Quantum entanglement is one of the key ingredients in various types of quantum information processing. A notable example is dense coding [1], which doubles the capacity of transmission of classical information assisted by an maximally entangled pair of qubits shared between the sender (Alice) and receiver (Bob). Several authors have studied the capacity of dense coding in various situations [2–6]. In this paper, the author derives an entanglement measure for any bipartite states in the light of the capacity of dense coding and investigates its properties systematically. First, the general scheme of dense coding with a mixed state on the Hilbert space $C^d \otimes C^d$ is described. Alice performs one of the local unitary transformations $U_i \in U(d)$ on her $d$-dimensional quantum system in order to put the initially shared entangled state $\rho$ in $\rho_i = (U_i \otimes I_d)\rho(U_i^\dagger \otimes I_d)$ with a priori probability $p_i$ ($i = 1, 2, \ldots, i_{\text{max}}$), and then she sends her quantum system to Bob. Upon receiving this quantum system, Bob performs a suitable measurement on $\rho_i$ to extract the signal. The optimal amount of information that can be conveyed is known to be bounded from above by the Holevo quantity [7],
\[
\chi = S(\mathcal{F}) - \sum_{i=1}^{i_{\text{max}}} p_i S(\rho_i),
\]
where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ denotes the von Neumann entropy and $\mathcal{F} = \sum_{i=1}^{i_{\text{max}}} p_i \rho_i$ is the average density matrix of the signal ensemble. Since the Holevo quantity is asymptotically achievable [8,9], Eq. (1) is used here as the definition of the capacity of dense coding. Capacity $\chi$ is maximized for signal states $\rho_i$ with mutually orthogonal unitary transformations, $\text{Tr} \left( U_i^\dagger U_j \right) = d \delta_{ij}$ and equal probabilities $p_i = d^{-2}$ ($i_{\text{max}} = d^2$) [4]. The optimal capacity is written as $\chi^*(\rho) = \log_2 d + I_B(\rho)$, where $I_B(\rho) = S(\rho^B) - S(\rho)$ is the coherent information with $\rho^B = \text{Tr}_A \rho$. Since max $[S(\rho^A) - S(\rho), (S(\rho^B) - S(\rho))] \leq E_R(\rho)$ [10],
\[
I_B(\rho) \leq E_R(\rho),
\]
and the capacity $\chi^*(\rho)$ satisfies $\chi^*(\rho) \leq \log_2 d + E_R(\rho)$ [4]. Here, $E_R(\rho)$ is the relative entropy of entanglement [11,12] for states $\rho$ defined as $E_R(\rho) = \min_{\sigma \in \mathcal{D}} S(\rho|\sigma)$, where $\mathcal{D}$ the set of states with positive partial transpose (PPT states) and $S(\rho|\sigma) = \text{Tr} \left[ \rho (\log_2 \rho - \log_2 \sigma) \right]$ is the quantum relative entropy of $\rho$ with respect to $\sigma$.

Note that this capacity is optimal in the sense that Alice and Bob uses a given mixed state $\rho$ as a resource for dense coding without any changes. If they are allowed to perform local quantum operations and classical communications (LQCC) on the initially shared mixed state $\rho$ prior to the dense coding, the capacity could be enhanced further. The maximally achievable capacity is
\[
\chi^*_\text{max}(\rho) = \log_2 d + \lim_{n \to \infty} \sup_{\Lambda_n} \frac{1}{n} I_B \left( \Lambda_n(\rho^\otimes n) \right) \equiv \log_2 d + E_{dc}(\rho),
\]
Namedly, $E_{dc}(\rho)$ is the asymptotic limit of the achievable (normalized) coherent information over the sequence of LQCC operations $\{\Lambda_n\}$ or an LQCC protocol.

Hereafter the properties of $E_{dc}(\rho)$ defined in Eq. (3) are examined. $E_{dc}(\rho)$ is the maximal dense coding capacity subtracted by the classically achievable capacity $\log_2 d$; it represents the maximal contribution of entanglement to the dense coding capacity. As shown in the following, $E_{dc}(\rho)$ is a measure of entanglement of $\rho$. Before proving this, the following inequalities must be proved.
\[
E_D(\rho) \leq E_{dc}(\rho) \leq E_R^{\infty}(\rho),
\]
where $E_D(\rho)$ and $E_R^{\infty}(\rho)$ are, respectively, the distillable entanglement [13] and the asymptotic relative entropy of entanglement [14] of $\rho$, both of which are asymptotic entanglement measures. $E_R^{\infty}(\rho)$ is defined as the average relative entropy of entanglement per copy:
\[
E_R^{\infty}(\rho) = \lim_{n \to \infty} \frac{E_R(\rho^\otimes n)}{n}.
\]
Noting the subadditivity of the relative entropy of entanglement, i.e., $E_R(\rho^\otimes n) \leq n E_R(\rho)$ [11], a weaker version of Eq. (4) is obtained:
\[
E_D(\rho) \leq E_{dc}(\rho) \leq E_R(\rho).
\]
Although the proof of the first part of Eq. (4) is essentially the same as that in [15], the proof is described here for completeness. It is always possible to consider that...
the distillation protocol is ended by $U \otimes U^*$ twirling [16] so that the final state is an isotropic state of the form

