Noncompact Heisenberg spin magnets from high-energy QCD
II. Quantization conditions and energy spectrum

S.É. Derkachov\textsuperscript{1}, G.P. Korchemsky\textsuperscript{2}, J. Kotański\textsuperscript{3} and A.N. Manashov\textsuperscript{4}\textsuperscript{*}

\textsuperscript{1} Department of Mathematics, St.-Petersburg Technology Institute, St.-Petersburg, Russia
\textsuperscript{2} Laboratoire de Physique Théorique\textsuperscript{†}, Université de Paris XI, 91405 Orsay Cédex, France
\textsuperscript{3} Institute of Physics, Jagellonian University, Reymonta 4, PL-30-059 Cracow, Poland
\textsuperscript{4} Department d’ECM, Universitat de Barcelona, 08028 Barcelona, Spain

Abstract:

We present a complete description of the spectrum of compound states of reggeized gluons in QCD in multi-colour limit. The analysis is based on the identification of these states as ground states of noncompact Heisenberg $SL(2,\mathbb{C})$ spin magnet. A unique feature of the magnet, leading to many unusual properties of its spectrum, is that the quantum space is infinite-dimensional and conventional methods, like the Algebraic Bethe Ansatz, are not applicable. Our solution relies on the method of the Baxter $\mathcal{Q}$–operator. Solving the Baxter equations, we obtained the explicit expressions for the eigenvalues of the $\mathcal{Q}$–operator. They allowed us to establish the quantization conditions for the integrals of motion and, finally, reconstruct the spectrum of the model. We found that intercept of the states built from even (odd) number of reggeized gluons, $N$, is bigger (smaller) than one and it decreases (increases) with $N$ approaching the same unit value for infinitely large $N$.

\textsuperscript{*}Permanent address: Department of Theoretical Physics, Sankt-Petersburg State University, St.-Petersburg, Russia
\textsuperscript{†}Unite Mixte de Recherche du CNRS (UMR 8627)
# Contents

1. Introduction .................................................. 2

2. Baxter $Q$--operator ......................................... 5

3. Quantization conditions ..................................... 7
   3.1. Solution around $z = 0$ ..................................... 8
   3.2. Solution around $z = 1$ ..................................... 11
   3.3. Transition matrices ........................................ 13
   3.4. Special case: $N = 2$ ....................................... 14

4. Eigenvalues of the Baxter $Q$--operator .................. 15
   4.1. Decomposition over the Baxter blocks ................. 17
   4.2. Properties of the blocks .................................. 19
   4.3. Quantization conditions from the $Q$--blocks .......... 20
   4.4. Energy spectrum from the $Q$--blocks ................... 22

5. Energy spectrum ............................................... 24
   5.1. Fine structure of the spectrum ......................... 26
   5.2. Quantum numbers of the $N = 3$ states ............... 27
   5.3. Quantum numbers of the $N = 4$ states ............... 30
   5.4. Quantum numbers of the states with higher $N$ ....... 34

6. Summary ......................................................... 37

A Appendix: Solution to the Baxter equation at $N = 2$ ... 39

B Appendix: Properties of the $Q$--blocks .................. 41

C Appendix: Contour integral representation ................ 43

D Appendix: Degenerate $Q$--blocks .......................... 46
1. Introduction

It has been recently realized that QCD possesses a hidden symmetry at high-energy [1, 2]. This symmetry is not seen at the level of classical QCD Lagrangian and it manifests itself through remarkable integrability properties of the Schrödinger equation for the partial waves of the scattering amplitudes in perturbative QCD in the so-called generalized leading logarithmic approximation (GLLA) [3, 4]. It turns out that in the multi-colour limit this equation coincides with the Schrödinger equation for two-dimensional quantum-mechanical completely integrable model, which was dubbed in [2, 5] as a noncompact Heisenberg spin magnet.

The asymptotics of the scattering amplitudes $A(s, t)$ at high energy, $s \gg -t$, is governed by the contribution of an infinite number of soft gluons exchanged between the scattered particles. In the GLLA approximation, the scattering amplitude is given by [3, 4, 6]

$$A(s, t) \sim -is \sum_{N=2}^{\infty} (i\tilde{\alpha}_s)^N \frac{s^{-\tilde{\alpha}_s E_N/4}}{(\tilde{\alpha}_s \sigma_N \ln s)^{1/2}} \xi_{1,N}(t) \xi_{2,N}(t),$$  \hspace{1cm} (1.1)

where $\tilde{\alpha}_s = \alpha_s N_c/\pi$ is the QCD coupling constant and the sum goes over an arbitrary number of reggeized gluons exchanged in the $t$–channel, $N = 2, 3, \ldots$. In the GLLA approximation, the reggeized gluons interact with each other elastically and form colour-singlet compound states [7]. These states can be defined as solutions to the Bartels-Kwiecinski-Praszalowicz (BKP) equation [3, 8]

$$H_N \Psi(z_1, z_2, ..., z_N) = E_N \Psi(z_1, z_2, ..., z_N).$$  \hspace{1cm} (1.2)

Here, the effective QCD Hamiltonian $H_N$ acts on the colour $SU(N_c)$ charges of $N$ reggeized gluons and their two-dimensional transverse coordinates, $z_k \; (k = 1, ..., N)$ belonging to the hyperplane orthogonal to the momenta of two scattered particles. The contribution to the scattering amplitude (1.1) of the compound states $\Psi(z_1, z_2, ..., z_N)$ built from $N$ reggeized gluons has the standard Regge form $\sim s^{-\tilde{\alpha}_s E_N/4}$ and it is dominated at large $s$ by the contribution of the ground state. Obviously, it increases (or decreases) with $s$ if the energy of the ground state $E_N$ for Eq. (1.2) is negative (or positive). As we will show below, the spectrum of (1.2) is gapless (see Eq. (5.6) below) and, as a consequence, one has to include in (1.1) the contribution of the excited states next to the ground state. This amounts to appearance of the additional factor $(\tilde{\alpha}_s \sigma_N \ln s)^{-1/2}$ in the r.h.s. of (1.1). The residue factors $\xi_{1(2),N}(t)$ measure the overlap of $\Psi(z_1, z_2, ..., z_N)$ with the wave functions of the scattered particles. They depend, in general, on the momentum transferred, $t$, and the colour factor, $1/N_c^2$.

Calculation of the spectrum of the compound states (1.2) for arbitrary number of reggeized gluons $N$ and eventual resummation of their contribution to the scattering amplitude (1.1) is a longstanding problem in high-energy QCD [9, 10, 11, 12]. At $N = 2$, the solution to (1.2) has been found a long time ago – the well-known Balitsky-Fadin-Kuraev-Lipatov (BFKL) Pomeron [7]. At $N = 3$ the solution to (1.2) – the Odderon state in QCD [13], was formulated only a few years ago by Janik and Wosiek [14] by making use of the remarkable integrability properties of the effective QCD Hamiltonian [1, 2]. This solution has been later verified in Refs. [15, 16]. The methods employed at $N = 3$ in [14] can not be generalized, however, to higher ($N \geq 4$) reggeized gluon compound states and, as a consequence, a very little is known about the solutions to (1.2) for $N \geq 4$. Recently, a significant progress has been made in solving the Schrödinger equation (1.2) for higher $N$ in the multi-colour limit, $N_c \to \infty$ and $\tilde{\alpha}_s = \text{fixed}$ [5, 16]. The first results of the calculation of the ground state energy $E_N$ for higher reggeized gluon compound states in
multi-colour QCD were reported in a letter format [17]. In this paper, we shall provide a detailed account on the approach used in [17] and present new results for the spectrum of the Schrödinger equation (1.2). The reader interested in learning more about the latter could skip the first part of the paper and go directly to Section 5.

Our approach to solving the BKP equation (1.2) is based on the equivalence of the effective Hamiltonian $H_N$ in the multi-colour QCD with the Hamiltonian of a completely integrable two-dimensional noncompact Heisenberg spin magnet [1, 2]. The latter model describes the nearest neighbour interaction between spins of $N$ particles “living” on a two-dimensional plane of transverse coordinates $\vec{z} = (x, y)$. The corresponding spin operators $\vec{S}_k$ and $\vec{\bar{S}}_k$ (with $k = 1, \ldots, N$) are the generators of the unitary principal series representation of the $SL(2, \mathbb{C})$ group specified by a pair of complex spins $(s, \bar{s})$. They act on the $\vec{z}$–plane as the differential operators\(^3\)

$$\begin{align*}
S_k^0 &= z_k \partial_{z_k} + s, \quad S_k^- = -\partial_{z_k}, \quad S_k^+ = z_k^2 \partial_{z_k} + 2sz_k. 
\end{align*}$$

(1.3)

The operators $\vec{S}_k^\pm, 0$ are given by similar expressions with $z_k$ and $s$ replaced by $\bar{z}_k$ and $\bar{s}$, respectively. Here, the notation was introduced for the (anti)holomorphic coordinates on a two-dimensional $\vec{z}$–plane, $z_k = x_k + i y_k$ and $\bar{z}_k = z_k^*$, so that $d^2z_k = dz_k d\bar{z}_k/2$. By the definition, $\vec{S}_k^2 = s(s - 1)$, $\vec{\bar{S}}_k^2 = \bar{s}(\bar{s} - 1)$ and $[\vec{S}_k^a, \vec{S}_k^{b\dagger}] = 0$ for $a, b = \pm, 0$.

For the principal series of the $SL(2, \mathbb{C})$, the possible values of the complex spins $(s, \bar{s})$ take the form [18]

$$s = \frac{1 + n_s}{2} + i \nu_s, \quad \bar{s} = 1 - s^* = \frac{1 - n_s}{2} + i \nu_s$$

(1.4)

with $n_s$ integer and $\nu_s$ real. For the reggeized gluon compound states, Eq. (1.2), the $SL(2, \mathbb{C})$ spins have to fixed as $s = 0$ and $\bar{s} = 1$, or equivalently $n_s = -1$ and $\nu_s = 0$.

The Hamiltonian of the noncompact $SL(2, \mathbb{C})$ Heisenberg spin magnet is given by [2, 5]

$$H_N = \sum_{k=1}^{N} \left[ H(J_{k,k+1}) + H(\bar{J}_{k,k+1}) \right],$$

(1.5)

where $H(J) = \psi(1 - J) + \psi(J) - 2\psi(1)$ with $\psi(x) = d\ln \Gamma(x)/dx$, $J_{k,k+1}$ is the sum of two $SL(2, \mathbb{C})$ spins, $J_{k,k+1}(J_{k,k+1} - 1) = (\vec{S}_k + \vec{\bar{S}}_{k+1})^2$ with $J_{N,N+1} \equiv J_{N,1}$, and similar for $\bar{J}_{k,k+1}$. The model possesses the set of mutually commuting conserved charges $q_k$ and $\bar{q}_k$ ($k = 2, \ldots, N$). Their number is large enough for the Schrödinger equation (1.2) to be completely integrable. The charges $q_k$ are given by the $k$–th order differential operators acting on the holomorphic coordinates of particles. They have a particularly simple form for the $SL(2, \mathbb{C})$ spins $s = 0$ and $\bar{s} = 1$ [1, 2]

$$q_k \bigg|_{s=0,\bar{s}=1} = i^k \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq N} \bar{z}_{j_1j_2\ldots j_k} \partial_{z_{j_1}} \cdots \partial_{z_{j_{k-1}}} \partial_{\bar{z}_{j_k}}$$

(1.6)

with $z_{j_k} \equiv z_j - z_k$. The charges $\bar{q}_k$ are given by similar expressions in the $\bar{z}$–sector. The eigenstates $\Psi(z_1, \bar{z}_2, \ldots, \bar{z}_N)$ have to diagonalize these operators and the corresponding eigenvalues $q \equiv \{q_k, \bar{q}_k = q_k^*\}$, with $k = 2, \ldots, N$, form the complete set of quantum numbers parameterizing the spectrum of the Schrödinger equation (1.2), $E_N = E_N(q, \bar{q})$. The eigenproblem for the operators (1.6) leads to a complicated system of $(N - 1)$–differential equations on $\Psi(z_1, \bar{z}_2, \ldots, \bar{z}_N)$, which was previously solved at $N = 2$ [7] and $N = 3$ [14]. For higher $N$, instead of dealing with

\(^3\)That is the isotopic “spin” space coincides with the coordinate space of $N$ particles.
this system, we apply the method developed in [5]. It represents an application of the Quantum Inverse Scattering Method [19] to noncompact Heisenberg spin magnet model.

The noncompact Heisenberg spin magnet, Eq. (1.5), can be considered as a generalization of the well-known spin—1/2 Heisenberg spin chain model (as well as its analogs for higher $SU(2)$ spins [20]) to arbitrary complex spins belonging to noncompact, unitary representations of the $SL(2, \mathbb{C})$ group. As we will demonstrate below, the noncompact and compact Heisenberg magnets have completely different properties, yet another manifestation of the fact that the quantum space of the model is infinite-dimensional in the former case. In particular, in distinction with the compact spins, the principal series of the $SL(2, \mathbb{C})$ group does not have the highest weight and, as a consequence, the conventional Algebraic Bethe Ansatz [21] is not applicable to diagonalization of the Hamiltonian (1.5). To solve the Schrödinger equation (1.2) for arbitrary number of particles $N$ we will rely instead on the method of the Baxter $Q$—operator [22].

