fruitful study into the relation of string theory and field theory in general and in low energy in particular [1, 2].

The lowest order in momentum expansion of the low energy string theory confirmed the interpretation of the massless string modes as the gauge bosons and gravitons of the respective theories, and the next order was a prediction for the corrections to the gauge theory and to the gravity[3, 4].

The realization of the significance of D-branes in string theory, correspondingly led to the study of the low energy effective dynamics of these objects in addition to the bulk dynamics [5, 6, 7, 8, 9, 10, 14]. The lowest order in the momentum expansion turned out to be the DBI action with higher order corrections involving nontrivial terms, both in the bosonic string and the superstring theory.

Discovery of noncommutativity in string theory in the presence of a nontrivial background [11] resulted in the dramatic resurgence of noncommutative gauge theories with their highly nontrivial novel properties emerging as deformation of the low energy action of open strings in the presence of a background. Higher order corrections to the noncommutative gauge theory have also been derived.

It is then natural to ask what kind of deformation the low energy effective action of closed strings suffers in the presence of constant background antisymmetric field. Noncommutative deformations of gravity have been proposed and studied recently [12]. However they have not been derived from string theory and are ad hoc in this sense. One of the motivations of the present study is to throw light on these deformations.

In this work we begin to address this problem by looking at the low energy amplitude of two closed strings with a D-brane in the presence of a constant background $B$ field. We find that as expected the bulk action is not modified, however, the effective action describing the dynamics of the D-brane is modified due to the presence of the constant B field on the D-brane. However the effective action is not obtainable, as a naive T-duality might indicate, from the action of vanishing $B$, by replacing the metric $g$ by $g + B$.

This might be understood on the basis of the fact that T-duality, at this order, mixes graviton scattering with antisymmetric field scattering and is highly nontrivial.

In section two we write down the amplitude of scattering of two gravitons with a D-brane in the presence of a B field and find its low energy behaviour. In section three we find the low energy effective action describing those amplitude to order $\alpha'$. 

1
2 String scattering from D-branes

In this section we compute the tree level bosonic string scattering amplitude of two massless closed string off a noncommutative D-brane. We take D-brane as a disk and conformalaly mapped on the upper half plane, with mixed boundary conditions and use the following notation for indices, on and off the D-brane

\[ \mu, \nu = 0, \ldots, 25 \quad a, b = 0, \ldots, p \quad i, j = p + 1, \ldots, 25. \]

(1)

For a D-brane localized at \( x_i^{p+1}, \ldots, x_{25} \), the boundary conditions are

\[ g_{ab}(\partial - \overline{\partial})X^a + B_{ab}(\partial + \overline{\partial})X^a \big|_{\bar{z} = z} = 0. \]

(2)

The two point correlator of string coordinates \( X^\mu(z, \overline{z}) \) on the D-brane is

\[ \langle X^\mu_i X^\nu_j \rangle = -\frac{\alpha'}{2} g_{\mu \nu} \log((z_i - z_j)(\overline{z}_i - \overline{z}_j)) + D_{\mu \nu}^{ij} \log(z_i - \overline{z}_j) + D_{i j}^{\mu \nu} \log(\overline{z}_i - z_j) \]

(3)

where

\[ D^{ab} = 2\left( \frac{1}{\eta + B} \right)^{ab} - \eta^{ab} \quad D^{ij} = -\delta^{ij} \quad D_{\alpha \beta}^{\mu \nu} = \eta^{\mu \nu} \]

(4)

with \( g \) the flat metric. Using these correlators we can compute the scattering amplitude of two closed strings off a D-brane,

\[ A = g_c^2 e^{-\lambda} \int d^2 z_1 d^2 z_2 \langle V(z_1, \overline{z}_1)V(z_2, \overline{z}_2) \rangle \]

(5)

with \( g_c \) the closed string coupling constant and \( \lambda \) the Euler number of the world sheet. The appropriate vertex operators for the closed strings are,

\[ V(z_i, \overline{z}_i) = \epsilon_{\mu \nu} : \partial X^\mu(z_i) \exp(ik_i.X(z_i)) : \partial \overline{X}^\nu(\overline{z}_i) \exp(ik_i.D.X(\overline{z}_i)) : \]

(6)

which the momenta \( k_i \) and polarizations \( \epsilon_{\mu \nu} \) of the massless closed strings satisfying

\[ \epsilon_{\mu \nu} k^\mu = \epsilon_{\mu \nu} k^\nu = 0 \quad k^2 = 0. \]

(7)

To go further we use the change of variable as follow,

\[ \check{X}^\mu(\overline{z}_i) = D^\mu_\xi X^\xi(z_i) \]

(8)

so that the closed string vertices change to

\[ V(z_i, \overline{z}_i) = \epsilon_{\mu \lambda} D^\lambda_\nu : \partial X^\mu(z_i) \exp(ik_i.X(z_i)) : \partial \overline{X}^\nu(\overline{z}_i) \exp(ik_i.D.X(\overline{z}_i)) : \]

(9)

Now inserting these vertices into (5) and going through the calculation of the correlation functions we find

\[ A = g_c^2 e^{-\lambda} \epsilon_1 \epsilon_\mu \lambda D^\lambda_\nu \epsilon_{2 \alpha \rho} D^\rho_\beta \int d^2 z_1 d^2 z_2 A^{\mu \alpha \beta} \]