$$\rho(F_n, d_n) = p P_+(C^d_n) + (1 - p) \frac{1}{d_n^2} I_{d_n},$$  \hspace{1cm} (7)

where $F_n = \text{Tr} \left[\rho(F_n, d_n) P_+(C^d_n)\right]$ is the fidelity, $I_{d_n}$ is the identity of dimensions $d_n$, and

$$P_+(C^d) = |\psi_+(C^d)\rangle \langle \psi_+(C^d)|$$  \hspace{1cm} (8)

is the maximally entangled state of a $C^d \otimes C^d$ system. In Eq. (8), $|\psi_+(C^d)\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ with $|i\rangle$ are basis vectors in $C^d$. Because the protocol mentioned above is not necessarily optimal for $E_{dc}$, $E_{dc}(\rho) \geq E_{DC}(\rho)$ (see below) of $\rho(F_n, d_n)$). The coherent information for $\rho(F_n, d_n)$ is easily calculated as

$$I_B(\rho(F_n, d_n)) = \log_2 d_n + F_n \log_2 F_n + (1 - F_n) \log_2 \frac{1 - F_n}{d_n^2 - 1}.$$  \hspace{1cm} (9)

By definition of the distillable entanglement [13], $F_n \rightarrow 1$ and $\frac{\log_2 d_n}{d_n^2} \rightarrow E_D(\rho)$ for $n \rightarrow \infty$. Therefore, $E_{dc}(\rho) \geq E_D(\rho)$. The proof of the second part of Eq. (4) is as follows. Equation (2) and the weak monotonicity (see below) of $E_R$ [12] give

$$E_{dc}(\rho) \leq \lim_{n \rightarrow \infty} \sup_{\Lambda_n} \frac{1}{n} E_R (\Lambda_n(\rho^{\otimes n})) \leq \lim_{n \rightarrow \infty} \frac{1}{n} E_R (\rho^{\otimes n}).$$  \hspace{1cm} (10)

The right-hand side is, by definition, $E_R^{\infty}(\rho)$. Therefore, $E_{dc}(\rho) \leq E_R^{\infty}(\rho)$.

The quantity thus defined is an entanglement measure; namely, it satisfies the following properties [15,17,18].

(i) $E_{dc}(\rho) = 0$ for any separable state $\rho$.

(ii) $E_{dc}(\rho) \geq 0$.

(iii) For a pure state $|\phi\rangle \langle \phi|$, $E_{dc}$ is the von Neumann entropy of the reduced density matrix, e.g., the entropy of entanglement:

$$E_{dc}(|\phi\rangle \langle \phi|) = S(\text{Tr}_A(|\phi\rangle \langle \phi|)) = S(\text{Tr}_B(|\phi\rangle \langle \phi|)).$$

In particular, $E_{dc}(P_+(C^d)) = \log_2 d$, where $P_+(C^d)$ is the maximally entangled state of a $C^d \otimes C^d$ system [Eq. (8)].

(iv) Partial additivity: $E_{dc}(\rho^{\otimes n}) = n E_{dc}(\rho)$.

(v) Weak monotonicity: $E_{dc}(\Lambda(\rho)) \leq E_{dc}(\rho)$, where $\Lambda$ is an LOCC operation. This is the most important property required of the entanglement measure.

(vi) Convexity on pure state decomposition:

$$E_{dc} \left( \sum_i p_i |\phi_i\rangle \langle \phi_i| \right) \leq \sum_i p_i E_{dc}(|\phi_i\rangle \langle \phi_i|),$$

with $\sum_i p_i = 1$ and $p_i \geq 0$.