In this method, the Hamiltonian (1.5), the integrals of motion (1.2) and, in general, all transfer matrices of the noncompact Heisenberg spin magnet are expressed in terms of a single operator $Q(u, \bar{u})$, which acts on the quantum space of the model and depends on a pair of spectral parameters, $u$ and $\bar{u}$. As a result, the Schrödinger equation (1.2) turns out to be equivalent to the eigenproblem for the Baxter $Q$—operator. The explicit form of the operator $Q(u, \bar{u})$ for noncompact Heisenberg spin magnet was found in [5]. In this paper, we will calculate the eigenvalues of the Baxter $Q$—operator. They will allows us to establish the quantization conditions for the integrals of motion (1.6) and, finally, obtain a complete description of the spectrum of the Schrödinger equation (1.2).

As we will show below, the system (1.2) has many features in common with two-dimensional conformal field theories (CFT) [23]. Since the Hamiltonian (1.5) is given by the sum of two mutually commuting operators acting in the $z$— and $\bar{z}$—sectors, the dynamics in the two sectors is independent on each other. As a consequence, the solutions to the Schrödinger equation (1.2) have the chiral structure similar to that of correlation functions in the CFT. Namely, the eigenstates $\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$ can be factorized into the product of “conformal blocks” depending on the (anti)holomorphic coordinates and the conserved charges $q$. Similar factorization holds for the eigenvalues of the Baxter operator $Q(u, \bar{u})$. For $\Psi(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N)$ to be a single-valued function on the two-dimensional $\vec{z}$—plane, the conserved charges $q$ have to satisfy the quantization conditions. The charges $q$ play the rôle analogous to that of the conformal weights of primary fields in the CFT. As we will show, the spectrum of their quantized values becomes very similar to the Kac spectrum of the conformal weights in the minimal CFT [23].

The paper is organized as follows. In Section 2, we summarize the main properties of the Baxter $Q$—operator for noncompact Heisenberg spin magnets. In Section 3, we show that the problem of finding the eigenvalues of the operator $Q(u, \bar{u})$ can be reduced to solving the $N$th order Fuchsian differential equation. Its solution leads to the set of consistency conditions which can be satisfied only if the integrals of motion $q$ take quantized values. In Section 4, we calculate the eigenvalues of the operator $Q(u, \bar{u})$ and demonstrate that they are factorized into a product of “conformal blocks” depending separately on the (anti)holomorphic spectral parameters, $u$ and $\bar{u}$. Using these expressions, it becomes straightforward to determine the exact spectrum of the model for arbitrary number of particles, $N$, and complex $SL(2, \mathbb{C})$ spins, $s$ and $\bar{s}$. In Section 5, we present the results of our calculations for the special values of the $SL(2, \mathbb{C})$ spins, $s = 0$ and $\bar{s} = 1$. The obtained expressions define the spectrum of the compound states of reggeized gluons in multi-colour QCD. Section 6 contains the concluding remarks. The details of the calculations are summarized in the Appendices.
2. Baxter $\mathbb{Q}$–operator

In this Section we shall describe, following [5], the general properties of the Baxter $\mathbb{Q}$–operator for the noncompact Heisenberg spin magnet. We assume that the number of particles $N$ is arbitrary and the complex $SL(2, \mathbb{C})$ spins $s$ and $\bar{s}$ are given by (1.4). The operator $\mathbb{Q}(u, \bar{u})$ acts on the quantum space of the model $V_N = V \otimes \ldots \otimes V$, with $V \equiv V^{(s, \bar{s})}$ being the representation space of the principal series of the $SL(2, \mathbb{C})$ group, and depends on two complex spectral parameters $u$ and $\bar{u}$. It commutes with the Hamiltonian of the model (1.5) and shares the common set of the eigenstates

$$\mathbb{Q}(u, \bar{u}) \Psi_{\vec{p}(q, \bar{q})}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N) = Q_{q, \bar{q}}(u, \bar{u}) \Psi_{\vec{p}(q, \bar{q})}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N),$$  \hspace{1cm} (2.1)

with $\vec{p}$ being the total two-dimensional momentum of the state and $\{q, \bar{q}\}$ denoting the total set of the quantum numbers, $q_k$ and $\bar{q}_k$ with $k = 2, \ldots, N$. For $\mathbb{Q}(u, \bar{u})$ to be a well-defined operator on $V_N$, the spectral parameters have to satisfy the additional condition

$$i(u - \bar{u}) = n$$  \hspace{1cm} (2.2)

with $n$ being an arbitrary integer.

The Baxter operator plays the central role in our analysis as the energy spectrum of the model can be expressed in terms of its eigenvalues, $Q_{q, \bar{q}}(u, \bar{u})$. Indeed, there exists the following two (equivalent) relations between $Q_{q, \bar{q}}(u, \bar{u})$ and the energy $E_N = E_N(q, \bar{q})$

$$E_{N}(q, \bar{q}) = \varepsilon_{N} + i \frac{d}{du} \ln \left[ Q_{q, \bar{q}}(u + is, u + i\bar{s}) (Q_{q, \bar{q}}(u - is, u - i\bar{s}))^{*} \right] \Big|_{u = 0},$$  \hspace{1cm} (2.3)

where $\varepsilon_N = 2N \text{Re} [\psi(2s) + \psi(2 - 2s) - 2\psi(1)]$ and

$$E_{N}(q, \bar{q}) = - \text{Im} \frac{d}{du} \ln \left[ u^{2N} Q_{q, \bar{q}}(u + i(1 - s), u + i(1 - \bar{s})) (Q_{q, \bar{q}}(u + i(1 - s), u + i(1 - \bar{s})))^{*} \right] \Big|_{u = 0}.$$  \hspace{1cm} (2.4)

The Hamiltonian (1.5) is invariant under cyclic permutations of particles $[\mathbb{P} \Psi](\vec{z}_1, \ldots, \vec{z}_{N-1}, \vec{z}_N) = \Psi(\vec{z}_2, \ldots, \vec{z}_N, \vec{z}_1)$ and, as a consequence, its eigenstates possess a definite value of the quasimomentum defined as

$$[\mathbb{P} \Psi_{\vec{p}(q, \bar{q})}](z_1, \ldots, z_N) = e^{iqN(q, \bar{q})} \Psi_{\vec{p}(q, \bar{q})}(z_1, \ldots, z_N), \hspace{1cm} \theta_N(q, \bar{q}) = 2\pi k/N,$$  \hspace{1cm} (2.5)

with $k$ integer in virtue of $\mathbb{P}^{N} = 1$. The quasimomentum can be expressed in terms of the eigenvalues of the Baxter operator as

$$\theta_N(q, \bar{q}) = i \ln \frac{Q_{q, \bar{q}}(is, i\bar{s})}{Q_{q, \bar{q}}(-is, -i\bar{s})}.$$  \hspace{1cm} (2.6)

Thus, the Schrödinger equation (1.2) turns out to be equivalent to the eigenproblem for the Baxter $\mathbb{Q}$–operator, Eq. (2.1).

The eigenvalues of the Baxter operator, $Q_{q, \bar{q}}(u, \bar{u})$, have to fulfil the following three conditions:

(i) Baxter equations:

The function $Q_{q, \bar{q}}(u, \bar{u})$ has to satisfy the holomorphic Baxter equation

$$t_{N}(u) Q_{q, \bar{q}}(u, \bar{u}) = (u + is)^{N} Q_{q, \bar{q}}(u + i, \bar{u}) + (u - is)^{N} Q_{q, \bar{q}}(u - i, \bar{u}),$$  \hspace{1cm} (2.7)

where $t_{N}(u)$ is
where

\[ t_N(u) = 2u^N + q_2u^{N-2} + \cdots + q_N \]  

(2.8)

is the eigenvalue of the auxiliary transfer matrix with \( q \equiv (q_2, \ldots, q_N) \) denoting the eigenvalues of the holomorphic integrals of motion. The “lowest” integral of motion, \( q_2 \), is related to the total \( SL(2, \mathbb{C}) \) spin, \( h \), of the system of \( N \) particles

\[ q_2 = -h(h-1) + Ns(s-1), \quad h = (1 + n_h)/2 + i\nu_h \]  

(2.9)

with \( n_h \) integer and \( \nu_h \) real. In addition, \( Q_{q,q}(u, \bar{u}) \) obeys the equation similar to (2.7) in the \( \bar{u} \)-sector with \( s, h \), and \( q_k \) replaced, respectively, by

\[ \bar{s} = 1 - s^*, \quad \bar{h} = 1 - h^*, \quad \bar{q}_k = q_k^* \]  

(2.10)

with \( k = 2, \ldots, N \). The function \( Q_{q,q}(u, \bar{u}) \) does not depend on the total momentum of the state, \( p = -i \sum_k S_k^+ \) and \( \bar{p} = -i \sum_k S_k^- \), and it is invariant under \( h \to 1 - h \) and \( \bar{h} \to 1 - \bar{h} \). Indeed, the Baxter operator \( Q(u, \bar{u}) \) commutes with the generators of the \( SL(2, \mathbb{C}) \) group and, as a consequence, its eigenvalue \( Q_{q,q}(u, \bar{u}) \) depends only on the \( SL(2, \mathbb{C}) \) Casimir operators, like \( q_2 \), which are symmetric under the above transformation of the total spin.

(ii) Analytical properties:

The possible values of the spectral parameters satisfying (2.2) can be parameterized as \( u = \lambda - in/2 \) and \( \bar{u} = \lambda + in/2 \), with \( n \) arbitrary integer and \( \lambda \) complex. Then, \( Q_{q,q}(u, \bar{u}) \) should be a meromorphic function of \( \lambda \) with an infinite set of poles of the order not higher than \( N \) situated at the points

\[ \{ u_m^\pm = \pm i(s - m), \quad \bar{u}_m^\pm = \pm i(\bar{s} - \bar{m}) \}, \quad m, \bar{m} = 1, 2, \ldots \]  

(2.11)

The behaviour of \( Q_{q,q}(u, \bar{u}) \) in the vicinity of the pole at \( m = \bar{m} = 1 \) can be parameterized as

\[ Q_{q,q}(u_1^\pm + \epsilon, \bar{u}_1^\pm + \epsilon) = R^\pm(q, \bar{q}) \left[ \frac{1}{\epsilon^N} + \frac{iE^\pm(q, \bar{q})}{\epsilon^{N-1}} + \cdots \right]. \]  

(2.12)

The functions \( R^\pm(q) \) fix an overall normalization of the Baxter operator, while the residue functions \( E^\pm(q) \) define the energy of the system (see Eqs. (2.15) and (2.18) below).

(iii) Asymptotic behaviour:

In the above parameterization of the spectral parameters, \( Q_{q,q}(u, \bar{u}) \) should have the following asymptotic behaviour for \( |\text{Im} \lambda| < 1/2 \) and \( \text{Re} \lambda \to \infty \)

\[ Q_{q,q}(\lambda - in/2, \lambda + in/2) \sim e^{i\Theta_h(q, \bar{q})} \chi^{h + \bar{h} - N(s + \bar{s})} + e^{-i\Theta_h(q, \bar{q})} \chi^{1-h+1-\bar{h}-N(s+\bar{s})}, \]  

(2.13)

with the phase \( \Theta_h(q, \bar{q}) \) depending on the quantum numbers of the state and the total \( SL(2, \mathbb{C}) \) spins \( h \) and \( \bar{h} \) defined in (2.9) and (2.10).

As we will show in Section 3, the Baxter equation (2.7) supplemented by the additional conditions on the pole structure of its solutions, Eq. (2.11), and asymptotic behaviour at infinity, Eq. (2.13), fixes uniquely the eigenvalues of the Baxter operator, \( Q_{q,q}(u, \bar{u}) \), and, therefore, allow us to determine the spectrum of the model.

Additional properties of the function \( Q_{q,q}(u, \bar{u}) \) can be deduced from the symmetry of the model under permutations of particles. Apart from the cycle symmetry, Eq. (2.5), the Hamiltonian \( \mathcal{H}_N \) is also invariant under mirror permutation \( M \Psi(z_1, \ldots, z_{N-1}, \bar{z}_N) = \Psi(z_N, \ldots, \bar{z}_2, \bar{z}_1) \) [5].
Let us construct the solution to the Baxter equation (2.7) satisfying the additional conditions

\[ [\mathcal{M} \Psi_{q,q}] (\bar{z}_1, \ldots, \bar{z}_{N-1}, \bar{z}_N) = \Psi_{-q,-\bar{q}} (\bar{z}_1, \ldots, \bar{z}_{N-1}, \bar{z}_N) \]  

(2.14)

with \(-q \equiv (q_2, -q_3, \ldots, (-1)^n q_n)\) and similar for \(\bar{q}\). This property leads to the following parity relations for the residue functions \(R^+(q, \bar{q})\) defined in (2.12)

\[ R^+(q, \bar{q}) / R^+(-q, -\bar{q}) = e^{2i\delta_N(q,q)} \]  

(2.15)

and for the eigenvalues of the Baxter operator

\[ Q_{q,q} (-u, -\bar{u}) = e^{i\delta_N(q,\bar{q})} Q_{-q,-\bar{q}} (u, \bar{u}) . \]  

(2.16)

We recall that the spectral parameters \(u\) and \(\bar{u}\) have to satisfy (2.2). Examining the behaviour of the both sides of (2.16) around the pole at \(u = u_1^\pm\) and \(\bar{u} = \bar{u}_1^\pm\) and making use of Eq. (2.12) one gets

\[ R^\pm (q, \bar{q}) = (-1)^N e^{i\delta_N(q,\bar{q})} R^\mp (-q, -\bar{q}) , \quad E^\pm (q, \bar{q}) = -E^\mp (-q, -\bar{q}) . \]  

(2.17)

To obtain the expression for the energy \(E_N(q, \bar{q})\), we apply (2.4). Calculating the logarithmic derivative of \(Q_{q,q}\) in the r.h.s. of (2.4), we replace the function \(Q_{q,q}(u \pm i(1 - s), u \pm i(1 - \bar{s}))\) by its pole expansion (2.12). Then, applying the second relation in (2.17), one finds

\[ E_N(q, \bar{q}) = E^+(-q, -\bar{q}) + (E^+(q, \bar{q}))^* = \text{Re} \left[ E^+(-q, -\bar{q}) + E^+(q, \bar{q}) \right] , \]  

(2.18)

where the last relation follows from hermiticity of the Hamiltonian (1.5). We conclude from (2.18), that in order to calculate the energy \(E_N(q, \bar{q})\) it is enough to find the residue of \(Q_{q,q}(u, \bar{u})\) at the \((N - 1)\)th order pole at \(u = i(s - 1)\) and \(\bar{u} = i(\bar{s} - 1)\).