(10)
\[ A^{\mu \nu \alpha \beta} = i C^X_{D_2}(2\pi)^{p+1}q_{k_1.k_2.D}(k_1 + k_2.D + k_2.D) \exp \left( \frac{\alpha}{2} \right) \frac{1}{k_1.D.k_1.log(z_1 - z_1)} \]

\[ + k_2.D.k_2.log(z_2 - z_2) + k_1.k_2.log(z_1 - z_2) + k_1.k_2.log(z_1 - z_2) \]

\[ + k_2.D.k_1.log(z_1 - z_2) + k_1.D.k_2.log(z_1 - z_2) \]

\[ \{ q_{11}^{\mu \nu} q_{12}^{\alpha \beta} + q_{12}^{\mu \nu} q_{12}^{\alpha \beta} + q_{12}^{\mu \nu} q_{12}^{\alpha \beta} + q_{12}^{\mu \nu} q_{12}^{\alpha \beta} + q_{12}^{\mu \nu} f_1^{\alpha \beta} + q_{12}^{\mu \nu} f_2^{\alpha \beta} \}
\]

\[ + q_{12}^{\mu \nu} f_1^{\alpha \beta} + q_{12}^{\mu \nu} f_2^{\alpha \beta} + q_{12}^{\mu \nu} f_2^{\alpha \beta} + q_{12}^{\mu \nu} f_2^{\alpha \beta} \].

(11)

in which \( C^X_{D_2} \) is the functional determinant for the string fields and the delta function ensures conservation of momentum parallel to the D-brane. Here \( f \)’s are terms that come from correlation function between derivative and exponential terms while the \( q \)’s are between the derivative terms, and are

\[ q_{ij}^{\mu \nu} = \frac{\eta^{\mu \nu}}{2 (z_i - z_j)^2} \quad f_i^\mu = -i \frac{\alpha'}{2} \sum_{j \neq i} \frac{k_i^\mu}{z_i - z_j} \]

(12)

with the indices as

\[ i, j = 1, \bar{1}, 2, \bar{2} \quad z_1 = \bar{z}_1 \quad z_2 = \bar{z}_2. \]

(13)

It is straightforward to see that the amplitude is SL(2,R) invariant. To fix this invariance we let \( z_1 = iy \) and \( z_2 = i \)

\[ d^2 z_1 d^2 z_2 \rightarrow 2(1 - y^2) dy \]

(14)

The \( f \) and \( q \) terms then become

\[ f_1^\mu = \frac{1}{2iy} \left( \frac{y + 1}{y - 1} k_1^\mu + \frac{y - 1}{y + 1} k_2.D^\mu \right) \quad f_1^\nu = -i \frac{1}{2iy} \left( \frac{y - 1}{y + 1} k_2^\mu + \frac{y + 1}{y - 1} k_2.D^\nu \right) \]

\[ f_2^\alpha = \frac{1}{2i} \left( \frac{y + 1}{y - 1} k_1^\alpha + \frac{y - 1}{y + 1} k_1.D^\alpha \right) \quad f_2^\beta = i \frac{1}{2i} \left( \frac{y - 1}{y + 1} k_1^\beta + \frac{y + 1}{y - 1} k_1.D^\beta \right) \]

\[ q_{11}^{\mu \nu} = -\frac{\eta^{\mu \nu}}{4y^2} \quad q_{12}^{\mu \nu} = -\frac{\eta^{\mu \nu}}{(y - 1)^2} \quad q_{12}^{\mu \nu} = \frac{\eta^{\mu \nu}}{(y + 1)^2} \]

\[ q_{12}^{\mu \nu} = -\frac{\eta^{\mu \nu}}{(y + 1)^2} \quad q_{12}^{\mu \nu} = -\frac{\eta^{\mu \nu}}{(y - 1)^2} \quad q_{22}^{\alpha \beta} = -\frac{\eta^{\alpha \beta}}{4} \]

(15)

We have dropped terms which will be canceled when inserted in (10) using the physical condition for closed strings. Inserting (15) into (11) we get

\[ A = 2i C^X_{D_2} \left( k_1 D_{sk_1} + k_2 D_{sk_2} \right) \int_0^1 dy (1 - y^2)(y - 1)^2 k_1 D_{sk_1}(y - 1) k_2 (y + 1) k_1 D_{sk_2} \]

\[ \{ \left( \frac{\alpha'}{2} \right)^3 \left[ \frac{a_1}{16y^2} + \frac{a_2}{(y - 1)^4} + \frac{a_3}{(y + 1)^4} \right] + \left( \frac{\alpha'}{2} \right)^3 \left[ -\frac{a_4}{16y^2} + \frac{a_5}{4(y + 1)^2} \right] \]
because of momentum conservation parallel to the D-brane we have are given in the appendix A. We choose the Mandelstam variables, $\epsilon$ where $\beta$ come from the Beta functions of the amplitude (19). There are two types of singularities in this amplitude that in the low energy effective theory of the string, we must analyze the pole structure of the amplitude. We recover the results of scattering amplitude for commutative D-branes $D_A$ and $D_S$ are the antisymmetric and symmetric parts of $D$ matrix. Notice that because of momentum conservation parallel to the D-brane we have $k_1 D_A k_2 = 0$. The constants $a_n$'s depend on momentum and polarization of external states and are given in the appendix A. We choose the Mandelstam variables,

$$s = - \frac{1}{2}(y + D)k_1)^2 = - \frac{1}{2}k_1 D_S k_1 = - \frac{1}{2}k_2 D_S k_2$$

