Properties of a beam splitter entangler with Gaussian input states

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Abstract

An explicit formula is given for the quantity of entanglement in the output state of a beam splitter, given the squeezed vacuum states input in each mode.

For the general Gaussian states input, the necessary and sufficient condition for the inseparability of the output state from a beam splitter is given.

The beam splitter is one of the few quantum devices that may act as the entangler. The entangler properties of a beam splitter have been studied in the past [1–7].

In laboratory, coherent states and squeezed states are two popularly existing robust states. It is well known that no entanglement is produced if the input states are coherent states. Therefore it is important to know the entanglement property when squeezed states are used as the input. The output entanglement quantity is studied in ref [4] given the squeezed state input. In particular, an explicit formula expressing the output state in the form of two mode squeezed states are given. However, the result there is limited to a type of rather specific case. For example, the beam splitter there is limited to the 50:50 beam splitter, the input squeezed states can only have the real squeezing parameters and so on. In this paper, we shall investigate this problem in a rather general background. We will give an explicit formula for the entanglement quantity of the output state.

In general, the input state could be a mixed state. So far there is no good measure for the

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entanglement quantity in the most general case. However, it’s possible to investigate the
problem of the necessary or necessary and sufficient conditions for a state to be an entangled
state. It is shown that [7] in order to obtain an entangled output state, a necessary condition
is that the input state should be non-classical. However, this is only a necessary condition
to obtain the entangled output state, it is not a sufficient condition in general. Since a beam
splitter operator is unitary, it is reversible. It has been shown in ref [4] that nonclassical
separable input state can be changed to an entangled state in the output. The inverse of
such a process makes examples that even though the input state is nonclassical, the output
could be still separable. Some specific examples are given in [6]. The necessary and sufficient
condition for an inseparable output state is not given so far. It is possible to obtain the
necessary and sufficient condition for inseparability of the output state given the Gaussian
input state. We will give this condition explicitly in this paper.

Consider a loseless beam splitter(see figure 1). We can distinguish the field mode \(a\) and mode
\(b\) by the different propagating direction. Most generally, the property of a beam splitter
operator \(\hat{B}\) in Schrodinger picture can be summarized by the following equations(see e.g.,
ref [8])

\[
\rho_{\text{out}} = \hat{B}\rho_{\text{in}}\hat{B}^{-1}, \tag{1}
\]

\[
\hat{B}^\dagger = \hat{B}^{-1}, \tag{2}
\]

\[
\hat{B}\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}\hat{B}^{-1} = M_B\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \tag{3}
\]

\[
M_B = \begin{pmatrix} 
\cos \theta e^{i\phi_0} & \sin \theta e^{i\phi_1} \\
-\sin \theta e^{-i\phi_1} & \cos \theta e^{-i\phi_0}
\end{pmatrix}, \tag{4}
\]

\[
\hat{B}|00\rangle = |00\rangle. \tag{5}
\]

Here \(\rho_{\text{in}}\) and \(\rho_{\text{out}}\) are the density operator for the input and output states respectively. Both
of them are two mode states including mode \(a\) and mode \(b\). The elements in the matrix \(M_B\)
are determined by the beam splitter itself, \( a, b \) are the annihilation operators for mode \( a \) and mode \( b \) respective, \(|00\rangle\) is the vacuum state for both mode. Equation (5) is due to the simple fact of no input no output.

Suppose the input states are the squeezed vacuum states in each mode, i.e.

\[
\rho_{in} = \hat{S}_a(\zeta_a)\hat{S}_b(\zeta_b)|00\rangle\langle 00|\hat{S}_a^\dagger(\zeta_a)\hat{S}_b^\dagger(\zeta_b),
\]

where

\[
\hat{S}_a(\zeta_a) = \exp \left( \frac{1}{2} \zeta_a^* \hat{a}^2 - \frac{1}{2} \zeta_a \hat{a}^\dagger \right);
\]

\[
\hat{S}_b(\zeta_b) = \exp \left( \frac{1}{2} \zeta_b^* \hat{b}^2 - \frac{1}{2} \zeta_b \hat{b}^\dagger \right).
\]

For simplicity, we use the characteristic function for the input state \( \rho_{in} \) and the output state \( \rho_{out} \).