3. Quantization conditions

Let us construct the solution to the Baxter equation (2.7) satisfying the additional conditions (2.12) and (2.13). It proves convenient to use the following integral representation for \(Q_{q,q}(u, \bar{u})\)

\[ Q_{q,q}(u, \bar{u}) = \int \frac{d^2z}{z \bar{z}} z^{-iu} \bar{z}^{-iu} Q(z, \bar{z}) , \]  

(3.1)

where integration goes over the two-dimensional \(z\) plane with \(z = z^*\). This ansatz is advantageous in many respects. Firstly, the condition (2.2) is automatically satisfied since it is only for these values of the spectral parameters that the \(z\)-integral in the r.h.s. of (3.1) is well-defined. Secondly, the functional Baxter equation on \(Q_{q,q}(u, \bar{u})\) is translated into the \(N\)th order differential equation on the function \(Q(z, \bar{z})\). The derivation of this equation is based on the identity

\[ P(u) Q_{q,q}(u, \bar{u}) = \int \frac{d^2z}{z \bar{z}} z^{-iu} \bar{z}^{-iu} P(-iz \partial_z) Q(z, \bar{z}) , \]  

(3.2)

with \(P(u)\) being a polynomial in \(u\). Substituting (3.1) into the Baxter equation (2.7) and applying this identity, one arrives at

\[ \left[ z^s (z \partial_z)^N z^{1-s} + z^{-s} (z \partial_z)^N z^{s-1} - 2(z \partial_z)^N - \sum_{k=2}^{N} i^k q_k (z \partial_z)^{N-k} \right] Q(z, \bar{z}) = 0 . \]  

(3.3)
The \(\bar{z}\)-dependence of \(Q(z, \bar{z})\) is constrained by a similar equation in the antiholomorphic sector with \(s\) and \(q_k\) replaced by \(\bar{s} = 1 - s^*\) and \(\bar{q}_k = q_k^*\), respectively. Finally, as we will show below, the remaining two conditions on the analytical properties and asymptotic behaviour of \(Q_{q,q}(u, \bar{u})\), Eqs. (2.12) and (2.13), become equivalent to a requirement for \(Q(z, \bar{z} = z^*)\) to be a single-valued function on the complex \(z\)-plane.

Going through a standard analysis [24], one finds that the differential equation (3.3) is of Fuchsian type with three regular singular points located at \(z = 0\), \(z = 1\) and \(z = \infty\). Defining \(N\) linear independent solutions to Eq. (3.3), \(Q_a(z)\), and their antiholomorphic counterparts, \(\overline{Q}_b(\bar{z})\), we construct the general expression for the function \(Q(z, \bar{z})\) as

\[
Q(z, \bar{z}) = \sum_{a, b = 1}^{N} Q_a(z) C_{ab} \overline{Q}_b(\bar{z}),
\]

with \(C_{ab}\) being an arbitrary mixing matrix. The functions \(Q_a(z)\) and \(\overline{Q}_b(\bar{z})\) acquire a nontrivial monodromy around three singular points, \(z, \bar{z} = 0, 1\) and \(\infty\). For \(Q(z, \bar{z} = z^*)\) to be well-defined on the whole plane, the monodromy should cancel in the r.h.s. of (3.4). This requirement leads to the set of nontrivial conditions on the matrix \(C_{ab}\). Solving them, we will be able not only to obtain the values of the mixing coefficients, \(C_{ab}\), but also determine the quantized values of the integrals of motion \(q_k\).

We would like to point out that our approach to defining the function \(Q(z, \bar{z})\) in Eq. (3.4) is similar in many respects to the well-known approach to constructing the correlation functions in the minimal CFT [25]. There, \(Q(z, \bar{z})\) plays the rôle of four-point correlation functions depending on the anharmonic ratios of the coordinates, \(z\) and \(\bar{z}\). The latter satisfy the differential equations (“the null vector condition”) similar to (3.3), in which the integrals of the motion \(q_k\) are replaced by some combinations of the conformal weights of the primary fields. In the minimal CFT, the conformal blocks \(Q_a(z)\) and \(\overline{Q}_b(\bar{z})\) are given by multiple contour integrals and their monodromy around the singular points \(z = 0, 1\) and \(\infty\) can be found in a closed form. Going over to Eq. (3.3), one finds (see Section 3.4 below) that similar representation exists only at \(N = 2\) and it remains unclear whether it can be generalized for arbitrary number of particles \(N\). Our subsequent analysis does not rely on such representation.

To determine the function \(Q(z, \bar{z})\), we shall construct the r.h.s. of (3.4) at the vicinity of three singular points, \(z = 0, 1\) and \(\infty\), and analytically continue the obtained expressions onto the whole \(z\)-plane. Additional simplification occurs due to the symmetry of the differential equation (3.3) under the transformation \(z \to 1/z\) and \(q_k \to (-1)^k q_k\). This symmetry is a manifestation of a general property of the eigenvalues of the Baxter operator, Eq. (2.16), which leads to

\[
Q_{q,q}(z, \bar{z}) = e^{i\theta} Q_{q,q}(1/z, 1/\bar{z}).
\]

Here, we indicated explicitly the dependence of the function \(Q(z, \bar{z})\) on the integrals of motion. Applying (3.5), one can define \(Q(z, \bar{z})\) around \(z = \infty\) from the solution at \(z = 0\).

### 3.1. Solution around \(z = 0\)

Looking for the solution to (3.3) around \(z = 0\) in the form \(Q(z) \sim z^a\) we find that the exponent \(a\) satisfies the indicial equation

\[
(a - 1 + s)^N = 0.
\]

8
Since its solution, \( a = 1 - s \), is \( N \)-times degenerate, the small-\( z \) asymptotics of \( Q(z) \) contains terms \( \sim (\ln z)^k \) with \( k \leq N - 1 \). Let us define the fundamental set of linear independent solutions to (3.3) around \( z = 0 \) as

\[
Q^{(0)}_m(z) = z^{1-s}u_m(z),
\]

\[
Q^{(0)}_m(z) = z^{1-s}\left[u_1(z)(\ln z)^{m-1} + \sum_{k=1}^{m-1} c_{m-1}^k u_{k+1}(z)(\ln z)^{m-k-1}\right], \tag{3.7}
\]

with \( 2 \leq m \leq N \) and the binomial coefficients \( c_{m-1}^k = (m-1)!/(k!(m-k-1)! \) inserted for later convenience. The functions \( u_m(z) \) are defined by power series

\[
u_m(z) = 1 + \sum_{n=1}^{\infty} z^n u_m^{(m)}(q), \tag{3.8}
\]

which converge uniformly inside the region \(|z| < 1\). Inserting (3.7) and (3.8) into (3.3), one finds that the expansion coefficients \( u_m^{(m)}(q) \) satisfy the three-term (nonhomogeneous) recurrence relations with respect to \( n \). To save space, we do not present here their explicit form.

The fundamental set of solutions to the antiholomorphic differential equation, \( \overline{Q}^{(0)}_m(\bar{z}) \), can be obtained from (3.7) by replacing \( s \) and \( q_k \) by \( \bar{s} = 1 - s^* \) and \( \bar{q}_k = q_k^* \), respectively. Then, the general solution for \( Q(z, \bar{z}) \) around \( z = 0 \) is given by

\[
Q(z, \bar{z}) \mid_{z=0} = \sum_{m=1}^{N} \sum_{m=1}^{N} Q^{(0)}_m(z) C^{(0)}_{nm} \overline{Q}^{(0)}_m(\bar{z}). \tag{3.9}
\]

The mixing matrix \( C^{(0)}_{nm} \) has to be chosen in such a way that \( Q(z, \bar{z}) \) should be single-valued at \( z = 0 \), or equivalently, the monodromy of \( Q^{(0)}_m(z) \) and \( \overline{Q}^{(0)}_m(\bar{z} = z^*) \) around \( z = 0 \) should cancel each other in the r.h.s. of (3.9). According to (3.7), the monodromy of \( Q^{(0)}_m(z) \) is due to \( z^{1-s} \)-factor and \( \ln z \)-terms. Taking into account that \( s - \bar{s} = n_\sigma \) is an integer, we find that the factor \( z^{1-s}\bar{z}^{1-s} \) does not affect the monodromy of the r.h.s. of (3.9). Then, for \( Q(z, \bar{z}) \) to be single valued, it should depend only on \( \ln(zz) \) rather than on \( \ln z \) and \( \ln \bar{z} \) separately. It is straightforward to verify that this condition is satisfied provided that the matrix elements \( C^{(0)}_{nm} \) vanish below the main anti-diagonal, that is for \( n + m > N + 1 \), and have the following form for \( n + m \leq N + 1 \)

\[
C^{(0)}_{nm} = \frac{\sigma}{(n-1)!(m-1)!} \left[ \sum_{k=0}^{N-n-m-1} (-2)^k \frac{(-1)^k}{k!} \alpha_{k+n+m-1} \right]. \tag{3.10}
\]

with \( \sigma, \alpha_1, ..., \alpha_{N-1} \) being arbitrary complex parameters and \( \alpha_N = 1 \).

The mixing matrix \( C^{(0)}_{nm} \) depends on \( N \) arbitrary complex parameters \( \sigma \) and \( \alpha_k \). We can fix the normalization of the function \( Q(z, \bar{z}) \) by choosing the value of \( \sigma \). Since the resulting expression for the eigenvalue of the Baxter operator has to satisfy simultaneously two parity relations,

\[
4The monodromy matrix, defined as \( Q^{(0)}_n(z^{e^{2\pi i}}) = M_{nn}Q^{(0)}_n(z) \), has a Jordan structure and, therefore, it can not be brought to a diagonal form upon redefinition of the fundamental basis. It is interesting to note that similar situation occurs in the operator algebra of primary fields in the so-called Logarithmic CFT [26].

\[
5To cancel the monodromy, it is enough to require that the sum over \( k \) should depend only on the sum \( n + m \). We have chosen the sum in this particular form for later convenience (see Eq. (3.11)).
the relation (2.15) is satisfied for \( \sigma \) over the region of large \( \rho \). As we will see in a moment, the relation (2.15) is satisfied for \( \sigma = \exp(i\theta_N(q, \bar{q})) \), with \( \theta_N(q, \bar{q}) \) being the quasimomentum. As to the second relation, Eq. (3.5), it leads to the quantization of the quasimomentum. Later, we will use (3.5) to calculate the eigenvalues of \( \theta_N(q, \bar{q}) \) (see Eq. (3.24)).

Substituting (3.10) into (3.9) and taking into account (3.7), one can obtain the small–z expansion of the function \( Q(z, \bar{z}) \). The leading asymptotic behaviour for \( z \rightarrow 0 \) can be obtained by neglecting \( O(z) \) corrections to (3.8). In this way, one finds

\[
Q_{q, \bar{q}}(z, \bar{z}) = z^{1-s}z^{1-\bar{s}}e^{i\theta_N(q, \bar{q})} \left[ \frac{\ln^{N-1}(z\bar{z})}{(N-1)!} + \frac{\ln^{N-2}(z\bar{z})}{(N-2)!} \alpha_{N-1} + \ldots + \frac{\ln(z\bar{z})}{\alpha_2 + \alpha_1} \right] (1 + O(z, \bar{z})).
\]

(3.11)

Using this expression, we can calculate the contribution of the small–z region to the eigenvalue of the Baxter operator, Eq. (3.1). Introducing a cut-off, \( \rho \ll 1 \), and replacing \( Q(z, \bar{z}) \) in Eq. (3.1) by its expansion (3.11), one integrates term by term over the region \( |z| < \rho \) by making use of the identity

\[
\int_{|z| < \rho} \frac{d^2z}{z\bar{z}} z^{-iu}z^{-i\bar{u}} \ln^n(z\bar{z}) \ln^{m-s} \ln^{m-\bar{s}} = \pi \delta_{m-s-iu, \bar{m}-\bar{s}-i\bar{u}} \left[ \frac{(-1)^n n!}{(m-s-iu)^{n+1}} + O((m-s-iu)^n) \right],
\]

with \( m \) and \( \bar{m} \) positive integers. We find that, in agreement with the general properties of the Baxter operator, Eqs. (2.11) and (2.12), the function \( Q_{q, \bar{q}}(u, \bar{u}) \) has poles of the order \( N \) at the points \( u = i(s - m) \) and \( \bar{u} = i(\bar{s} - \bar{m}) \). At \( m = \bar{m} = 1 \) one finds from (3.11)

\[
Q_{q, \bar{q}}(u_1^+, \bar{u}_1^+, \epsilon) = -\frac{\pi e^{i\theta_N(q, \bar{q})}}{(i\epsilon)^N} \left[ 1 + i\epsilon \alpha_{N-1} + \ldots + (i\epsilon)^N \alpha_2 + (i\epsilon)^{N-1} \alpha_1 + O(\epsilon^N) \right].
\]

(3.12)

with \( u_1^+ \) and \( \bar{u}_1^+ \) defined in (2.11). It is easy to see, using (3.5) and (3.11), that the remaining poles of \( Q_{q, \bar{q}}(u, \bar{u}) \) located at \( u = -i(s - m) \) and \( \bar{u} = -i(\bar{s} - \bar{m}) \) originate from the integration in (3.1) over the region of large \( |z| > 1/\rho \).