Finally making the change of variable $y = \frac{1 - \sqrt{2}}{1 + \sqrt{2}}$ we find

$$A = 2i C \alpha' \frac{1}{2} \{a_1 B(-\frac{\alpha' t}{4} + 1, -\alpha' s - 1) + a_2 B(-\frac{\alpha' t}{4} - 1, -\alpha' s + 1)$$

$$+ a_3 B(-\frac{\alpha' t}{4} + 1, -\alpha' s + 1) - \frac{\alpha'}{2} a_4 B(-\frac{\alpha' t}{4} + 1, -\alpha' s - 1)$$

$$- a_5 B(-\frac{\alpha' t}{4} + 1, -\alpha' s + 1) + a_6 B(-\frac{\alpha' t}{4}, -\alpha' s) + a_7 B(-\frac{\alpha' t}{4}, -\alpha' s - 1)$$

$$+ a_8 B(-\frac{\alpha' t}{4} + 2, -\alpha' s - 1) - a_9 B(-\frac{\alpha' t}{4} + 2, -\alpha' s)$$

$$+ a_{10} B(-\frac{\alpha' t}{4} - 1, -\alpha' s - 1) + a_{11} B(-\frac{\alpha' t}{4} + 1, -\alpha' s - 1)$$

$$+ a_{12} B(-\frac{\alpha' t}{4}, -\alpha' s - 1) + a_{13} B(-\frac{\alpha' t}{4} + 2, -\alpha' s - 1)$$

$$+ a_{14} B(-\frac{\alpha' t}{4} - 1, -\alpha' s - 1) + a_{15} B(-\frac{\alpha' t}{4} + 3, -\alpha' s - 1)\}$$

where $B$'s are beta functions. Note that as a test of our calculation replacing every polarization of closed strings by momentum $\epsilon_{\mu \nu} \rightarrow k_{\mu} k_{\nu}$ gives zero value for the amplitude. We recover the results of scattering amplitude for commutative D-branes found in [14] by turning the $B$ field off (see Appendix B). As we will be interested in the low energy effective theory of the string, we must analyze the pole structure of the amplitude (19). There are two types of singularities in this amplitude that come from the Beta functions

$$-\frac{\alpha' t}{4} - 1 = -n \quad -\alpha' s - 1 = -n' \quad n, n' = 0, 1, \ldots$$

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which in terms of \( t \) and \( s \) of (18) are
\[
t = m_{\text{closed}}^2 = -\frac{4}{\alpha'}(-n + 1) \\
s = m_{\text{open}}^2 = -\frac{1}{\alpha'}(-n' + 1),
\]
(21)

exhibiting the mass spectrum of the intermediate closed string propagators in the t-channel and the mass spectrum of the open strings in the s-channel. Now we may study the low energy limit of the amplitude i.e. \( s \), \( t \to 0 \). There are then three types of terms to consider

\textbf{t-channel}:
\[
A_t = i \frac{C \alpha'^2}{t} \left\{ -2a_2s + a_6 + a_7 + a_{10} - \alpha'(-a_{10}s + \frac{1}{2}a_{12} + a_{14}) + \mathcal{O}(\alpha'^2) \right\}
\]
(22)

\textbf{s-channel}:
\[
A_s = \frac{iC \alpha'^2}{-4s} \left\{ \frac{1}{2}a_1t + a_5 - a_6 - a_7 + a_8 + a_9 - a_{10} - \frac{\alpha'}{4}[(a_4 + a_7 + a_8)t \\
- 2a_{12} + 2a_{13} - 4a_{14} + 4a_{15}] + \mathcal{O}(\alpha'^2) \right\}
\]
(23)

\textbf{contact-terms}:
\[
A_c = i \frac{C \alpha'^2}{2} \left\{ -(a_1 + a_2 - a_3) + \frac{\alpha'}{2}[a_4 + a_7 + a_8 - a_9 + a_{10} \\
- (2s - \frac{t}{2})(a_1 + a_2 + a_3)] + \mathcal{O}(\alpha'^2) \right\}
\]
(24)

\section{Low energy effective action}

In this section we will derive the low energy action for the above amplitude to the first two leading orders of \( \alpha' \). For scattering of two gravitons we use the physical condition for them as,
\[
\epsilon_{\mu\nu} = \epsilon_{\nu\mu} \quad \epsilon^\mu_{\mu} = 0.
\]
(25)

In the t-channel we have two gravitons which interact in the bulk space and produce another graviton or dilaton field which in turn is absorbed by the D-brane. This easily seen that the massless antisymmetric field does not contribute to the amplitude. The t-channel amplitude to be reproduced by inserting for the coefficients \( a_i \) of appendix A,
\[
A_t = i \frac{C \alpha'^2}{t} \left\{ (k_1 D_s k_1) Tr(\epsilon_1 \epsilon_2) - 2k_1 D_s \epsilon_1 \epsilon_2 k_1 - 2k_2 D_s \epsilon_2 \epsilon_1 k_2 - 2k_1 \epsilon_2 \epsilon_1 k_1 \right. \\
- 2k_1 \epsilon_2 \epsilon_1 k_2 + Tr(\epsilon_1 D_s) (k_1 \epsilon_2 k_1) + Tr(\epsilon_2 D_s) (k_2 \epsilon_1 k_2) \\
- \left. \alpha'[(k_1 \epsilon_2 \epsilon_1 k_2)(k_1 D_s k_1) + (k_1 \epsilon_2 k_1)(k_2 \epsilon_1 k_2) + (k_1 \epsilon_2 k_1)(k_2 \epsilon_1 D_s k_2) + (k_2 \epsilon_1 k_2)(k_1 \epsilon_2 D_s k_1)] \right\} = A_0^t + A_1^t.
\]
(26)
It is not difficult to see that the action which describes this amplitude to order of $\alpha'^0$ is nothing but the DBI and Einstein-Hilbert action. The DBI action in the Einstein frame is,

$$S_{D-brane}^0 = -T_p \int d^{p+1} x e^{-\Phi} \sqrt{-\text{det}(e^{-\gamma \Phi} g + \mathcal{B} + f)}_{ab}$$

(27)

where $g$ is the induced metric on the D-brane and $\mathcal{B}_{ab} = B_{ab} - 2\kappa b_{ab}$ is the pull back of the antisymmetric field along the D-brane with $B$ constant and $f_{ab}$ is the gauge field strength on the D-brane and $\gamma = -\frac{4}{d-2}$. Expanding $g_{ab}$ around the Minkowski metric, $g_{ab} = \eta_{ab} + 2\kappa h_{ab}$ we get for the action to the first order of $h$,

$$S_{D-brane}^0 = -\kappa T_p c \int d^{p+1} x h_{ab} V^{ab}$$

(28)

where

$$V^{ab} = \frac{1}{2}(\eta + D)^{ba} \quad c = \sqrt{-\text{det}(\eta + B)}.$$  

(29)