\[
C_{in}(\xi_a, \xi_b) = \text{tr} \left[ \exp \left( \xi_a \hat{a} + \xi_b \hat{b} - \xi_a^* \hat{a}^\dagger - \xi_b^* \hat{b}^\dagger \right) \rho_{in} \right]
\]

\[
= \text{tr} \left\{ \exp \left[ i \sqrt{2} (\xi_a^I \hat{x}_a + \xi_a^R \hat{p}_a + \xi_b^I \hat{x}_b + \xi_b^R \hat{p}_b) \right] \rho_{in} \right\},
\]

where the parameters \( \xi_{a,b} = \xi_{a,b}^R + i \xi_{a,b}^I \), \((\hat{x}_a, \hat{p}_a) = N(\hat{a}^\dagger, \hat{a})^T \), \((\hat{x}_b, \hat{p}_b) = N(\hat{b}^\dagger, \hat{b})^T \) and

\[
N = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \]

In the case of squeezed states input, the characteristic function is

\[
C_{in}(\xi_a, \xi_b) = \exp \left[ -\frac{1}{2} \left( |\xi_a| \cosh r_a - \xi_a^* e^{i\chi_a} \sinh r_a |^2 + |\xi_b| \cosh r_b - \xi_b^* e^{i\chi_b} \sinh r_b |^2 \right) \right],
\]

where \( r_{a,b} = |\zeta_{a,b}| \) and \( \chi_{a,b} = \zeta_{a,b} / r_{a,b} \). The characteristic function of the output state is

\[
C_{out}(\xi_a, \xi_b) = \exp \left[ -\frac{1}{2} \left( |\xi_a| \cosh r_a - \xi_a^* e^{i\phi_a} \sinh r_a \right) \cosh r_a - (\xi_a^* \cos \theta e^{i\phi_0} + \xi_b^* \sin \theta e^{i\phi_1} e^{i\chi_a} \sinh r_a |^2 \right)
\]

\[
\cdot \exp \left[ -\frac{1}{2} \left( |\xi_a \sin \theta e^{i\phi_0} + \xi_b \cos \theta e^{i\phi_1} \cosh r_b - (\xi_a^* \sin \theta e^{-i\phi_1} + \xi_b^* \cos \theta e^{-i\phi_0}) e^{i\chi_b} \sinh r_b |^2 \right) \right].
\]

Denote the output state of mode \( a \) being \( \rho_{oa} \). The quantity of entanglement for the output state between mode \( a \) and mode \( b \) is

\[
E(\rho_{oa}) = \text{tr}(\rho_{oa} \ln \rho_{oa}).
\]
The characteristic function for the output state in mode $a$ is

$$C_{oa}(\xi_a) = C_{out}(\xi_a, \xi_b = 0) = \exp\left[-\frac{1}{2} \left| \xi_a \cos \theta e^{-i\phi_0} \cosh r_a - \xi_a^* \cos \theta e^{i\phi_0 + i\chi_a} \sinh r_a \right|^2 \right]$$

$$\cdot \exp\left[-\frac{1}{2} \left| \xi_a \sin \theta e^{i\phi_1} \cosh r_b - \xi_a^* \sin \theta e^{-i\phi_1 + i\chi_b} \sinh r_b \right|^2 \right]$$

(12)

This can be written in the form in $\xi_R$ and $\xi_I$ where $\xi_R + i\xi_I = \xi_a$, i.e.

$$C_{oa} = \exp\left[-\frac{1}{2}(\xi_R, \xi_I)M_{oa}(\xi_R, \xi_I)^T \right] .$$

(13)

Here $M_{oa}$ is the $2 \times 2$ covariance matrix as $M_{oa} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$. After calculation we obtain the matrix elements

$$m_{11} = \cosh^2 r_a + \sinh^2 r_a - 2 \sinh r_a \cosh r_a \cos(2\phi_0 + \chi_a)$$

$$+ \cosh^2 r_b + \sinh^2 r_b - 2 \sinh r_b \cosh r_b \cos(2\phi_1 - \chi_b),$$

(14)

$$m_{12} = m_{21} = 2 [\sinh r_b \cosh r_b \sin(2\phi_1 - \chi_b) - \sinh r_a \cosh r_a \sin(2\phi_0 + \chi_a)]$$

(15)

and

$$m_{22} = \cosh^2 r_a + \sinh^2 r_a + 2 \sinh r_a \cosh r_a \cos(2\phi_0 + \chi_a)$$

$$+ \cosh^2 r_b + \sinh^2 r_b + 2 \sinh r_b \cosh r_b \cos(2\phi_1 - \chi_b).$$

(16)

We can choose an appropriate unitary transformation to $\rho_{oa}$ to obtain another density operator $\rho'_{oa}$ whose characteristic function is

$$C'_{oa}(\xi_a) = \exp\left[-\frac{1}{2}(\xi_R, \xi_I) \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}(\xi_R, \xi_I)^T \right]$$