Matching (3.12) into (2.12) one gets

\[
R^+(q, \bar{q}) = -\frac{\pi}{iN} e^{i\theta_N(q, \bar{q})}, \quad E^+(q, \bar{q}) = \alpha_{N-1}(q, \bar{q}).
\]

(3.13)

We verify, using \( \theta_N(q, \bar{q}) = -\theta_N(-q, -\bar{q}) \), that the obtained expression for \( R^+(q, \bar{q}) \) satisfies the parity relation (2.15). According to their definition, Eq. (3.10), \( \alpha_n \) are arbitrary complex parameters. We anticipate however that their values will be fixed by the quantization conditions to be discussed below. This is the reason why we indicated in (3.13) the dependence of the \( \alpha \)-parameters on the integrals of motion. Insertion of the second relation in (3.13) into (2.18) leads to the following remarkable expression for the energy

\[
E_N(q, \bar{q}) = \text{Re} [\alpha_{N-1}(q, \bar{q}) + \alpha_{N-1}(q, \bar{q})],
\]

(3.14)

where \( q = \{q_k\} \), \( -q = \{-(-1)^k q_k\} \) and similar for \( \bar{q} \).

We conclude that the small–z asymptotics (3.11) reproduces correctly the analytical properties of the eigenvalues of the Baxter operator, Eq. (2.11). Moreover, the energy of the system, \( E_N(q, \bar{q}) \), is related to the matrix elements of the mixing matrix (3.10) in the fundamental basis (3.7).
3.2. Solution around $z = 1$

Substituting $Q(z) \sim (z - 1)^b$ into (3.3) one obtains after some calculation the following indicial equation

$$
(b + 1 + h - Ns)(b + 2 - h - Ns) \prod_{k=0}^{N-3} (b - k) = 0 ,
$$

(3.15)

with the total $SL(2, \mathbb{C})$ spin $h$ defined in (2.9). Since the solutions $b = k$ with $k = 0,...,N - 3$ differ from each other by an integer, one expects to encounter logarithmically enhanced terms $\sim \ln(1 - z)$. However, a close examination of (3.3) reveals that the solutions to (3.3) do not contain such terms provided that $h \neq (1 + n_h)/2$, or equivalently $\Im h \neq 0$ (see Eq. (2.9)). At $h = (1 + n_h)/2$ the additional degeneracy occurs between the solutions to (3.15), $b = Nj - h - 1$ and $b = Nj + h - 2$, that leads to the appearance of the terms $\sim \ln(1 - z)$ in the asymptotics of $Q(z)$ for $z \to 1$. Obviously, similar relations hold in the $\bar{z}$-sector.

The fundamental set of solutions to Eq. (3.3) around $z = 1$ is defined similarly to (3.7). For $\Im h \neq 0$ it has the form

$$
Q_1^{(1)}(z) = z^{1-s}(1 - z)^{Ns-h-1}v_1(z) ,
$$

$$
Q_2^{(1)}(z) = z^{1-s}(1 - z)^{Ns+h-2}v_2(z) ,
$$

$$
Q_m^{(1)}(z) = z^{1-s}(1 - z)^{m-3}v_m(z) ,
$$

(3.16)

with $m = 3,...,N$. The functions $v_i(z)$ ($i = 1, 2$) and $v_m(z)$ are given by the power series

$$
v_i(z) = 1 + \sum_{n=1}^{\infty} (1 - z)^n v_n^{(i)}(q) ,
$$

$$
v_m(z) = 1 + \sum_{n=N-m+1}^{\infty} (1 - z)^n v_n^{(m)}(q) ,
$$

(3.17)

which converge uniformly inside the region $|1 - z| < 1$. We notice that $Q_2^{(1)}(z)$ can be obtained from the function $Q_1^{(1)}(z)$ by replacing $h \to 1 - h$. Substituting (3.16) into (3.3), one finds that the expansion coefficients $v_n^{(i)}$ and $v_n^{(m)}$ satisfy the $N$-term homogenous recurrence relations with respect to the index $n$.\textsuperscript{6} As was already mentioned, at $h = (1 + n_h)/2$ the solutions $Q_{1,2}^{(1)}(z)$ become degenerate and one of them, $Q_1^{(1)}(z)$ for $n_h \geq 0$, has to be redefined to include the additional $\ln(1 - z)-$term

$$
Q_1^{(1)}(z) |_{h=(1+n_h)/2} = z^{1-s}(1 - z)^{Ns-(n_h+3)/2} [(1 - z)^{n_h} \ln(1 - z) v_2(z) + \bar{v}_1(z) ,
$$

(3.18)

where the function $v_2(z)$ is the same as before, while $\bar{v}_1(z) = \sum_{k=0}^{\infty} \bar{v}_k z^k$ and the coefficients $\bar{v}_k$ satisfy the $N-$term recurrence relations with the boundary condition $\bar{v}_{n_h} = 1$.

The fundamental basis in the antiholomorphic sector, $Q_n^{(1)}(\bar{z})$, is obtained from the functions $Q_n^{(1)}(z)$ by replacing $s$ and $h$ by $\bar{s} = 1 - s^*$ and $\bar{h} = 1 - h^*$, respectively. Among all functions in the fundamental set (3.16) only two, $Q_1^{(1)}(z)$ and $Q_2^{(1)}(z)$, are not analytical at $z = 1$. As a consequence, a general solution for $Q(z, \bar{z})$ possessing a trivial monodromy around $z = 1$ can be constructed as

$$
Q(z, \bar{z}) |_{|z|<1} = \beta_h Q_1^{(1)}(z)Q_1^{(1)}(\bar{z}) + \beta_{-h} Q_2^{(1)}(z)Q_2^{(1)}(\bar{z}) + \sum_{m,m=3}^{N} Q_m^{(1)}(z) \gamma_{mm} Q_m^{(1)}(\bar{z}) .
$$

(3.19)

The factor $z^{1-s}$ was included in the r.h.s. of (3.16) and (3.18) to simplify the form of the recurrence relations. Without this factor, the recursion will involve $N + 1$ terms.
Here, the $\beta-$coefficients depend, in general, on the total spin $h$ (and $\bar{h} = 1 - h^*$). They are chosen in (3.19) in such a way that the symmetry of the eigenvalues of the Baxter operator under $h \to 1 - h$ becomes manifest. It is convenient to rewrite (3.19) in a matrix form as

$$Q(z, \bar{z}) = \overline{Q^{(1)}} \cdot C^{(1)} \cdot \overline{Q^{(1)}^\dagger}, \quad C^{(1)} = \begin{pmatrix} \beta_h & 0 \\ 0 & \beta_{1-h} \end{pmatrix} \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \quad (3.20)$$

with $\gamma \equiv \gamma_{mn\bar{m}}$. The expansion (3.19) is valid only for $\text{Im} \, h \neq 0$. For $h = (1 + n_h)/2$ the first two terms in the r.h.s. of (3.19) look differently in virtue of (3.18)

$$Q(z, \bar{z}) \big|_{h=(1+n_h)/2} = \beta_1 \left[ Q_1^{(1)}(z) \overline{Q_2^{(1)}}(\bar{z}) + Q_2^{(1)}(z) \overline{Q_1^{(1)}}(\bar{z}) \right] + \beta_2 Q_2^{(1)}(z) \overline{Q_2^{(1)}}(\bar{z}) + \ldots, \quad (3.21)$$

where ellipses denote the remaining terms. Substituting (3.19) into (3.1) and performing integration over the region of $|1 - z| \ll 1$, one can find the asymptotic behaviour of $Q(u, \bar{u})$ at large $u$. As we will show in Section 4, it turns out to be in agreement with the general properties of the Baxter operator, Eq. (2.13).

The mixing matrix $C^{(1)}$ defined in (3.20) has a block-diagonal structure. It depends on $2 + (N - 2)^2$ complex parameters $\beta_h$, $\beta_{1-h}$ and $\gamma_{mn\bar{m}}$ which, in general, are some functions of the integrals of motion $(q, \bar{q})$ to be fixed by the quantization conditions. Let us take into account that the function $Q(z, \bar{z})$ has to satisfy the duality relation (3.5). For $|z| \to 1$, one can apply (3.19) to evaluate the both sides of (3.5) in terms of the mixing matrices $C^{(1)}(q, \bar{q})$ and $C^{(1)}(-q, -\bar{q})$. This leads to the set of relations on the functions $\beta(q, \bar{q})$ and $\gamma_{mn\bar{m}}(q, \bar{q})$. Their derivation is based on the following property of the fundamental basis (3.16)

$$Q_a^{(1)}(1/z; -q) = \sum_{b=1}^N S_{ab} Q_b^{(1)}(z; q), \quad (3.22)$$

where $\text{Im}(1/z) > 0$. Here, we indicated explicitly the dependence on the integrals of motion. Since the $Q-$functions in the both sides of this relation satisfy the same differential equation (3.3), the $S-$matrix does not depend on $z$. As a consequence, one can evaluate the matrix elements $S_{ab}$ by examining the leading asymptotic behaviour of the both sides of (3.22) for $z \to 1$. In this way, applying (3.16) and (3.17), one finds that the only nonvanishing matrix elements are given by

$$S_{11} = e^{-i\pi(Ns-h-1)}, \quad S_{22} = e^{-i\pi(N\bar{s}+h-2)}, \quad S_{k,k+m} = (-1)^{k-3} \frac{(k-2s-1)m}{m!} \quad (3.23)$$

with $(x)_m \equiv \Gamma(x+m)/\Gamma(x)$, $3 \leq k \leq N$ and $0 \leq m \leq N - k$. Similar relations hold in the anti-holomorphic sector, $\overline{S}_{11} = e^{i\pi(N\bar{s}-h-1)}, \overline{S}_{22} = e^{i\pi(N\bar{s}+h-2)}$ and $\overline{S}_{k,k+m} = (-1)^{k-3} (k-2\bar{s}-1)m/m!$.

Finally, substituting (3.19) and (3.22) into (3.5), we find

$$\beta_h(q, \bar{q}) = e^{i\theta_N(q, \bar{q})} (-1)^{Nn_s+n_h} \beta_h(-q, -\bar{q}),$$

$$\gamma_{mn\bar{m}}(q, \bar{q}) = e^{i\theta_N(q, \bar{q})} \sum_{n,\bar{n} \geq 3} S_{nm} \gamma_{\bar{n}m}(q, -\bar{q}) \overline{S_{\bar{n}}}. \quad (3.24)$$

These relations imply that, similar to the energy, Eq. (3.14), the eigenvalues of the quasimomentum, $\theta_N(q, \bar{q})$, can be calculated from the mixing matrix at $z = 1$. In particular, it follows from
the first relation in (3.24) that for the eigenstates with \( q_{2k+1} = \bar{q}_{2k+1} = 0 \) \((k = 1, 2, \ldots)\), so that

\[
\beta_n(q, \bar{q}) = \beta_n(-q, -\bar{q}),
\]

the quasimomentum for \( N \geq 3 \) is equal to

\[
e^{i\theta_n(q, \bar{q})} = (-1)^{Nn_s + nh}.
\]

At \( N = 2 \) one finds from (3.24) that \( e^{i\theta_2} = (-1)^{nh} \).

### 3.3. Transition matrices

Eqs. (3.9) and (3.19) provide the solution for the function \( Q(z, \bar{z}) \) at the vicinity of \( z = 0 \) and \( z = 1 \), respectively. To obtain the eigenvalues of the Baxter \( Q \)-operator, Eq. (3.1), one has to sew (3.9) and (3.19) inside the region \( |1 - z| < 1, |z| < 1 \) and, then, analytically continue the resulting expression for \( Q(z, \bar{z}) \) into the whole complex \( z \)-plane by making use of the duality relation (3.5). As we will see in a moment, this can be done only for the special values of integrals of motion \((q, \bar{q})\) satisfying the quantization conditions (see Eq. (3.28) below).