Equation (28) exhibits a source term for gravity on the D-brane,$^6$

$$(S_h)^{ab} = -\frac{1}{2} T_p \kappa C(\eta^{ab} + D^{ab}_S).$$  

(30)

The Einstein-Hilbert action in the bulk is

$$S_{bulk}^{(0)} = \frac{1}{2\kappa^2} \int d^{26} x \sqrt{-G R}.$$  

(31)

Using 26-dimensional propagator $(G^{hh})_{\mu\nu\lambda\rho}$ and the three point interaction vertex $(V_{h\epsilon_1\epsilon_2})^{\lambda\rho}$ for gravitons coming from this action [6], and the above source term, we find

$$i(S_h)^{\mu\nu}(G^{hh})_{\mu\nu\lambda\rho}(V_{h\epsilon_1\epsilon_2})^{\lambda\rho} = A^0_t + C^0_t$$

(32)

where $A^0_t$ is zeroth order term of the amplitude (26), and $C^0_t$ is a contact term, with no poles,

$$C^0_t = -icT_p\kappa^2(Tr(\epsilon_1\epsilon_2) + Tr(\epsilon_1 D S \epsilon_2)).$$

(33)

Here the tension of the D-branes, $T_p$ from (27) is written in terms of the coefficients of (26) i.e.

$$T_p = \frac{C\alpha'^0}{8\kappa^2}.$$   

(34)

We note that to this order of $\alpha'$ the above results are exactly the same as in the superstring theory [6]. To get the next order terms in the string amplitude (26), we include the next order of $\alpha'$ gravitational action in the bulk, [3, 4]

$$S_{bulk}^{(1)} = \frac{\alpha'}{8\kappa^2} \int d^{26} x e^{\gamma \Phi} \sqrt{-G} R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} - 4R^{\mu\nu} R_{\mu\nu} + R^2.$$  

(35)

$^5$For simplicity in writing we have dropped the coefficient of $f$ i.e. $2\pi\alpha'f \rightarrow f$

$^6$Note that we have expanded the action (27) around the background $\eta_{ab} + B_{ab}$ using

$$\sqrt{\text{det}(M_0 + M)} = \sqrt{\text{det}M_0}(1 + \frac{1}{2}Tr(M_0^{-1}M) + ...)$$
There is a new three point interaction for gravitons $V_{h\epsilon_1\epsilon_2}$ in this Lagrangian. In addition it contains the interaction between one dilaton and two gravitons $V_{\Phi\epsilon_1\epsilon_2}$. So there are two source terms, gravitational source $S_h$ and dilatonic source $S_\Phi$ and we find

$$ iS_h G^{hh'} V_{h\epsilon_1\epsilon_2} + iS_\Phi G^{\Phi\Phi} V_{\Phi\epsilon_1\epsilon_2} = A_1^1 + C_1^1, \quad (36) $$

where $A_1^1$ is the first order term of the amplitude (26) and

$$ C_1^1 = \frac{ie T_p k^2 \alpha'}{2} \{ -k_1 D_S k_1 \text{Tr}(\epsilon_1 \epsilon_2) + k_1 k_2 \text{Tr}(\epsilon_1 D_S \epsilon_2) - k_1 \epsilon_2 \epsilon_1 D_S k_2 \\
+ k_2 D_S \epsilon_2 \epsilon_1 k_2 - k_1 \epsilon_2 D_S \epsilon_1 k_2 \} \quad (37) $$

is a contact term. This calculation exhibits similar gravitational source term as in the order of $\alpha'^0$.

Next we consider the s-channel amplitude up to the first two powers of $\alpha'$. We may write this amplitude, (23), as

$$ A_s = -\frac{i c k^2 T_p}{4 s} \left\{ -\frac{1}{2} k_1 k_2 \text{Tr}(\epsilon_1 D_S) \text{Tr}(\epsilon_2 D_S) - k_2 D_S \epsilon_2 \epsilon_1 D_S k_1 + k_2 D_A \epsilon_2 \epsilon_1 D_A k_1 \\
+ k_2 D_A \epsilon_2 D_S \epsilon_1 D_A k_1 + k_2 D_S \epsilon_2 D_S \epsilon_1 D_S k_1 + 2 k_2 D_A \epsilon_2 D_A \epsilon_1 D_S k_1 \\
+ \text{Tr}(\epsilon_1 D_S)(k_1 \epsilon_2 D_S k_2 - k_1 D_S \epsilon_2 D_S k_2 - k_1 D_A \epsilon_2 D_A k_2) \\
+ \left( \frac{\alpha'}{2} \right)[k_1 k_2 \text{Tr}(\epsilon_1 D_S)(k_2 D_S \epsilon_2 D_S k_2) + (k_2 \epsilon_1 D_S k_1)(k_1 \epsilon_2 D_S k_2) \\
- (k_1 D_S \epsilon_1 D_S k_2)(k_2 D_S \epsilon_2 D_S k_1) + 2(k_1 D_A \epsilon_2 D_A k_1)(k_1 D_S \epsilon_1 D_S k_2) \\
- (k_1 D_A \epsilon_2 D_A k_1)(k_2 D_A \epsilon_1 D_A k_2)] \right\} + (1 \leftrightarrow 2) = A_s^0 + A_s^1. \quad (38) $$