(vii) Weak continuity: For any sequence of the pure state $|\psi_n\rangle$ and the mixed state $\rho_n$ of a system $C^d_n \otimes C^d_n$ such that $\|\rho_n - |\psi_n\rangle \langle \psi_n|| \rightarrow 0$ and $d_n \rightarrow \infty$ for $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} \frac{E_{dc}(\rho_n) - E_{dc}(|\psi_n\rangle \langle \psi_n|)}{\log d_n} = 0.$$  \hspace{1cm} (11)

Here, $\|A\|$ denotes the trace norm of $A$; $\|A\| = \text{Tr} \sqrt{A^\dagger A}$.

Properties (i)-(iii) are obvious from Eq. (6). The proof of property (iv) is as follows.

$$E_{dc}(\rho^{\otimes m}) = \lim_{n \rightarrow \infty} \sup_{\Lambda_n} \frac{1}{n} E_R (\Lambda_n(\rho^{\otimes nm}))$$

$$= m \lim_{n \rightarrow \infty} \sup_{\Lambda_n} \frac{1}{n} E_R (\Lambda_n(\rho^{\otimes n'}))$$

$$= m E_{dc}(\rho).$$

Property (v) is obvious because $E_{dc}(\rho)$ is the optimized quantity with respect to LOCC protocols by definition and the tensor product of an LOCC operation is also an LOCC operation. Property (vi) follows from Eq. (6) and the fact that both $E_{dc}(\rho)$ and $E_R(\rho)$ coincide on pure states;

$$E_{dc} \left( \sum_i p_i |\phi_i\rangle \langle \phi_i| \right) \leq E_R \left( \sum_i p_i |\phi_i\rangle \langle \phi_i| \right) \leq \sum_i p_i E_R (|\phi_i\rangle \langle \phi_i|) = \sum_i p_i E_{dc}(|\phi_i\rangle \langle \phi_i|).$$  \hspace{1cm} (12)

The proof of property (vii) is given as follows. Noting the fact that $I_B$, $E_{dc}$, and $E_R$ coincide on pure states, the inequalities, $I_B(\rho_n) \leq E_{dc}(\rho_n) \leq E_R(\rho_n)$ give

$$I_B(\rho_n) - I_B(|\psi_n\rangle \langle \psi_n|) \leq E_{dc}(\rho_n) - E_{dc}(|\psi_n\rangle \langle \psi_n|) \leq E_R(\rho_n) - E_R(|\psi_n\rangle \langle \psi_n|).$$  \hspace{1cm} (13)

Firstly,

$$\lim_{n \rightarrow \infty} \frac{E_R(\rho_n) - E_R(|\psi_n\rangle \langle \psi_n|)}{\log d_n} = 0,$$  \hspace{1cm} (14)

because $E_R$ is continuous. Secondly, Fannes' inequality [19],

$$|S(\rho) - S(\sigma)| \leq \|\rho - \sigma\|_1 \log_2 \text{dim} \mathcal{H} + \eta(\|\rho - \sigma\|_1),$$  \hspace{1cm} (15)
plays a key role. It holds for any two states $\rho$ and $\sigma$ acting on the Hilbert space $\mathcal{H}$ provided that $\|\rho - \sigma\|_1 \leq 1/e$.

In Eq. (15), $\eta(s) = -s\log_2 s$. Noting the fact that the partial trace does not increase the trace norm and $\eta(s)$ is a monotonically increasing function for $0 \leq s \leq 1/e$, Fannes’ inequality [Eq. (15)] gives

$$|I_B(\rho_n) - I_B(\psi_n)\rangle\langle\psi_n|\rangle| \leq 3\|\rho_n - |\psi_n\rangle\langle\psi_n||_1 \log_2 d_n + 2\eta(\|\rho_n - |\psi_n\rangle\langle\psi_n||_1), \quad (16)$$

Therefore,

$$\lim_{n \to \infty} \frac{|I_B(\rho_n) - I_B(\psi_n)\rangle\langle\psi_n|\rangle|}{\log d_n} = 0. \quad (17)$$

From Eqs. (13), (14), and (17), the following equation is obtained:

$$\lim_{n \to \infty} \frac{E_{dc}(\rho_n) - E_{dc}(\psi_n)\rangle\langle\psi_n|\rangle|}{\log d_n} = 0. \quad (18)$$

Namely, $E_{dc}$ is weakly continuous.