The sewing procedure is based on the relation between two fundamental sets of solutions, Eqs. (3.7) and (3.16). Choosing \( z \) to be inside the region of convergence of the both series, Eqs. (3.8) and (3.17), we define the transition matrices \( \Omega(q) \) and \( \Omega(q) \)

\[
Q^{(0)}(z) = \sum_{m=1}^{N} \Omega_{nm}(q) Q^{(1)}(z), \quad \tilde{Q}^{(0)}(z) = \sum_{m=1}^{N} \tilde{\Omega}_{nm}(\bar{q}) \tilde{Q}^{(1)}(z).
\]

Since the functions \( Q^{(0)}(z) \) and \( Q^{(1)}(z) \) satisfy the same differential equation (3.3), the transition matrices are \( z \)-independent. For the fundamental set of solutions, Eqs. (3.7) and (3.16), these matrices are uniquely fixed and they can be calculated as \([14]\)

\[
\Omega(q) = W^{(0)}[W^{(1)}]^{-1}, \quad W^{(j)}_{nk} = \partial_{z_0} Q^{(j)}(z_0)
\]

with \( j = 0, 1 \) and \( z_0 \) being some reference point, say \( z_0 = 1/2 \), and similar for \( \tilde{\Omega}(\bar{q}) \). The resulting expressions for the matrices \( \Omega(q) \) and \( \tilde{\Omega}(\bar{q}) \) take the form of infinite series in \( q \) and \( \bar{q} \), respectively. The transition matrices in two sectors are related to each other as \( \Omega(q) = \Omega_s(h, q) \) and \( \tilde{\Omega}(\bar{q}) = \Omega_s(\bar{h}, \bar{q}) \).

The transition matrices allow us to analytically continue the solutions (3.9) valid for \( |z| < 1 \) to the region \( |1 - z| < 1 \). Substituting (3.26) into (3.9) and matching the result into (3.19), we find that the two expressions for the function \( Q(z, \bar{z}) \), Eqs. (3.9) and (3.19), can be sewed together provided that the mixing matrices \( C^{(0)} \) and \( C^{(1)} \) satisfy the following relation

\[
C^{(1)}(q, \bar{q}) = [\Omega(q)]^T C^{(0)}(q, \bar{q}) \tilde{\Omega}(\bar{q}).
\]

This matrix equation provides the quantization conditions for the integrals of motion of the model, \( q_k \) and \( \bar{q}_k \) with \( k = 3, \ldots, N \). In addition, it allows us to determine the matrices \( C^{(0)} \) and \( C^{(1)} \) and, as a consequence, evaluate the eigenvalues of the Baxter \( Q \)-operator, Eq. (3.1). Indeed, replacing in (3.28) the mixing matrices by their expressions, Eqs. (3.9) and (3.19), we obtain the system of \( N^2 \) equations involving \( (N - 1) \alpha \)-parameters inside the matrix \( C^{(0)} \), \( 2 + (N - 2)^2 \) parameters \( \beta_{1,2} \) and \( \gamma_{mn} \) inside the matrix \( C^{(1)} \), as well as \( (N - 2) \) integrals of motion \( q_3, \ldots, q_N \) (we recall that \( q_k = q_k^* \)). Thus, the system (3.28) is overdetermined. It allows us to determine all parameters including the quantized \( q \) and, in addition, it provides \((2N - 3)\) nontrivial consistency conditions on the obtained solutions. Additional consistency conditions follow from (3.24).

The solutions to the quantization conditions (3.28) for different number of particles \( N \) and the emerging properties of the spectrum of the model will be described in details in Section 5.
3.4. Special case: $N = 2$

As was already mentioned, the solution to the differential equation (3.3) for $N = 2$ admit representation in the form of contour integrals and, as a result, the quantization conditions (3.28) can be solved exactly. At $N = 2$, after the change of variables $z = (x - 1)/x$ and $Q(z) = [x(1 - x)]^{1-s}y(x)$, Eq. (3.3) takes the form of the Legendre’s differential equation [24]

$$\left[ \frac{1}{dx}x(1-x) \frac{d}{dx} + h(h-1) \right]y(x) = 0.$$  (3.29)

Its general solution is well-known as $y(x) = \int_{C_w} dw w^{h-1}(w-1)^{h-1}(w-x)^{-h}$, where the integration contour $C_w$ has to be chosen in such a way that the integrand resumes its original value after encircling $C_w$. Then, two linear independent solutions to (3.29) are given by the Legendre’s functions of the first and second kind, $P_{-h}(2x - 1)$ and $Q_{-h}(2x - 1)$, respectively. Using the relation between these functions

$$-\pi \cot(\pi h) P_{-h}(2x - 1) = Q_{-h}(2x - 1) - Q_{h-1}(2x - 1)$$  (3.30)

and going back to the $z$-representation, we choose the fundamental basis of solutions to Eq. (3.3) as $Q_{s}(z; h)$ and $Q_{s}(z; 1 - h)$, where the notation was introduced for the function\(^{7}\)

$$Q_{s}(z; h) \equiv \left[ \frac{z}{(1-z)^2} \right]^{1-s} Q_{-h}\left( \frac{1+z}{1-z} \right).$$  (3.31)

The properties of the function $Q_{s}(z; h)$ including its relation to the fundamental basis (3.7) and (3.16) can be found in the Appendix A.

Following (3.4), we construct $Q(z, \bar{z})$ as a bilinear combination of the functions $Q_{s}(z; h)$ and $Q_{s}(z; 1 - h)$ and their antiholomorphic counterparts, $Q_{s}(\bar{z}; \bar{h})$ and $Q_{s}(\bar{z}; 1 - \bar{h})$. Requiring $Q(z, \bar{z})$ to have a trivial monodromy around $z = 1$ and taking into account that $Q_{s}(z; h) \sim \text{const} \times (1-z)^{2s-h-1}$ for $z \to 1$ (see Eq. (A.2)), one gets

$$Q(z, \bar{z}) = c_{h} Q_{s}(z; h) Q_{s}(\bar{z}; \bar{h}) + c_{1-h} Q_{s}(z; 1 - h) Q_{s}(\bar{z}; 1 - \bar{h}).$$  (3.32)

To fix the coefficients $c_{h}$ and $c_{1-h}$, one examines the small-$z$ asymptotics of (3.32) with a help of Eq. (A.1) and requires that the terms $\sim \ln z \ln \bar{z}$ should cancel and the coefficients in front of $\ln z$ and $\ln \bar{z}$ should be the same. Applying the identity $\psi(1-h) - \psi(h) = \pi \cot(\pi h)$ one finds that the both conditions are fulfilled provided that $c_{h} = -c_{1-h}$ and $\cot(\pi h) = \cot(\pi \bar{h})$. The second relation is automatically satisfied thanks to the property of the $SL(2, \mathbb{C})$ spins of the principal series, $h - \bar{h} = n_{h}$ with $n_{h}$ integer. Choosing

$$c_{h} = \frac{2}{\pi} \tan(\pi h)(-1)^{n_{h}},$$  (3.33)

we obtain from (3.32) and (A.1) the small-$z$ behaviour of the function $Q(z, \bar{z})$ as

$$Q(z, \bar{z}) = z^{1-s} z^{1-\bar{s}} (-1)^{n_{h}} \left\{ \ln(z \bar{z}) + 2 \Re [\psi(h) + \psi(1-h) - 2\psi(1)] + O(z, \bar{z}) \right\}.$$  (3.34)

---

\(^{7}\)Obviously, this definition is ambiguous. Instead of using the $Q$-functions, one may define the fundamental basis entirely in terms of the $P$-functions (see Eq. (A.9) below).
Taking into account that the quasimomentum of the $N = 2$ states is equal to $e^{i\theta_2} = (-1)^{n_h}$, Eq. (3.25), we find that this relation is in agreement with (3.11). Matching of (3.34) into (3.11) allows us to determine the $\alpha-$parameter

$$\alpha_1(h) = 2 \text{Re} [\psi(h) + \psi(1 - h) - 2\psi(1)].$$

(3.35)

Finally, applying (3.14) we calculate the energy at $N = 2$ as

$$E_2(h, \bar{h}) = 2\alpha_1(h) = 8 \text{Re} \left[ \psi \left( \frac{1 + |n_h|}{2} + i\nu_h \right) - \psi(1) \right].$$

(3.36)

The ground state corresponds to $h = \bar{h} = 1/2$, or equivalently $n_h = \nu_h = 0$. Its energy defines the intercept of the BFKL Pomeron [7]

$$\min E_2(h, \bar{h}) = -16 \ln 2.$$ 

(3.37)

The obtained exact solution was based on the properties of the Legendre functions, Eq. (3.31). Going over to the systems with the number of particles $N \geq 3$, one encounters the following difficulties. Firstly, representation for the solution to (3.3) in the form of contour integrals does not exist or, at least, it is not warranted. Secondly, for $N \geq 3$ the function $Q(z, \bar{z})$ depends on the integrals of motion, $q$ and $\bar{q}$, whose values should be determined by solving the quantization conditions (3.28). As we will show in Section 5, both problems can be solved by using the power series representation for the fundamental set of solutions, Eqs. (3.7) and (3.16).

In this Section we have demonstrated that the eigenvalues of the Baxter $Q-$operator possess the prescribed properties, Eqs. (2.7) – (2.13) only if the integrals of motion satisfy the quantization conditions (3.28). For such values of the integrals of motion one can construct the function $Q(z, \bar{z})$ at the vicinity of the singular points, $z = 0$, $z = 1$ and $z = \infty$, and analytically continue it onto the whole $z-$plane with a help of the transition matrices (3.26). The spectrum of the model – the energy and the quasimomentum, can be obtained from the mixing matrices $C^{(0)}$ and $C^{(1)}$, defining the asymptotic behaviour of $Q(z, \bar{z})$ around $z = 0$ and $z = 1$, respectively. In terms of the Baxter $Q-$operator, these two matrices control the behaviour of $Q_{q,\bar{q}}(u, \bar{u})$ around the pole, $u = i(s - 1)$ and $\bar{u} = i(\bar{s} - 1)$, and the asymptotics of $Q_{q,\bar{q}}(u, \bar{u})$ at infinity.

4. Eigenvalues of the Baxter $Q-$operator

In the previous Section, we established the quantization conditions for the integrals of motion $q$ and obtained the expression for the energy $E_N$. Let us now construct the corresponding eigenstates $\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}_1, \ldots, \vec{z}_N)$ defined in Eqs. (1.2) and (2.1).

The analysis is based on the method of Separated Variables (SoV) developed by Sklyanin [27]. It allows us to find the integral representation for the eigenstates of the model by going over to the representation of the separated coordinates $\vec{x} = (\vec{x}_1, \ldots, \vec{x}_{N-1})$ [5]

$$\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}) = \int d^{N-1}x \, \mu(\vec{x}) U_{\vec{p},\vec{x}}(\vec{z}) \Phi_{\{q,\bar{q}\}}(\vec{x})^*,$$

(4.1)

with $\vec{z} = (\vec{z}_1, \ldots, \vec{z}_N)$. Here, $U_{\vec{p},\vec{x}}(\vec{z})$ is the kernel of the unitary operator corresponding to this transformation. Its explicit expression of $U_{\vec{p},\vec{x}}(\vec{z})$ can be found in [5]. $\Phi_{\{q,\bar{q}\}}(\vec{x})$ is the wave
function in the separated coordinates. Remarkable property of the SoV representation is that \( \Phi_{(q,q)}(\vec{x}) \) is factorized into the product of the eigenvalues of the Baxter \( Q \)–operator depending on different separated coordinates

\[
(\Phi_{(q,q)}(\vec{x}))^* = \prod_{k=1}^{N-1} \left[ \frac{\Gamma(s + i\bar{x}_k)\Gamma(s - i\bar{x}_k)}{\Gamma(1 - s + i\bar{x}_k)\Gamma(1 - s - i\bar{x}_k)} \right]^N Q_{q,q}(x_k, \bar{x}_k). \tag{4.2}
\]

Contrary to the \( \vec{z} = (z, \bar{z}) \)–coordinates, the possible values of the separated coordinates \( x_k = (x_k, \bar{x}_k) \) are quantized as follows [5, 16]

\[
x_k = \nu_k - \frac{ik}{2}, \quad \bar{x}_k = \nu_k + \frac{ik}{2}, \tag{4.3}
\]

with \( \nu_k \) real and \( n_k \) integer. Integration on the space of separated variables, \( \int d\vec{x} \mu(\vec{x}) \), implies summation over integer \( n_k \) and integration over continuous \( \nu_k \)

\[
\int d^{N-1}\vec{x} = \prod_{k=1}^{N-1} \left( \sum_{n_k = -\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_k \right), \quad \mu(\vec{x}) = \frac{2\pi^{-N^2}}{(N-1)!} \prod_{j,k=1}^{N-1} |\vec{x}_k - \vec{x}_j|^2, \tag{4.4}
\]

where \( |\vec{x}_k - \vec{x}_j|^2 = (\nu_k - \nu_j)^2 + (n_k - n_j)^2 / 4 \).

Eqs. (4.1) and (4.2) allow us to calculate the eigenfunctions of the model in terms of the eigenvalues of the Baxter \( Q \)–operator. By the construction, the latter have poles specified in (2.11). One verifies, however, that they lie outside the integration axis in (4.4) and the integral in (4.1) is well-defined. Still, one can make use of the pole structure of \( Q_{q,q}(u, \bar{u}) \) by closing the integration contour over \( \nu_k \) into the upper (or lower) half-plane and calculating the asymptotics of the wave function \( \Psi_{\tilde{p},(q,q)}(\vec{z}) \) in the different regions of the \( \vec{z} \)–space.