In this channel we expect the two gravitons scatter from the D-brane and exchange a gauge or scalar field which propagate on the D-brane. These propagating fields are in fact the low energy limits of the massless open strings on the D-brane, scalars ($X^i$) and gauge bosons ($a_a$). We would like to write down a Lagrangian for these fields on the D-brane which produces the vertices for interactions between gravitons and these fields and finally gives the scattering amplitude of the string theory. For zeroth order of $\alpha'$ we again take the DBI action on D-brane (27) with gauge field strength defined as $f_{ab} = \partial_a a_b - \partial_b a_a$ then expand the induced metric $g_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$ on the D-brane around the flat space and choose the static gauge ($X^a = x^a$) to get

$$ g_{ab} = \eta_{ab} + 2\kappa h_{ab} + 2\kappa (h_{ia} \partial_a X^i + h_{bi} \partial_a X^i) + \partial_a X^i \partial_b X^i + 2\kappa h_{ij} \partial_a X^i \partial_b X^j. \quad (39) $$

Redefining the gauge and scalar fields by $A_a = 2\pi \alpha' \sqrt{T_p} a_a$ and $\lambda^i = \sqrt{T_p} X^i$ and expanding DBI action around the background field $\eta_{ab} + B_{ab}$ for the zeroth order of $\alpha'$ we find the following vertices,

$$ (V_{\epsilon_1 A})^a = \sqrt{T_p} \kappa c (\text{Tr}(\epsilon_1 V_S) k_1 V_A^a - 2k_1 V_S \epsilon_1 V_A^a - 2k_1 V_A \epsilon_1 V_S^a) $$

$$ (V_{\epsilon_1 \lambda})_i = \sqrt{T_p} \kappa c (\text{Tr}(\epsilon_1 V_S) k_1 i - 2k_1 V_S \epsilon_1 i) \quad (40) $$
and propagators,

\[(G^{\lambda\lambda})_{ij} = \frac{i}{cs} \eta^{ij} \quad (G^{AA})_{ab} = \frac{i}{cs} (V^{-1}_{S})_{ab}\]  \hspace{1cm} (41)

where \(V_A\) and \(V_S\) are the antisymmetric and symmetric part of the matrix \(V\). Note that in deriving the above vertices we have used the fact that the closed string fields in the DBI action appear as functionals of the transverse scalar fields \(X^i\) which should be Taylor expanded. Knowing these vertices and propagators we can compute the amplitude from the effective field theory on D-brane and show that to order of \(\alpha'\),

\[V_{\epsilon_1 \lambda} G^{\lambda\lambda} V_{\epsilon_2} + V_{\epsilon_1 A} G^{AA} V_{\epsilon_2} = A^0_s + C^0_s\]  \hspace{1cm} (42)

where \(A^0_s\) is the zeroth order term in (38) and \(C^0_s\) is a contact term,

\[C^0_s = \frac{-i\kappa^2 T_r c}{4} Tr(\epsilon_1 D_S) Tr(\epsilon_2 D_S).\]  \hspace{1cm} (43)

We have used the following useful relations in the derivation

\[(V_A V^{-1}_S V_A)_{ab} = -\frac{1}{2} \left( \eta - D_S \right)_{ab} \quad \eta^{ij} = \frac{1}{2} \left( \eta - D_S \right)^{ij}.\]  \hspace{1cm} (44)

It turns out that in order to describe the amplitude to the first order in \(\alpha'\) we need to add new terms to the DBI action as follow

\[S^1 = \frac{-\alpha' T_r}{2} \int d^{p+1}x \left\{ \sqrt{-\det(\eta + B + f)} R_{abcd} \left( \frac{1}{\eta + B + f} \right)^{ad} \left( \frac{1}{g + B + f} \right)^{bc} \right\}

- \sqrt{-\det(g + B + f)} \left( \Omega^i_{ac} \Omega_{ibd} - \Omega^i_{ad} \Omega_{ibc} \right) \left( \frac{1}{g + B + f} \right)^{ad} \left( \frac{1}{g + B + f} \right)^{bc} \right\} \]  \hspace{1cm} (45)

Which \(R_{abcd}\) is Riemann tensor constructed out of induced metric on the D-brane and \(\Omega\)'s are second fundamental forms,

\[R_{abcd} = \Gamma_{abd,c} - \Gamma_{abc,d} + \Gamma_{cad} \Gamma^{e}{}_{bc} - \Gamma_{eac} \Gamma^{e}{}_{bd}, \]

\[\Omega^i_{ab} = \kappa (-\partial_i h_{ab} + \partial_a h^i_b + \partial_b h^i_a) + \partial_a \partial_b X^i\]  \hspace{1cm} (46)

which \(a, b, c, d, e = 0, \ldots, p\) and \(i, j = p+1, \ldots d\). The \(\Gamma\)'s are the Christofel symbols constructed out of the induced metric \(g\). This action contains every vertex that we need for producing the next order of \(\alpha'\) in (38), in addition to certain contact terms which we will describe later. To reproduce the s-channel amplitude of string theory from such an action we need to find the graviton-scalar and graviton-gauge field vertices. We find the first one from the second fundamental form linear in terms of \(X\) and \(h\)\(^7\), and the second one from the expansion the above Lagrangian linear in

\(^7\)We must Taylor expand the linear terms in \(h\) of the above action, but because of momentum conservation relation these terms do not contribute to the vertex.
terms of $f$ and $h$. By summing these vertices corresponding to the above vertices with the previous terms which we found from DBI action to zeroth order of $\alpha'$, we find that

\[
(V_{\epsilon, A})^a = \frac{\kappa c \sqrt{\frac{T_p}{4}}}{4} \{2Tr(\epsilon_1 D_S)k_1 V_A^a - 4k_1 D_S \epsilon_1 V_A^a + 4k_1 D_A \epsilon_1 V_S^a - \alpha'[(k_1 D_S \epsilon_1 D_S k_1 + k_1 D_A \epsilon_1 D_A k_1)k_1 V_A^a - (k_1 D_S k_1)Tr(\epsilon_1 D_S)k_1 V_A^a]\}
\]

\[
(V_{\epsilon, \lambda})_i = \frac{\kappa c \sqrt{\frac{T_p}{4}}}{4} \{2Tr(\epsilon_1 D_S)k_1 i - 4k_1 D_S \epsilon_1 i - \alpha'[(k_1 D_S \epsilon_1 D_S k_1 + k_1 D_A \epsilon_1 D_A k_1)k_1 i - (k_1 D_S k_1)Tr(\epsilon_1 D_S)k_1 i]\}. \quad (47)
\]