(17)

and

$$\delta = 2\sqrt{m_{11}m_{22} - m_{12}^2} .$$

(18)

We know the Wigner characteristic function for a thermal state $(1 - e^{-\beta})e^{-\beta a^\dagger a}$ is [10]

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\( C_{th}(\xi) = \exp \left[ -\frac{1}{2} (\xi_R, \xi_I) \begin{pmatrix} \frac{1+e^{-\beta}}{2(1-e^{-\beta})} & 0 \\ 0 & \frac{1+e^{-\beta}}{2(1-e^{-\beta})} \end{pmatrix} (\xi_R, \xi_I) \right] \), \hspace{0.5cm} (19)

This is to say, the state defined by the characteristic function in equation(17) is a thermal state in the form

\[ \rho_{ao}' = (1 - e^{-\beta})e^{-\beta a^\dagger a} \] \hspace{0.5cm} (20)

with the parameter

\[ e^{-\beta} = \frac{2\delta - 1}{2\delta + 1}. \] \hspace{0.5cm} (21)

Since the trace value does not change under any unitary transformation, the entanglement quantity defined in equation(11) is

\[ E(\rho_{ao}) = \text{tr} \rho_{ao}' \ln \rho_{ao}' \] \hspace{0.5cm} (22)

For the thermal state defined by equation(20), calculation for the quantity \( \text{tr} \rho_{ao}' \ln \rho_{ao}' \) is straightforward. Thus we have the following result for the quantity of entanglement of the output state given the squeezed state input in each mode:

\[ E(\rho_{out}) = \ln(1 - e^{-\beta}) + \frac{\beta e^{-\beta}}{1 - e^{-\beta}} = \ln \frac{2}{2\delta + 1} - \frac{2\delta - 1}{2} \ln \frac{2\delta - 1}{2\delta + 1}, \] \hspace{0.5cm} (23)

with \( \delta \) being defined by equation(18) and equation(14,15,16). The above equation together with the previous equations for the definition of \( \delta \) gives a direct calculation formula for the entanglement quantity given the independent squeezed state as the input to each mode. In order to maximize the entanglement, we should maximize the value of \( \delta \). Therefore the following conditions are required for the maximum entanglement

\[ 2\phi_1 - \chi_b = 0; 2\phi_0 + \chi_0 = 0. \] \hspace{0.5cm} (24)

Note that even these conditions are satisfied, we should still take a further setting between the values of \( \cos^2 \theta \) and \( r_a, r_b \) to obtain the largest entanglement given the input state. That is to say, with our formula, one is not only able to know entanglement quantity of the output...
state but also able to maximize the entanglement through adjusting the parameters in the beam splitter given the input squeezed states or adjusting the parameters in the input states given the beam splitter.

We have studied the entanglement quantity given squeezed vacuum states input. Now we begin to study the criterion for the inseparability given arbitrary Gaussian state input. Actually, to a two mode Gaussian state, the necessary and sufficient condition for the inseparability has been given in the existing literature already, e.g., Simon’s work [9]. So to give the criterion for the entangled output state, we only need to give the characteristic function of the output state first and then use Simon’s result. Again we use the characteristic function for the input state. Without any loss of generality, we may assume the characteristic function for a Gaussian input state to be

\[ C_{in}(\xi_a, \xi_b) = \exp \left[ -\frac{1}{2} (\xi')^T M_{in} (\xi') \right], \]  
\( (\xi') = (\xi^I_a, \xi^R_a, \xi^I_b, \xi^R_B) \) 

(25)

and \( M_{in} \) is a \( 4 \times 4 \) real symmetric matrix. The characteristic function for the output state is

\[ C_{out}(\xi_a, \xi_b) = \text{tr} \left[ \exp \left( \xi_a \hat{a} - \xi_a^* \hat{a}^\dagger + \xi_b \hat{b} - \xi_b^* \hat{b}^\dagger \right) \hat{B} \rho_{in} \hat{B}^{-1} \right] 
\]

\[ = \text{tr} \left[ \hat{B}^{-1} \exp \left( \xi_a \hat{a} - \xi_a^* \hat{a}^\dagger + \xi_b \hat{b} - \xi_b^* \hat{b}^\dagger \right) \hat{B} \rho_{in} \right]. \]

\[ = \text{tr} \left\{ \hat{B}^{-1} \exp \left[ i\sqrt{2}(\xi_a^I \hat{x} + \xi_a^R \hat{p}_a + \xi_b^I \hat{x} + \xi_b^R \hat{p}_b) \right] \hat{B} \rho_{in} \right\}. \]  
\( (\hat{x}, \hat{p}_a, \hat{x}, \hat{p}_b) \) 