According to (3.1) and (3.4), the eigenvalue of the Baxter \( Q \)–operator is given by the following two-dimensional integral

\[
Q_{q,q}(u, \bar{u}) = \int \frac{d^2 \vec{z}}{\vec{z}\bar{\vec{z}}} \, z^{-iu} \bar{z}^{-i\bar{u}} \sum_{a,b=1}^{N} Q_a(z) C_{ab} \bar{Q}_b(\bar{z}). \tag{4.5}
\]

Similar integrals have already appeared in the calculation of correlation functions in two-dimensional CFT. Applying the results of [28], we can convert \( Q_{q,q}(u, \bar{u}) \) into a sum of products of holomorphic and antiholomorphic contour integrals, \( \int_C dz z^{-1-i\bar{u}} Q_a(z) \) and \( \int_{\bar{C}} \bar{d}z \bar{z}^{-1-iu} \bar{Q}_b(\bar{z}) \), respectively, with the integration contours \( C \) and \( \bar{C} \) starting and ending at one of the singular points \( z = 0, \bar{z} = 1 \) and \( z = \infty \). These contour integrals define the set of \( 2N \)–functions depending separately on the (anti)holomorphic spectral parameters \( u \) and \( \bar{u} \). In analogy with the CFT, we shall refer to them as to the \( Q \)–blocks. Then, the resulting expression for \( Q_{q,q}(u, \bar{u}) \), Eq. (4.5), will be given by a bilinear combination of \( N \) blocks belonging to two sectors.

We would like stress that, contrary to \( Q_{q,q}(u, \bar{u}) \), the definition of the \( Q \)–blocks is ambiguous. The eigenvalues of the Baxter operator, \( Q_{q,q}(u, \bar{u}) \), stay invariant if one replaces the \( Q \)–blocks by their linear combinations and redefines appropriately the expansion coefficients \( C_{ab} \). Making use of this ambiguity, one may look for the definition of the blocks, for which the expression for \( Q_{q,q}(u, \bar{u}) \) looks particularly simple. In this Section, we shall present such a definition. We will demonstrate that the eigenvalues of the Baxter operator can be expressed in terms of only two \( Q \)–blocks – one in each sector, defined below in Eq. (4.12). We will also show that the quantization conditions on the integrals of motion and the expressions for the energy and the quasimomentum, established in Section 3, can be reformulated in terms of the \( Q \)–blocks.
4.1. Decomposition over the Baxter blocks

To proceed with calculation of (4.5), we have to specify the mixing matrix $C_{ab}$ as well as the functions $Q_a(z)$ and $\overline{Q}_b(\bar{z})$. By the definition, these functions should be defined uniformly on the whole complex $z$–plane with cuts and their bilinear combination, Eq. (3.4), should match (3.9) and (3.19) for $z \to 0$ and $z \to \infty$, respectively.

Let us choose $C_{ab}$ to be the mixing matrix introduced in (3.19) and the functions $Q_a(z)$ to be defined at the vicinity of $z = 1$ as

$$C_{ab} = C_{ab}^{(1)}(q, \bar{q}), \quad Q_a(z; q)^{z=1} Q_a^{(1)}(z; q).$$

The analytical continuation of $Q_a(z; q)$ to the region $z \to 0$ and $z \to \infty$ can be obtained from (3.26) and (3.22) as

$$Q_a(z; q)^{z=0} \sum_{b=1}^{N} [\Omega^{-1}(q)]_{ab} Q_b^{(0)}(z; q),$$

$$Q_a(z; q)^{z=\infty} \sum_{b=1}^{N} [\Omega(-q) S]_{ab}^{-1} Q_b^{(0)}(1/z; -q),$$

with the matrix $S$ given by (3.23). The functions $\overline{Q}_b(\bar{z}; \bar{q})$ are defined similarly.

The functions $Q_a(z)$ defined in this way possess a nontrivial monodromy at $z = 1$. Encircling the point $z = 1$ on the complex $z$–plane in anticlockwise direction, one calculates from (3.16) the corresponding monodromy matrix as

$$Q_a(z) \mapsto M_{ab} Q_b(z), \quad M = \text{diag}(e^{2\pi i(Ns-h)}, e^{2\pi i(Ns+h)}, 1, ..., 1).$$

Here, unity entries in the monodromy matrix $M$ correspond to $(N - 2)$ functions in the fundamental set (3.16) analytical at $z = 1$. Following [28], the two-dimensional integral (4.5) can be evaluated as (see Appendix C for details)

$$Q_{a\bar{b}}(u, \bar{u}) = \frac{1}{2i} \sum_{a, b=1}^{N} [(1 - M^T) C^{(1)}]_{ab} \int_{1}^{\infty} \frac{dz}{z} z^{-iu} Q_a(z; q) \int_{0}^{1} \frac{d\bar{z}}{\bar{z}} \bar{z}^{-iu} \overline{Q}_b(\bar{z}; \bar{q})$$

with $M^T = M$ according to (4.8). As follows from their definition, Eq. (4.7), the functions $Q_a(z)$ satisfy the relation (3.22)

$$Q_a(1/z; -q) = \sum_{b=1}^{N} S_{ab} Q_b(z; q),$$

which holds for arbitrary $z$ such that $\text{Im}(1/z) > 0$. Changing the integration variable in (4.9) as $z \to 1/z$ and applying (4.10) one gets

$$Q_{a\bar{b}}(u, \bar{u}) = \frac{1}{2i} \sum_{a, b=1}^{N} [S^T (1 - M) C^{(1)}]_{ab} \int_{0}^{1} \frac{dz}{z} z^{-iu} Q_a(z; -q) \int_{0}^{1} \frac{d\bar{z}}{\bar{z}} \bar{z}^{-iu} \overline{Q}_b(\bar{z}; \bar{q}).$$

We recall that the matrices $S$, $M$ and $C^{(1)}$ were defined before in Eqs. (3.23), (4.8) and (3.19), respectively. Notice also that $M^{-1} = S^2$. Substituting the monodromy matrix (4.8) into (4.11) we
find that among $N^2-$terms in the r.h.s. of (4.11) only two (with $a = b = 1$ and $a = b = 2$) provide a nonvanishing contribution to $Q_{q,\bar{q}}(u, \bar{u})$. These two terms correspond to the $Q-$functions nonanalytical at $z = 1$, $Q_1(z) \sim (1 - z)^{N_s - h - 1}$ and $Q_2(z) \sim (1 - z)^{N_s + h - 2}$.

Let us show that the blocks

\[ Q(u; h, q) = \frac{1}{\Gamma(Ns - h)} \int_0^1 \frac{dz}{z} z^{iu} Q_1(z; -q), \]

\[ \overline{Q}(\bar{u}; \bar{h}, \bar{q}) = \frac{1}{\Gamma(Ns - h)} \int_0^1 \frac{d\bar{z}}{\bar{z}} \bar{z}^{-iu} \overline{Q}_1(\bar{z}; \bar{q}), \]

(4.12)

with the normalization factors chosen for later convenience (see Eq. (4.17) below). Taking into account Eqs. (3.23) and (3.19), one gets from (4.11)

\[ Q_{q,\bar{q}}(u, \bar{u}) = \pi \left[ \frac{\Gamma(Ns - \bar{h})}{\Gamma(1 - Ns + h)} \beta_h(q, \bar{q}) Q(u; h, q) \overline{Q}(\bar{u}; \bar{h}, \bar{q}) \right. \]

\[ + \frac{\Gamma(Ns - 1 + \bar{h})}{\Gamma(2 - Ns - h)} \beta_{1-h}(q, \bar{q}) Q(u; 1 - h, q) \overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}) \right]. \quad (4.13) \]

Here, we also used the fact that the functions $Q_1(z; q)$ and $Q_2(z; q)$ (as well as $\overline{Q}_1(\bar{z}; \bar{q})$ and $\overline{Q}_2(\bar{z}; \bar{q})$) can be obtained one from another through $h \leftrightarrow 1 - h$.

We notice that in Eq. (4.9) the integration contours over $z$ and $\bar{z}$ are different whereas in the original two-dimensional integral (4.5) the both variables appear on the same footing. Repeating the calculation of (4.5) and making use of the monodromy of the functions $\overline{Q}_b(\bar{z}; \bar{q})$ in the antiholomorphic sector

\[ \overline{Q}_a(\bar{z}) \mapsto \overline{M}_{ab} \overline{Q}_b(\bar{z}), \quad \overline{M} = \operatorname{diag}\left( e^{-2\pi i(Ns - \bar{h})}, e^{-2\pi i(Ns + \bar{h})}, 1, \ldots, 1 \right), \quad (4.14) \]

one arrives at another (through equivalent) expression for the eigenvalues of the Baxter operator

\[ Q_{q,\bar{q}}(u, \bar{u}) = \pi \left[ \frac{\Gamma(Ns - h)}{\Gamma(1 - Ns + h)} \beta_h(q, \bar{q}) Q(-u; h, -q) \overline{Q}(-\bar{u}; \bar{h}, -\bar{q}) \right. \]

\[ + \frac{\Gamma(Ns - 1 + h)}{\Gamma(2 - Ns - h)} \beta_{1-h}(q, \bar{q}) Q(-u; 1 - h, -q) \overline{Q}(-\bar{u}; 1 - \bar{h}, -\bar{q}) \right]. \quad (4.15) \]

To verify the equivalence of this expression with (4.13), one substitutes (4.15) into the l.h.s. of (2.16) and takes into account the second relation in (3.24).

Thus, for given set of the integrals of motion $(q, \bar{q})$, satisfying the quantization condition (3.28), the eigenvalue of the Baxter $Q-$operator, $Q_{q,\bar{q}}(u, \bar{u})$, is unique and it is expressed in terms of two chiral blocks defined in (4.12).

### 4.2. Properties of the blocks

Let us show that the blocks $Q(u; h, q)$ and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$, defined in (4.12), have the following three properties:

(i) $Q(u; h, q)$ satisfies the chiral Baxter equation (2.7) and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$ obeys similar equation in the antiholomorphic sector.
(ii) $Q(u; h, q)$ and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$ are meromorphic functions [29, 5, 16] on the complex $u-$ and $\bar{u}-$planes, respectively, with the only poles of the order not higher than $N$ located at the points $u_m = -i(s - m)$ and $\bar{u}_\bar{m} = i(\bar{s} - \bar{m})$, with $m$ and $\bar{m}$ positive integer. The same property can be expressed in a concise form as

$$Q(u; h, q) = \Gamma^N(1 - s + iu)f(u),$$

$$\overline{Q}(\bar{u}; \bar{h}, \bar{q}) = \Gamma^N(1 - \bar{s} - i\bar{u})\bar{f}(\bar{u})$$

(4.16)

with $f(u)$ and $\bar{f}(%(u)$ being entire functions.

(iii) At large $u$ and $\bar{u}$, away from the poles (4.16), the functions $Q(u; h, q)$ and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$ have the asymptotic behaviour

$$Q(u; h, q) \sim (iu)^{-N s + h} [1 + O(1/u)],$$

$$\overline{Q}(\bar{u}; \bar{h}, \bar{q}) \sim (-i\bar{u})^{-N \bar{s} + \bar{h}} [1 + O(1/\bar{u})].$$

(4.17)

By the definition, Eq. (4.12), the blocks $Q(u; h, q)$ and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$ are related to the same universal function calculated for different values of the parameters

$$Q(u; h, q) = Q_s(u; h, q), \quad \overline{Q}(\bar{u}; \bar{h}, \bar{q}) = Q_s(-\bar{u}; \bar{h}, -\bar{q}).$$

(4.18)

To verify these properties one uses the integral representation for the blocks, Eqs. (4.12). Then, the first property follows from the fact that the functions $Q_1(z)$ and $\overline{Q}_1(\bar{z})$ satisfy the differential equation (3.3). As to the second property, the poles of $Q(u; h, q)$ and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$ come from integration in (4.12) over the region of small $z$ and $\bar{z}$. The leading asymptotic behaviour of the functions $Q_1(z)$ and $\overline{Q}_1(\bar{z})$ in this region can be obtained from the first relation in (4.7) as $Q_s(z) \sim z^{1-s} \ln^N z$ and $\overline{Q}_s(\bar{z}) \sim (\bar{z})^{1-\bar{s}} \ln^N \bar{z}$. Finally, to obtain the asymptotics at infinity, Eq. (4.17), one integrates in (4.12) over the region $z \to 1$ and $\bar{z} \to 1$ and makes use of Eqs. (4.7) and (3.16) to replace $Q_1(z) \sim (1 - z)^{Ns - h - 1}$ and $\overline{Q}_1(\bar{z}) \sim (1 - \bar{z})^{N\bar{s} - \bar{h} - 1}$.

The above three properties define the blocks uniquely only for $\text{Im} h \neq 0$. As we have seen in Section 3.2 (see Eq. (3.18)), for $\text{Im} h = 0$ one of the fundamental solutions to the differential equation (3.3) has to be redefined in order to avoid a degeneracy. One encounters the same problem trying to define the block $Q(u; h, q)$ at $h = (1 + n_h)/2$. In this case, the linear combination $Q(u; h, q) + c Q(u; 1 - h, q)$ satisfies the three conditions on the $Q-$block for arbitrary $c$ and, as a consequence, the blocks $Q(u; h, q)$ and $Q(u; 1 - h, q)$ become degenerate. The expression for the blocks at $h = (1 + n_h)/2$ can be found in the Appendix D.