The check that these vertices are in fact correct we have calculated one closed and one open string scattering from D-brane and we have found similar vertices (see Appendix C). Now using these vertices we can find the first order of $\alpha'$ terms in $s$-channel from string theory. We have used the previous propagators for gauge and scalar fields, as (45) does not introduce any corrections to the propagators. We then find

\[
V_{\epsilon, \lambda} G_{\lambda \lambda} V_{\lambda \epsilon} + V_{\epsilon, A} G_{\lambda A} V_{\lambda \epsilon} = A_s^1 + C_s^1 \quad (48)
\]
in which $A_s^1$ is the first order of $\alpha'$ terms of (38) and $C_s^1$ is the first order in $\alpha'$ terms with no poles and can be written as

\[
C_s^1 = \frac{ic k^2 T_p}{4} \alpha' \{\frac{1}{2} Tr(\epsilon_1 D_S)Tr(\epsilon_2 D_S)(k_1 D_S k_2 - k_1 k_2) + \frac{1}{2}(k_1 D_S)(Tr(\epsilon_1 D_S)Tr(\epsilon_2 D_S) + Tr(\epsilon_1 \epsilon_2) + Tr(D \epsilon_1 D \epsilon_2))\} \quad (49)
\]

It remains to verify that all the above contact terms (33), (43) and contact terms from (27) at $\alpha^0$ and (37), (49) and contact terms from (45) at $\alpha'$ add up to the contact terms of the string amplitude (24). We can write (24) as

\[
A_c = \frac{ic k^2 T_p}{4} \{ -Tr(\epsilon_1 D) Tr(\epsilon_2 D) - Tr(\epsilon_1 \epsilon_2) + Tr(D \epsilon_1 D) + \alpha'[Tr(\epsilon_1 D)(k_2 D \epsilon_2 D k_2) - k_1 D \epsilon_2 D \epsilon_1 D k_2 + k_1 \epsilon_2 \epsilon_1 k_2] + \frac{1}{2}(k_1 D k_2)(Tr(\epsilon_1 D) Tr(\epsilon_2 D) + Tr(\epsilon_1 \epsilon_2) + Tr(D \epsilon_1 D \epsilon_2))\} \quad (50)
\]

If we expand DBI action (27) and find graviton-graviton contact terms from this expansion,

\[
A_{hh}^0 = -2i k^2 c T_p \{\frac{1}{2} Tr(\epsilon_1 V) Tr(\epsilon_2 V) - Tr(\epsilon_1 V \epsilon_2 V)\}. \quad (51)
\]

We can show that by adding (33) and (43) to the above equation

\[
A_{hh}^0 + C_t^0 + C_s^0 = \frac{ic k^2 T_p}{2} \{Tr(\epsilon_1 D) Tr(\epsilon_2 D) + Tr(\epsilon_1 \epsilon_2) - Tr(\epsilon_1 D \epsilon_2 D)\} \quad (52)
\]
we find exactly the $\alpha'^0$ term in equation (50). Again as before if we expand Lagrangian in equation (45), we find for graviton-graviton contact terms,

$$A^1_{hh} = -2i\alpha'\kappa^2 c T_p \{ -k_1 V\epsilon_2 V k_2 - k_1 V\epsilon_2 V_1 k_2 - k_1 \epsilon_2 V\epsilon_1 k_2 + k_1 V\epsilon_2 V\epsilon_1 k_2 $$

$$- \frac{1}{4} k_1 k_2 Tr(\epsilon_1 V) Tr(\epsilon_2 V) + \frac{1}{2} k_1 V k_2 Tr(\epsilon_1 V \epsilon_2) + \frac{1}{4} k_1 k_2 Tr(V\epsilon_1 V_2)$$

$$+ \frac{1}{2} k_1 V k_1 Tr(V\epsilon_1 V_2) + \frac{1}{2} k_1 V \epsilon_1 k_2 + \frac{1}{2} k_1 V \epsilon_1 k_1 Tr(\epsilon_2 V) \} + (1 \leftrightarrow 2) \quad (53)$$

we can show that by adding (37) and (49) to above equation,

$$A^1_{hh} + C^1_t + C^1_s = \frac{i\alpha'\kappa^2 c T_p}{4} \{ Tr(\epsilon_1 D)(k_2 D\epsilon_2 Dk_2) - k_1 D\epsilon_2 D\epsilon_1 Dk_2 + k_1 \epsilon_2 \epsilon_1 k_2 $$

$$+ \frac{1}{2} (k_1 Dk_2) [Tr(\epsilon_1 D) Tr(\epsilon_2 D) + Tr(\epsilon_1 \epsilon_2) + Tr(D\epsilon_1 D\epsilon_2)] \}$$

$$+ (1 \leftrightarrow 2) \quad (54)$$

we recover the $\alpha'$ term in (50). It is noticeable that in evaluating the t-channel poles in field theory we use the linear coupling of graviton to the D-brane using the DBI action (27) which is of zero order of $\alpha'$ whereas linear coupling of graviton to D-brane coming from action (45) is of first order of $\alpha'$ however using conservation of momentum this coupling has no effect in the t-channel.