(27)

Using the properties \( \hat{B} \) as listed in eq.(3) we have

\[ \hat{B}^{-1}(\hat{x}, \hat{p}_a, \hat{x}, \hat{p}_b)^T \hat{B} = \left( \begin{array}{cc} N & O \\ O & N \end{array} \right) \sigma \left( \begin{array}{cc} M_B & O \\ O & M_B \end{array} \right) \sigma \left( \begin{array}{cc} N^{-1} & O \\ O & N^{-1} \end{array} \right)(\hat{x}, \hat{p}_a, \hat{x}, \hat{p}_b)^T, \]  

(28)

where \( O \) is the \( 2 \times 2 \) matrix with all matrix elements being 0 and
\[ \sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{29} \]

Therefore the characteristic function of the output state is
\[ C_{out}(\xi_a, \xi_b) = \exp \left[ -\frac{1}{2} (\xi'')^T M_{in}(\xi'') \right] = \exp \left[ -\frac{1}{2} (\xi')^T M_{out}(\xi') \right] , \tag{30} \]

where
\[ (\xi'') = (\xi_a^I, s_a, \xi_b^I, \xi_B^R) Q , \tag{31} \]

and
\[ Q = \begin{pmatrix} N & O \\ O & N \end{pmatrix} \sigma \begin{pmatrix} M_B & O \\ O & M_B^* \end{pmatrix} \sigma \begin{pmatrix} N^{-1} & O \\ O & N^{-1} \end{pmatrix} . \tag{32} \]

From this we can see that the covariance matrix of the output state is
\[ M_{out} = Q M_{in} Q^T . \tag{33} \]

After calculation we obtain the following explicit form for the matrix \( Q \)
\[ Q = \begin{pmatrix} \cos \theta \cos \phi_0 & -\cos \theta \sin \phi_0 & \sin \theta \cos \phi_1 & -\sin \theta \sin \phi_1 \\ \cos \theta \sin \phi_0 & \cos \theta \cos \phi_0 & \sin \theta \sin \phi_1 & \sin \theta \cos \phi_1 \\ -\sin \theta \cos \phi_1 & -\sin \theta \sin \phi_1 & \cos \theta \cos \phi_0 & \cos \theta \sin \phi_0 \\ \sin \theta \sin \phi_1 & -\sin \theta \cos \phi_1 & -\cos \theta \sin \phi_0 & \cos \theta \cos \phi_0 \end{pmatrix} = \begin{pmatrix} X & Y \\ -Y & X^T \end{pmatrix} \tag{34} \]

where \( X = \begin{pmatrix} \cos \theta \cos \phi_0 & -\cos \theta \sin \phi_0 \\ \cos \theta \sin \phi_0 & \cos \theta \cos \phi_0 \end{pmatrix} \) and \( Y = \begin{pmatrix} \sin \theta \cos \phi_1 & -\sin \theta \sin \phi_1 \\ \sin \theta \sin \phi_1 & \sin \theta \cos \phi_1 \end{pmatrix} \). Suppose the covariance matrix of the input state can be decomposed in the following block form
\[ M_{in} = \begin{pmatrix} A_0 & C_0 \\ C_0^T & B_0 \end{pmatrix} . \tag{35} \]
Then the output covariance matrix can be written in the form of
\[
M_{out} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} = \begin{pmatrix} X & Y \\ -Y & X^T \end{pmatrix} \begin{pmatrix} A_0 & C_0 \\ C_0^T & B_0 \end{pmatrix} \begin{pmatrix} X & Y \\ -Y & X^T \end{pmatrix}^T.
\] (36)

Using the Simon’s result [9] we know that the necessary and sufficient condition for the inseparability of the output state is
\[
\det A \det B + \left( \frac{1}{4} - |\det C| \right) - \text{tr}(AJCJBC^TJ) < \frac{1}{4}(\det A + \det B)
\] (37)

and \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Together with equation (34) and equation (36), equation (37) gives the criterion to judge whether the output state is entangled or not given the beam splitter and arbitrary Gaussian form input state.

In summary, we have studied the following properties for a beam splitter entangler: In the case that the input state is a squeezed vacuum state in each mode, we have given the explicit formula for the entanglement quantity in the output state; in the case that the input state is a general Gaussian state, a necessary and sufficient condition is given for the inseparability of the output state.

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REFERENCES


FIG. 1. A schematic diagram for the beamsplitter operation. Both the input and the output are two mode states. The different mode is distinguished by the propagating direction of the field.