Using the definition of the blocks $Q(u; h, q)$ and $\overline{Q}(\bar{u}; \bar{h}, \bar{q})$, one can establish different useful relations between them. In particular, as shown in the Appendix B, the blocks in each sector satisfy nontrivial Wronskian relations (Eqs. (B.4) and (B.6)). Moreover, the blocks in two sectors are related to each other as (see Eq. (B.3))

$$Q(u; h, q) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^N Q(u^*; 1 - \bar{h}, \bar{q})^*.$$  

(4.19)

At $N = 2$ the eigenvalues of the Baxter $Q-$operator and the $Q-$blocks can be expressed in a concise form in terms of the $3F_2-$hypergeometric series of a unit argument. As was shown in
Section 3.4, at $N = 2$ the function $Q_1(z)$ entering (4.12) is equal (up to an overall normalization) to the Legendre function of the second kind. Substituting (3.31) into (4.12) and using integral representation of the Legendre functions [24], one obtains after some algebra

$$Q_s(u; h) = \frac{1}{2} \Gamma \left[ \begin{array}{c} 1-s+i\mu, 1 - s + i\nu, 1 - h \\ 1 + s - h + i\mu, 2 - s + i\nu - h \end{array} \right] _3 F_2 \left( \begin{array}{c} s + i\mu, 1 - s + i\nu, 1 - h \\ 1 + s - h + i\mu, 2 - s - h + i\mu \end{array} \right),$$

(4.20)

where $\Gamma[...]$ denotes the ratio of the products of the $\Gamma-$functions with the arguments in the upper and lower rows, respectively. Together with (4.13) and (A.7) this leads to the following expression for the eigenvalues of the Baxter operator at $N = 2$

$$Q(u, \bar{u}) = 2(-1)^{n_h} \tan(\pi h) \times \frac{\Gamma(2\bar{s} - \bar{h})}{\Gamma(1 - 2s + h) Q(u; h) \overline{Q(\bar{u}; h)}} - \frac{\Gamma(2\bar{s} - 1 + \bar{h})}{\Gamma(2 - 2s - h)} Q(u; 1 - h) \overline{Q(\bar{u}; 1 - \bar{h})},$$

(4.21)

where $Q(u; h) = Q_s(u; h)$ and $\overline{Q(\bar{u}; h)} = Q_s(-\bar{u}; \bar{h})$ are the (non-normalized) $Q-$blocks at $N = 2$.\footnote{Due to the additional factor $\Gamma^2(1 - h)/(2\Gamma(2 - 2h))$ in the r.h.s. of (A.4), this expression has asymptotics at infinity that differs from (4.17) by the same factor.}

For $N \geq 3$ one can calculate the blocks by replacing the functions $Q_1(z; q)$ in (4.12) by their expressions in terms of the fundamental solutions around $z = 0$ and $z = 1$, Eqs. (4.7) and (4.6), respectively. As shown in the Appendix B, this leads to two different series representations for the block $Q(u; h, q)$, Eqs. (B.9) and (B.11), which are valid in the different regions on the complex $u-$plane.

We are now in position to demonstrate that the eigenvalues of the Baxter operator constructed in this Section have correct asymptotic behaviour at infinity. Substituting (4.17) into (4.13), we verify that $Q_{q,\bar{q}}(u, \bar{u})$ satisfies (2.13) with the phase $\Theta_h(q, \bar{q})$ given by

$$e^{2i\Theta_h(q, \bar{q})} = (-1)^{n_h} \frac{\beta_h(q, \bar{q})}{\beta_{1-h}(q, \bar{q})} \frac{\Gamma(2 - Ns - h)\Gamma(N\bar{s} - \bar{h})}{\Gamma(1 - Ns + h)\Gamma(N\bar{s} - 1 + \bar{h})}. \quad (4.22)$$

In particular, for $h = \bar{h} = 1/2 + i\nu_h$ and $\nu_h \rightarrow 0$ one has $\beta_h(q, \bar{q}) \sim 1/\nu_h$ leading to $e^{2i\Theta_{1/2}(q, \bar{q})} = -1$.

### 4.3. Quantization conditions from the $Q-$blocks

As we have seen in the previous Section, the eigenvalues of the Baxter $Q-$operator can be expressed in terms of the $Q-$blocks satisfying the conditions (i)-(iii). In distinction with $Q_{q,\bar{q}}(u, \bar{u})$, the $Q-$blocks can be constructed for arbitrary values of the integrals of motion $q, \bar{q}$. In this Section, we will show that the requirement for $Q_{q,\bar{q}}(u, \bar{u})$ to have correct analytical properties leads to the quantization conditions for the integrals of motion which are equivalent to (3.28).

The general expression for the eigenvalue of the Baxter operator $Q_{q,\bar{q}}(u, \bar{u})$ in terms of the blocks $Q(u; h, q)$ and $Q(u; 1 - h, q)$ and their antiholomorphic counterparts looks like

$$Q_{q,\bar{q}}(u, \bar{u}) = c_h Q(u; h, q) \overline{Q(\bar{u}; \bar{h}, \bar{q})} + c_{1-h} Q(u; 1 - h, q) \overline{Q(\bar{u}; 1 - \bar{h}, \bar{q})}$$

(4.23)

with $c_h$ (and $c_{1-h}$) being arbitrary function of $h$ and the integrals of motion. By the construction, $Q_{q,\bar{q}}(u, \bar{u})$ is symmetric under $h \rightarrow 1 - h$ and $\bar{h} \rightarrow 1 - \bar{h}$, it satisfies the Baxter equations in the holomorphic and antiholomorphic sectors, and its asymptotic behaviour at infinity is in agreement
with Eqs. (2.13). Therefore, it remains to show that $Q_{q,q}(u, \bar{u})$ has the correct structure of the poles, Eq. (2.11). To this end, one applies (B.3) and rewrites (4.23) in two equivalent forms

$$Q_{q,q}(u, \bar{u}) = \left[ \frac{\Gamma(1 - s - i\bar{u})}{\Gamma(s - i\bar{u})} \right]^N \left\{ c_h Q(u; h, q)(Q(\bar{u}^*; 1 - h, q))^* + c_{1-h} Q(u; 1 - h, q)(Q(\bar{u}^*; h, q))^* \right\}$$

$$= \left[ \frac{\Gamma(1 - s + i\bar{u})}{\Gamma(s + i\bar{u})} \right]^N \left\{ c_h Q(\bar{u}; \bar{h}, \bar{q})(Q(u^*; 1 - \bar{h}, \bar{q}))^* + c_{1-h} Q(\bar{u}; 1 - \bar{h}, \bar{q})(Q(u^*; \bar{h}, \bar{q}))^* \right\}$$ (4.24)

The main advantage of such representation is that the analytical properties of $Q_{q,q}(u, \bar{u})$ are manifest – the poles of $Q_{q,q}(u, \bar{u})$ in $\bar{u}$ and $u$ are generated by the $\Gamma$–functions in the first and the second relation, respectively. One deduces from (4.24) that for arbitrary $c_h$ and $c_{1-h}$ the function $Q_{q,q}(u, \bar{u})$ has the $N$–th order poles at $\bar{u} = i(s - \bar{m})$ and, separately, at $u = -i(s - m)$ with $m, \bar{m} \in \mathbb{Z}_+$.

Let us now require that the analytical properties of (4.24) should match similar properties of the Baxter $Q$–operator, Eq. (2.11). We remind that the operator $Q(u, \bar{u})$ is well-defined only if the spectral parameters $u$ and $\bar{u}$ satisfy (2.2). Imposing this condition, we find that (4.24) has the prescribed poles, (2.11), plus additional “spurious” $N$–th order poles located at

$$\{ u = i(s + m - 1), \ \bar{u} = i(s - \bar{m}) \},$$

$$\{ u = -i(s - m), \ \bar{u} = -i(s + \bar{m} - 1) \},$$

with $m, \bar{m} = 1, 2, \ldots$. Thus, the coefficients $c_h$ (and $c_{1-h}$) and the integrals of motions $(q, \bar{q})$ have to be chosen in such a way that (4.24) has to have vanishing residues at the poles (4.25).

Introducing the functions

$$\Phi(\epsilon) = \frac{Q(i(s + \epsilon); h, q)}{Q(i(s + \epsilon); 1 - h, q)}, \quad \overline{\Phi}(\epsilon) = \frac{\overline{Q}(-i(s + \epsilon); \bar{h}, \bar{q})}{\overline{Q}(-i(s + \epsilon); 1 - \bar{h}, \bar{q})}$$ (4.26)

and examining the first and the second relation in (4.24) for $\{ u = i(s + m - 1 + \epsilon), \ \bar{u} = i(s - \bar{m} + \epsilon) \}$ and $\{ u = -i(s - m + \epsilon), \ \bar{u} = i(s + \bar{m} - 1 + \epsilon) \}$, respectively, as $\epsilon \to 0$, one finds that this requirement leads to

$$c_h \Phi(m - 1 + \epsilon) + c_{1-h} (\Phi(m - 1 + \epsilon))^* = \mathcal{O}(\epsilon^N)$$

$$c_h \overline{\Phi}(m - 1 + \epsilon) + c_{1-h} (\overline{\Phi}(m - 1 + \epsilon))^* = \mathcal{O}(\epsilon^N),$$

(4.27)

with $m$ and $\bar{m}$ being positive integer. It follows from the Wronskian relations (see Appendix B, Eq. (B.7)), that the infinite system of equations (4.27) becomes equivalent to a single condition at $m = \bar{m} = 1$

$$\frac{c_{1-h}}{c_h} (\Phi(\epsilon))^* = -1 + \mathcal{O}(\epsilon^N), \quad \frac{c_{1-h}}{c_h} (\overline{\Phi}(\epsilon))^* = -1 + \mathcal{O}(\epsilon^N),$$

(4.28)

with the functions $\Phi(\epsilon)$ and $\overline{\Phi}(\epsilon)$ defined in (4.26).

We find from (4.28) that $c_h/c_{1-h}$ is a pure phase. Its value can be obtained from the matching of (4.23) into (2.13) at large $u$ and $\bar{u}$

$$\frac{c_{1-h}}{c_h} = (-1)^{n_h} e^{-2i\Theta_h(q, \bar{q})},$$

(4.29)
with $\Theta_{h}(q, \bar{q})$ given by (4.22). Then, recalling the definition of the function $\Phi(\epsilon)$, Eq. (4.26), we obtain from (4.28)

$$\arg \left[ \frac{Q(i(s + \epsilon); h, q)}{Q(i(s + \epsilon); 1 - h, q)} \right] = \pi \left( \frac{n_{h} + 1}{2} + \ell \right) - \Theta_{h}(q, \bar{q}) + \mathcal{O}(\epsilon^{N})$$  \hspace{1cm} (4.30)

and similar relation for the antiholomorphic block

$$\arg \left[ \frac{\overline{Q}(-i(\bar{s} + \epsilon); \bar{h}, \bar{q})}{Q(-i(\bar{s} + \epsilon); 1 - \bar{h}, \bar{q})} \right] = \pi \left( \frac{n_{h} + 1}{2} + \bar{\ell} \right) - \Theta_{h}(q, \bar{q}) + \mathcal{O}(\epsilon^{N}),$$  \hspace{1cm} (4.31)

with $\ell$ and $\bar{\ell}$ being integers such that, in general, $\bar{\ell} \neq \ell$. Applying (4.19), one can express (4.31) entirely in terms of the holomorphic blocks.

The blocks entering the relations (4.30) and (4.31) can be calculated from (B.9). Expanding the both sides of (4.30) and (4.31) in powers of $\epsilon$, one obtains the system of $2N$ real quantization conditions on $N - 2$ complex charges $q_{3}, ..., q_{N}$ and real phase $\Theta_{h}(q, \bar{q})$. In Section 5, we will verify that their solutions are consistent with the quantization conditions (3.28).

### 4.4. Energy spectrum from the $Q$–blocks

Let us show that the energy, $E_{N}(q, \bar{q})$, and quasimomentum, $\theta_{N}(q, \bar{q})$, admit a simple representation in terms of the $Q$–blocks. To this end, let us introduce new blocks

$$Q_{0}(u; q) = a_{h}Q(u; h, q) + a_{1-h}Q(u; 1 - h, q),$$

$$\overline{Q}_{0}(\bar{u}; \bar{q}) = \bar{a}_{h}\overline{Q}(\bar{u}; \bar{h}, \bar{q}) + \bar{a}_{1-h}\overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}),$$  \hspace{1cm} (4.32)

symmetric under $h \rightarrow 1 - h$ and $\bar{h} \rightarrow 1 - \bar{h}$. We require that $Q_{0}(u; q)$ and $\overline{Q}_{0}(\bar{u}; \bar{q})$ should have the same poles as the $Q$–blocks, Eq. (4.16), but of the order not be higher than $N - 1$.

Applying (4.19), it is straightforward to verify that the linear combination of the $Q$–blocks in the r.h.s. of (4.32) has a vanishing residues at the $N$th pole at $u_{N}^{\pm} = -i(s - 1)$ and $\bar{u}_{N}^{\pm} = i(\bar{s} - 1)$ provided that (up to an overall normalization)

$$a_{h} = \tan(\pi h)(\overline{Q}(-i\bar{s}; \bar{h}, \bar{q}))^{*}, \quad \bar{a}_{h} = \tan(\pi \bar{h})(Q(is; h, q))^{*}.\hspace{1cm} (4.33)$$

Then, the residues at the remaining $N$th order poles, $\{u_{m}, \bar{u}_{m}^{\pm}\}$, vanish automatically, since otherwise $Q_{0}(u; q)$ and $\overline{Q}_{0}(\bar{u}; \bar{q})$ will not satisfy the Baxter equations (2.7).