It should be noted that when we turn off the $B$ field on the D-brane we find the Einstein-Hilbert action as we expected [14]. To see this, note that as we turn off the $B$ field on the D-brane we see, $V^{ab} = \eta^{ab}$ where upon (45) changes up to some total derivatives to,

$$S^1_{D-brane} = \frac{\alpha' T_p}{2} \int d^{p+1}x \sqrt{-\det g_{ab}} (R + \Omega^i_a \Omega_i^b - \Omega^i_{ab} \Omega_i^b)$$

$$R = R_{abcd} g^{ac} g^{bd} \quad (55)$$

which $a, b, c, d = 0, \ldots, p$ and $i, j = p + 1, \ldots, d$. Where we have ignored gauge fields because they do not any contraction with gravitons because of the antisymmetry of $f_{ab}$. In deriving the above equation we have used the following relation at $O(h^2)$,

$$\sqrt{-\det \eta R_{abcd} \eta^{ad} g^{bc}} = \sqrt{-\det g R_{abcd} g^{ac} g^{bd}} + (T.D), \quad (56)$$

Which by momentum conservation relation, total derivative terms have no effect in scattering amplitude.
4 Discussion

In this work we have found the effective action corrections to the DBI action to order of $\alpha'$, (45), for the description of the graviton-graviton-D-brane scattering in bosonic string theory in the presence of a constant $B$ field.

It is known that D-branes in the presence of background $B$ field become noncommutative and the DBI action in terms of the noncommutative open string fields involve the $\star$ product and a Wilson line operator. In our calculation we have found that only couplings with at most one open string field arise so that $\star$ products and Wilson line operators don’t play any role. The principle modification introduced in the effective action (45), due to the presence of the $B$ field, is in the form of the modified metrics $\frac{1}{\eta + B}$ and $\frac{1}{g + B}$ contracting with the curvature tensor made out of the induced bulk gravity.

We have also calculated the effective action for dilaton and antisymmetric field scattering with a D-brane. The results are not qualitatively different and will be reported separately.

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Appendix A

Coefficients for Noncommutative D-brane

Here we presented the coefficient for scattering of two closed string from D-brane in the presence of B field.

\begin{align}
    a_1 &= Tr(\epsilon_1 D)Tr(\epsilon_2 D) \\
    a_2 &= Tr(\epsilon_1^T \epsilon_2) \\
    a_3 &= Tr(D\epsilon_1 D\epsilon_2) \\
    a_4 &= Tr(\epsilon_1 D)(k_1 \epsilon_2 Dk_1 + k_1 D\epsilon_2 k_1) + (1 \leftrightarrow 2) \\
    a_5 &= k_1 \epsilon_2 D\epsilon_1 Dk_2 + k_2 \epsilon_1 D\epsilon_2 Dk_1 + k_1 D\epsilon_2 D\epsilon_1 k_2 + k_2 D\epsilon_1 D\epsilon_2 k_1 - k_2 D^T \epsilon_1^T \epsilon_2 Dk_1 \\
        &\quad - k_2 D\epsilon_1 \epsilon_2^T D^T k_1 \\
    a_6 &= k_2 D^T \epsilon_1^T \epsilon_2 k_1 + k_2 \epsilon_1^T \epsilon_2 Dk_1 - k_1 \epsilon_2 D\epsilon_1 k_2 - k_2 \epsilon_1 D\epsilon_2 k_1 + k_2 \epsilon_1^T D^T k_1 \\
    a_7 &= Tr(\epsilon_1 D)(k_1 \epsilon_2 k_1) + (1 \leftrightarrow 2) \\
    a_8 &= Tr(\epsilon_1 D)(k_1 D\epsilon_2 Dk_1) + (1 \leftrightarrow 2) \\
    a_9 &= k_1 D\epsilon_2 D\epsilon_1 Dk_2 + (1 \leftrightarrow 2) \\
    a_{10} &= k_2 \epsilon_1^T \epsilon_2 k_1 + k_2 \epsilon_1 \epsilon_2^T k_1 \\
    a_{11} &= (k_1 \epsilon_2 Dk_1 + k_1 D\epsilon_2 k_1)(k_2 \epsilon_1 Dk_2 + k_2 D\epsilon_1 k_2) \\
        &\quad + (k_1 \epsilon_2 k_1)(k_2 D\epsilon_1 Dk_2) + (k_1 D\epsilon_2 Dk_1)(k_2 \epsilon_1 k_2) \\
    a_{12} &= (k_1 \epsilon_2 k_1)(k_2 \epsilon_2 Dk_2 + k_2 D\epsilon_1 k_2) + (1 \leftrightarrow 2) \\
    a_{13} &= (k_1 D\epsilon_2 Dk_1)(k_2 \epsilon_1 Dk_2 + k_2 D\epsilon_1 k_2) + (1 \leftrightarrow 2) \\
    a_{14} &= (k_1 \epsilon_2 k_1)(k_2 \epsilon_1 k_2) \\
    a_{15} &= (k_1 D\epsilon_2 Dk_1)(k_2 D\epsilon_1 Dk_2) \\
\end{align}

(57)