In addition to properties (i)-(vii), $E_{dc}(\rho)$ exhibits superadditivity. Namely, $E_{dc}(\rho \otimes \sigma) \geq E_{dc}(\rho) + E_{dc}(\sigma)$. The proof is as follows. Because of the additivity of the coherent information,

$$E_{dc}(\rho) + E_{dc}(\sigma) = \lim_{n \to \infty} \sup_{\Lambda_n^\otimes \otimes \Lambda_n^\otimes} \frac{1}{n} I_B \left( (\Lambda_n^\otimes \otimes \Lambda_n^\otimes) (\rho^\otimes_n \otimes \sigma^\otimes_n) \right). \quad (19)$$

Here, even if the protocol $\{\Lambda_n^\otimes \otimes \Lambda_n^\otimes\}$ is optimized, it is not necessarily the optimal one for $\{(\rho \otimes \sigma)^\otimes_n\}$. Therefore,

$$E_{dc}(\rho) + E_{dc}(\sigma) \leq \lim_{n \to \infty} \sup_{\Lambda_n^\otimes \otimes \Lambda_n^\otimes} \frac{1}{n} I_B \left( \Lambda_n^\otimes \otimes \Lambda_n^\otimes (\rho \otimes \sigma)^\otimes_n \right) = E_{dc}(\rho \otimes \sigma). \quad (20)$$

It is not clear at present if the equality (full additivity) holds. The convexity of the general form,

$$E_{dc} \left( \sum_i p_i \rho_i \right) \leq \sum_i p_i E_{dc}(\rho_i), \quad (21)$$

is also doubtful. However, it should be noted that the breakdown of the full additivity and the general convexity is not a drawback; it is argued that these two requirements are too strong for asymptotic entanglement measures [15,17].

Although it is in general quite difficult to calculate $E_{dc}(\rho)$, there are some special mixed states in which $E_{dc}(\rho)$ is obtained explicitly.

**Example 1**—This is the example by Rains [20,21]. It is called the maximally correlated state of a $C^d \otimes C^d$ system, and takes the form

$$\rho = \sum_{i,j=1}^d \alpha_{ij} |ii\rangle\langle jj| \cdot (22)$$

The relative entropy of entanglement is calculated as

$$E_R(\rho) = I_B(\rho) = H(\alpha_{11}, \alpha_{22}, \cdots) - S(\rho), \quad (23)$$

where $H(\alpha_{11}, \alpha_{22}, \cdots)$ denotes the Shannon entropy of probability distribution $\{\alpha_{ij}\}$. From $E_R(\rho) = I_B(\rho)$, $E_{dc}(\rho) = E_{dc}^R(\rho) = E_R(\rho)$, which is proved as follows:

$$E_R(\rho) \leq E_{dc}(\rho) \leq \lim_{n \to \infty} \sup_{\Lambda_n^\otimes \otimes \Lambda_n^\otimes} \frac{1}{n \Lambda_n^\otimes} E_R \left( \Lambda_n^\otimes (\rho^\otimes_n) \right) \leq \lim_{n \to \infty} \frac{1}{n \Lambda_n^\otimes} E_R(\rho^\otimes_n) = E_R^\otimes(\rho) \leq E_R(\rho). \quad (24)$$

The first inequality is obvious because $I_B(\rho) \leq E_{dc}(\rho)$. The second inequality is a consequence of Eq. (2), and the third inequality follows from the weak monotonicity of $E_R$. The last inequality is a result of the subadditivity of $E_R$. The optimal LQCC operation for $E_{dc}$ is simply $\Lambda_n^I = I_{d^n}$. It has been shown that $E_R(\rho)$ is exactly the PPT distillable entanglement (distillable entanglement with respect to positive partial transpose operations [22]). Since the set of LQCC operations is a subset of the set of PPT operations, $E_D(\rho)$ is the lower bound on the PPT distillable entanglement. Therefore, $E_{dc}(\rho) = E_{dc}^R(\rho) = E_R(\rho) = I_B(\rho) \geq E_D(\rho)$. When $d = 2$, the maximally correlated state is equivalent to a mixture of two Bell states (a Bell diagonal state of rank two) if $\text{Tr}_{ABC} = I_{12}/2$. For this state the hashing protocol of distillation yields the value of $E_R(\rho) = 1 - S(\rho)$ for the distillable entanglement [13] so that $E_{dc}(\rho) = E_D(\rho)$.