Appendix B

Coefficients for commutative D-brane

When we turn the B field off we must only change $D_S^{ab} = D_0^{ab} = \eta^{ab}$ and $D_S^{ij} = D_0^{ij} = -\delta^{ij}$ and $D_A = 0$ again we find the same amplitude as (19) by including these changes and replacing $a_n$ coefficients with $b_n$, which

\begin{align}
    b_1 &= Tr(\epsilon_1 D_0)Tr(\epsilon_2 D_0) \\
    b_2 &= Tr(\epsilon_1^T \epsilon_2) \\
    b_3 &= Tr(D_0 \epsilon_1 D_0 \epsilon_2) \\
    b_4 &= Tr(\epsilon_1 D_0)(k_1 \epsilon_2 D_0 k_1 + k_1 D_0 \epsilon_2 k_1) + (1 \leftrightarrow 2) \\
    b_5 &= k_1 \epsilon_2 D_0 \epsilon_1 D_0 k_2 + k_2 \epsilon_1 D_0 \epsilon_2 D_0 k_1 + k_1 D_0 \epsilon_2 D_0 \epsilon_1 k_2 + k_2 D_0 \epsilon_1 D_0 \epsilon_2 k_1 \\
        &\quad - k_2 D_0 \epsilon_1 \epsilon_2 D_0 k_1 - k_2 D_0 \epsilon_1 \epsilon_2^T D_0 k_1 \\
    b_6 &= k_2 D_0 \epsilon_1^T \epsilon_2 k_1 + k_2 \epsilon_1^T \epsilon_2 D_0 k_1 - k_1 \epsilon_2 D_0 \epsilon_1 k_2 - k_2 \epsilon_1 D_0 \epsilon_2 k_1 \\
        &\quad + k_2 \epsilon_1 \epsilon_2^T D_0 k_1 + k_2 D_0 \epsilon_1 \epsilon_2^T k_1 \\
\end{align}
b_7 = Tr(\epsilon_1 D_0)(k_1 \epsilon_2 k_1) + (1 \leftrightarrow 2)
b_8 = Tr(\epsilon_1 D_0)(k_1 D_0 \epsilon_2 D_0 k_1) + (1 \leftrightarrow 2)
b_9 = k_1 D_0 \epsilon_2 D_0 \epsilon_1 D_0 k_2 + (1 \leftrightarrow 2)
b_{10} = k_2 \epsilon_2 T \epsilon_2 k_1 + k_2 \epsilon_1 \epsilon_2 k_1
b_{11} = (k_1 \epsilon_2 D_0 k_1 + k_1 D_0 \epsilon_2 k_1)(k_2 \epsilon_1 D_0 k_2 + k_2 D_0 \epsilon_1 k_2) + (k_1 \epsilon_2 k_1)(k_2 D_0 \epsilon_1 D_0 k_2)
+ (k_1 D_0 \epsilon_2 D_0 k_1)(k_2 \epsilon_1 k_2)
b_{12} = (k_1 \epsilon_2 k_1)(k_2 \epsilon_2 D_0 k_2 + k_2 D_0 \epsilon_1 k_2) + (1 \leftrightarrow 2)
b_{13} = (k_1 D_0 \epsilon_2 D_0 k_1)(k_2 \epsilon_2 D_0 k_2 + k_2 D_0 \epsilon_1 k_2) + (1 \leftrightarrow 2)
b_{14} = (k_1 \epsilon_2 k_1)(k_2 \epsilon_2 k_2)
b_{15} = (k_1 D_0 \epsilon_2 D_0 k_1)(k_2 D_0 \epsilon_1 D_0 k_2)

(58)

Appendix C
One open and one closed bosonic string scattering
In this appendix we want to calculate scattering of one massless open string with
one massless closed graviton string. In the bosonic case vertices for open and closed
string are,

\[ V_c = \epsilon_{\mu \lambda} D_\lambda \partial X^\mu (z_1) e^{i k_1 X(z_1)} ; \partial X^\nu (\bar{z}_1) e^{i k_1 D X(\bar{z}_1)} ; \]

\[ V_o = \xi_{\alpha} (V^T)^{\alpha}_{\rho} \partial X^\rho (y) e^{2 i k_2 V^T X(y)} ; \]

(59)

Which again we have used doubling trick for writing them. For momentum conservation relation we have,

\[ k_1^\mu + k_1 D^\mu + 2 k_2 V^T \mu = 0 \]

and physical condition for momentum and polarizations are,

\[ k_1^\mu = 0 \quad k_1^2 = 0 \quad \epsilon_\mu = 0 \]
\[ k_2 V^T \xi = 0 \quad m_{open}^2 = -(2k_2 V^T)^2 = 0 \]

(61)

By using these two last equations and \( V^T V = \frac{1}{2}(\eta + D_S) \) we find,

\[ k_1 D_S k_1 = -\frac{1}{2} m_{open}^2 = 0 \]

(62)

by fixing SL(2,R) invariant we find for amplitude of open-closed scattering from D-brane,

\[ A_{OCD} = g_c g_0 e^{-\lambda} \int d^2 z_1 dy (V_c V_o) \big|_{z=i, y=0} \]
\[ d^2 z_1 dy \to (z_1 - y)(\bar{z}_1 - y)(z_1 - \bar{z}_1) \]

(63)

with \( (DV)^{ab} = (V^T)^{ab} \) and \( (DV)^{ij} = -\eta^{ij} \) after a little computation we find graviton-gauge field and graviton-scalar scattering. From this amplitude we read
vertices as,

\[
(V_{\epsilon_1 A})^a \sim \frac{\kappa c \sqrt{T_p}}{4} (2 \text{Tr}(\epsilon_1 D_S) k_1 V_A^a - 4 k_1 D_S \epsilon_1 V_A^a + 4 k_1 D_A \epsilon_1 V_S^a \\
- \alpha' (k_1 D_S \epsilon_1 D_S k_1 + k_1 D_A \epsilon_1 D_A k_1) k_1 V_A^a)
\]

\[
(V_{\epsilon_1 \lambda})_i \sim \frac{\kappa c \sqrt{T_p}}{4} (2 \text{Tr}(\epsilon_1 D_S) k_{1i} - 4 k_1 D_S \epsilon_{1i} \\
- \alpha' (k_1 D_S \epsilon_1 D_S k_1 + k_1 D_A \epsilon_1 D_A k_1) k_{1i})
\]

which are exactly the same as (47) if one uses the on shell condition for the states in (47). The last terms in the vertices in (47) vanishes upon using the massless condition for open strings i.e. (62).
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