**Example 2**—This is the example by Eisert et al. [23]. Suppose that Alice and Bob share initially $N = 2J$ ($J = 1, 2, \cdots$) pair of qubits each in the same state $|\phi\rangle = \alpha |00\rangle + \beta |11\rangle$. Hereafter in this example $\alpha = \beta = 1/\sqrt{2}$ is assumed for simplicity. After the complete loss of the order of Bob’s particles, the initially shared pure state $|\phi^\otimes_N\rangle$ becomes a mixed state of the form

$$\rho = \sum_{j=0}^J \sum_{\alpha, \beta=1}^d p_j |\psi_j(\alpha_j, \beta_j\rangle\langle\psi_j(\alpha_j, \beta_j)\rangle, \quad (25)$$

where

$$|\psi_j(\alpha_j, \beta_j\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} |j, m, \alpha_j \rangle |j, m, \beta_j\rangle, \quad (26)$$

$p_j = (2j + 1)/(d_j 2^{2j})$, and $d_j = 2j + 1$ is the multiplicity of the $j$—representation in $SU(2)^{\otimes N}$. It is easy to calculate the coherent information;

$$I_B(\rho) = \sum_{j=0}^d d_j^2 p_j \log_2 (2j + 1) - \log_2 d_j \cdot (27)$$
On the other hand, the relative entropy of entanglement and the distillable entanglement are calculated as [23]

\[ E_R(\rho) = E_D(\rho) = \sum_{j=0}^{J} d_j^2 p_j \log_2(2j + 1) \]  

so that \( E_{dc}(\rho) = E_R^*(\rho) = E_R(\rho) = E_D(\rho) \geq I_B(\rho) \). The first three equalities follow from Eqs. (4) and (6) and the subadditivity of \( E_R \). The last equality holds only for \( J = 1 \) (\( d_0 = d_1 = 1 \)). The optimal distillation is the optimal LQCC protocol for \( E_{dc} \).

Two examples described above show that it is reasonable to conjecture that the optimal protocol for \( E_{dc}(\rho) \) is either the identity \( E_{dc}(\rho) = I_B(\rho) \geq E_D(\rho) \) or the optimal distillation protocols \( E_{dc}(\rho) = E_D(\rho) \geq I_B(\rho) \). However, Horodecki et al. have conjectured that \( I_B(\rho) \) does not exceed the one-way distillable entanglement (distillable entanglement with local operations plus one-way classical communications) for any state \( \rho \) [24]. If this conjecture (hashing inequality) is true,

\[ E_D(\rho) \geq I_B(\rho) \]

holds for any state \( \rho \). This inequality implies that \( E_D \) is weakly continuous, which is not proved yet. The proof of weak continuity follows from the same arguments of the proof of property (vii) of \( E_{dc} \). Equation (29) also implies

\[ E_{dc}(\rho) = E_D(\rho). \]

The proof of Eq. (30) is simple and essentially the same as that in [24]. The partial additivity and the weak monotonicity of \( E_D \) [15, 17] give

\[
E_D(\rho) = \lim_{n \to \infty} \frac{1}{n} E_D(\rho^\otimes n) \\
\geq \lim_{n \to \infty} \sup_{\Lambda_n} \frac{1}{n} E_D(\Lambda_n(\rho^\otimes n)) \\
\geq \lim_{n \to \infty} \sup_{\Lambda_n} \frac{1}{n} I_B(\Lambda_n(\rho^\otimes n)) = E_{dc}(\rho).
\]

From Eqs. (4) and (31), Eq. (30) is obtained. This is a satisfactory result. It strengthens the information-theoretic meaning of the distillable entanglement; namely, \( E_D \) is the ultimate measure of resources for dense coding. Furthermore, the optimal entanglement distillation seems to be the best strategy to maximize the coherent information since it increases \( S(\text{Tr}_A(\Lambda_n(\rho^\otimes n))) \) on one hand and decreases \( S(\Lambda_n(\rho^\otimes n)) \) on the other hand while keeping the dimension of \( \Lambda_n(\rho^\otimes n) \) as large as possible. According to the above reasonings, it is most likely that \( E_{dc}(\rho) = E_D(\rho) \). Unfortunately, the assumed inequality \( E_D(\rho) \geq I_B(\rho) \), which is also a consequence of the equalities \( E_{dc}(\rho) = E_D(\rho) \) [Eq. (30)], is not proven yet. One of the possible counter-examples is Example 1. However, Rains has conjectured that for any maximally correlated state \( \rho \) both the PPT distillable entanglement and the one-way distillable entanglement coincide [21], so \( E_D(\rho) = I_B(\rho) \). It should be noted that this conjecture is also a consequence of the hypothetical hashing inequality.

In summary, in the light of the dense coding capacity optimized with respect to LQCC, an asymptotic entanglement measure \( E_{dc} \) for any bipartite states was derived and its properties was investigated. Some examples of \( E_{dc} \) with explicit forms were also given. Finally, it was argued that \( E_{dc} \) coincides with the distillable entanglement. A possible counter-example for this conjecture was also given.

This work was supported by CREST of Japan Science and Technology Corporation (JST).

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