Let us now consider the following expressions

$$Q_{1}(u; q) = \left[ \frac{\Gamma(1 - s + i\epsilon)}{\Gamma(s + i\epsilon)} \right]^{N} (Q_{0}(u^{*}; \bar{q}))^{*} = (a_{1-h})^{*}Q(u; h, q) + (\bar{a}_{h})^{*}Q(u; 1 - h, q),$$

$$\overline{Q}_{1}(\bar{u}; \bar{q}) = \left[ \frac{\Gamma(1 - \bar{s} - i\epsilon)}{\Gamma(\bar{s} - i\epsilon)} \right]^{N} (Q_{0}(\bar{u}^{*}; q))^{*} = (a_{1-h})^{*}\overline{Q}(\bar{u}; \bar{h}, \bar{q}) + (\bar{a}_{h})^{*}\overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}),$$  \hspace{1cm} (4.34)

where in the r.h.s. we used (B.3). In distinction with (4.19), this transformation maps $Q_{0}(u; q)$ into another block $Q_{1}(u; q)$ and similar for $\overline{Q}_{0}(\bar{u}; \bar{q})$. The ratio of the $\Gamma$–functions in the r.h.s. of (4.34) ensures that $Q_{1}(u; q)$ and $\overline{Q}_{1}(\bar{u}; \bar{q})$ have the same pole structure as the $Q$–block, Eq. (4.16) and, in addition,

$$Q_{1}(i(s + n) + \epsilon; q) \sim \epsilon, \quad \overline{Q}_{1}(-i(\bar{s} + n) + \epsilon; \bar{q}) \sim \epsilon$$  \hspace{1cm} (4.35)
for $\epsilon \to 0$ and $n$ nonnegative integer. This property plays a crucial rôle in our analysis.

Combining together (4.32) and (4.34), one can express the $Q-$blocks in terms of new blocks $Q_0(u; q)$ and $Q_1(u; q)$. Substitution of the resulting expressions into (4.23) yields

$$Q_{q,q}(u, \bar{u}) = A_{q,q} Q_0(u; q) \overline{Q_0(\bar{u}; q)} + B_{q,q} Q_1(u; q) \overline{Q_1(\bar{u}; q)}.$$  \hspace{1cm} (4.36)

Notice that the cross-terms, like $Q_0(u; q) \overline{Q_1(\bar{u}; q)}$, do not appear in this expression. They have nonvanishing residues at the $N$th order spurious poles (4.25) and their appearance in the r.h.s. of (4.36) is protected by the quantization conditions (4.30) and (4.31). The coefficients $A_{q,q}$ and $B_{q,q}$ can be expressed in terms of $c_h, a_h$ and $\bar{a}_h$ defined in Eqs. (4.29) and (4.33), but we will not need them for our purposes.

To find the quasimomentum (2.6), one has to evaluate (4.36) at $u = \pm i s$ and $\bar{u} = \pm i \bar{s}$. Since the second term in (4.36) vanishes at these points due to (4.35), one gets

$$\theta_N(q, \bar{q}) = i \ln \frac{Q_0(is; q)}{Q_0(-is; q)} + i \ln \frac{\overline{Q_0(is; \bar{q})}}{\overline{Q_0(-is; \bar{q})}}.$$  \hspace{1cm} (4.37)

The calculation of the energy (2.3) is more cumbersome. One finds from (4.36) and (4.35) that

$$\frac{d}{du} \ln Q_{q,q}(u - is, u - i\bar{s}) \bigg|_{u=0} = (\ln Q_0(-is; q))' + (\ln \overline{Q_0(-is; \bar{q})})' + \Delta,$$  \hspace{1cm} (4.38)

where prime stands for derivative with respect to the spectral parameter and

$$\Delta = \frac{B_{q,q} Q_1(-is) \overline{Q_1(-i\bar{s})}}{A_{q,q} Q_0(-is) \overline{Q_0(-i\bar{s})}} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \frac{Q_1(-is) Q_0(-i(s - 1 + \epsilon))}{Q_0(-is) Q_1(-i(s - 1 + \epsilon))}.$$  \hspace{1cm} (4.39)

Here, the second relation follows from the requirement for (4.36) to have vanishing residues at the “spurious” poles (4.25), or equivalently $Q_{q,q}(-i(s - 1 + \epsilon), -i(\bar{s} + \epsilon))$ to be finite as $\epsilon \to 0$. The residue of $Q_1(-i(s - 1 + \epsilon))$ at the $N$th order pole at $\epsilon = 0$ can be obtained from the first relation in (4.34). To calculate the residue of $Q_0(-i(s - 1 + \epsilon))$ at the $(N - 1)$th order pole at $\epsilon = 0$ one applies the Wronskian relation

$$Q_0(u + i; q) Q_1(u; q) - Q_0(u; q) Q_1(u + i; q) = \text{const} \times \left[ \frac{\Gamma(iu - s)}{\Gamma(iu + s)} \right]^N,$$  \hspace{1cm} (4.40)

which follows from (B.4) and the definition of the $Q_0-$ and $Q_1-$blocks. The normalization constant in the r.h.s. of (4.40) can be obtained for $u = -i(s + \epsilon)$ from the comparison of the residues of the both sides of (4.40) at the $N$th order pole in $\epsilon$. In this way, one arrives at

$$\frac{d}{du} \ln Q_{q,q}(u - is, u - i\bar{s}) \bigg|_{u=0} = \frac{iN}{1 - 2s} + (\ln \overline{Q_0(-i\bar{s}; \bar{q})})' - (\ln \overline{Q_0(-i\bar{s}; \bar{q})'})' \bigg|_{u=0}.$$  \hspace{1cm} (4.41)

Repeating similar calculation of the logarithmic derivative of (4.36) at $u = is$ and $\bar{u} = i\bar{s}$ we find

$$\frac{d}{du} \ln Q_{q,q}(u + is, u + i\bar{s}) \bigg|_{u=0} = -\frac{iN}{1 - 2\bar{s}} + (\ln Q_0(is; q))' - (\ln \overline{Q_0(is; q)})' - \epsilon_N.$$  \hspace{1cm} (4.42)

Finally, we substitute the last two relations into (2.3) and obtain the following expression for the energy

$$E_N(q, \bar{q}) = -2 \text{Im} (\ln Q_0(is; q))' + 2 \text{Im} (\ln \overline{Q_0(-i\bar{s}; \bar{q})})' + \epsilon_N.$$  \hspace{1cm} (4.43)
where $\varepsilon_N = 2N \text{Re} [\psi(2s) + \psi(2 - 2s) - 2\psi(1)]$. One can further simplify this expression and bring it to the form, in which the symmetry $E_N(q, \bar{q}) = E_N(-q, -\bar{q})$ becomes manifest

$$E_N(q, \bar{q}) = -2 \text{Im} (\ln Q_0(is; q))' - 2 \text{Im} (\ln Q_0(is; -q))' + E_N^{(0)},$$

(4.44)

where $E_N^{(0)} = 2N \text{Re} [\psi(2s) + \psi(1 - 2s) - 2\psi(1)]$. To get this expression one uses the relation $(\ln Q_{q,\bar{q}}(-is, -i\bar{s}))' = -(\ln Q_{-q,-\bar{q}}(is, i\bar{s}))'$ that follows from (2.16).

The results obtained in Sections 3 and 4 provide the basis for calculating the energy spectrum of the model. We have demonstrated that

- The eigenvalues of the Baxter operator, $Q_{q,\bar{q}}(u, \bar{u})$, possessing the correct analytical properties and asymptotic behaviour at infinity (see Section 2) can be constructed only for the special values of the integrals of motion $q$ and $\bar{q}$ satisfying the quantization conditions, Eqs. (3.28), (4.30) and (4.31).

- The functions $Q_{q,\bar{q}}(u, \bar{u})$ can be decomposed over the $Q$–blocks depending on the spectral parameters and the quantum numbers in the (anti)holomorphic sector, Eq. (4.13) and (4.36). In distinction with $Q_{q,\bar{q}}(u, \bar{u})$, the definition of the $Q$–blocks is ambiguous and they play in our analysis the rôle of auxiliary building blocks.

- Once the quantization conditions for the integrals of motion are fulfilled and the eigenvalues of the Baxter operator are constructed, the energy spectrum of the model is uniquely fixed and it can be calculated from (3.14), (4.43) and (4.44).

We would like to mention that we disagree on these points with the approach of Ref. [16], in which the Baxter equation for noncompact magnet of spin $s = 1$ has been investigated.

5. Energy spectrum

In this Section we solve the quantization conditions (3.28) and describe the spectrum of the Schrödinger equation (1.2) for different number of particles $N$.

We recall that the Hamiltonian of the noncompact Heisenberg magnet (1.5) depends on the number of particles involved, $N$, and their $SL(2, \mathbb{C})$ spins $(s, \bar{s})$, defined in (1.4). For arbitrary $N$, thanks to a complete integrability of the Schrödinger equation (1.2), the spectrum of the model – the energy, $E_N(q, \bar{q})$, and the corresponding eigenstates, $\Psi_{\vec{p}, \{q, \bar{q}\}}(\vec{Z})$, are uniquely specified by the total set of quantum numbers in the holomorphic and antiholomorphic sectors, $q = (q_2, q_3, \ldots, q_N)$ and $\bar{q} = (\bar{q}_2, \bar{q}_3, \ldots, \bar{q}_N)$, respectively.

Since $q_k = \bar{q}_k$, the total number of independent complex valued quantum numbers is equal to $(N - 1)$. One of them, $q_2$, fixes the total $SL(2, \mathbb{C})$ spin of the state $(h, \bar{h})$ defined in (2.9). According to Eq. (2.9), the quantized values of $h$, or equivalently $q_2$, are parameterized by integer $n_h$ and real $\nu_h$

$$q_2(n_h, \nu_h) = \frac{1}{4} - \left(\frac{n_h}{2} + i\nu_h\right)^2 + Ns(s - 1).$$

(5.1)

At $N = 2$ this becomes the only quantum number parameterizing the spectrum of the model (3.36). For $N \geq 3$ one has to deal with a bigger set of the quantum numbers.

To find the spectrum of quantized $(q, \bar{q})$ for $N \geq 3$, one has to solve the overdetermined system of the quantization conditions (3.28). As was explained in Section 3, their solutions give
us the expressions for the $\alpha-$, $\beta-$ and $\gamma-$parameters entering the mixing matrices $C^{(0)}$ and $C^{(1)}$, Eqs. (3.9) and (3.19). Then, the energy, $E_N(q, \bar{q})$, and the quasimomentum, $\theta_N(q, \bar{q})$, of the eigenstates are calculated by inserting these expressions into (3.14) and (3.24), respectively. In distinction with the $N = 2$ case, Eqs. (3.36) and (3.35), the resulting expressions for the spectrum of the model can not be expressed for arbitrary $N$ in a closed analytical form. Nevertheless, the results that we are going to present in this Section exhibit a remarkable regularity and they suggest that there should exist a WKB-like description of the spectrum.

To proceed with solving the quantization conditions (3.28), one has to specify the values of the $SL(2, \mathbb{C})$ spins $(s, \bar{s})$. Since our main motivation for studying the noncompact Heisenberg spin magnets came from high-energy QCD, we shall fix their values to be the same as for the $N-$reggeized gluons compound state, Eqs. (1.1) and (1.2),

$$s = 0, \quad \bar{s} = 1.$$  \hspace{1cm} (5.2)

Another specifics of QCD is that, solving the Schrödinger equation (1.2), one is mainly interested in finding the ground state. It is this state that provides the dominant contribution to the asymptotic behaviour of the scattering amplitudes (1.1) at high-energy. Additional constraints on the solutions to (1.2) are imposed in QCD by the Bose symmetry of the $N-$reggeized gluon compound states. As we will argue below, they select among all eigenstates only those with the quasimomentum $\exp(i \theta_N(q, \bar{q})) = \pm 1$. This condition is automatically satisfied for the ground state for which one expects that $\theta_N(q, \bar{q}) = 0$.

To analyze the quantization conditions (3.28), one needs the expressions for the transition matrices $\Omega(q)$ and $\overline{\Omega}(\bar{q})$, defined in (3.26). For $N = 2$ they can be determined exactly from the properties of the Legendre functions (see Appendix A). For $N \geq 3$ the calculation of these matrices is based on the relation (3.27), which in turn relies on the power series representation of the fundamental solutions, Eqs. (3.7) and (3.16). In general, the resulting expressions for the transition matrices take the form of an infinite series in $q$. Performing numerical solutions of the quantization conditions, we truncate infinite series in Eqs. (3.8) and (3.17) and retain a large enough number of terms ($n_{\text{max}} \sim 10^3$).

As was already mentioned, the system of $N^2$ quantization conditions (3.28) is overdetermined. Using only $(N^2 - 3N + 5)$ of them, we can express the $\alpha-$, $\beta-$ and $\gamma-$parameters in terms of $q$ and define the following test function

$$f(q_2, q_3, ..., q_N) = \text{Tr}(TT^\dagger), \quad T = C^{(1)}(q, \bar{q}) - [\Omega(q)]^T C^{(0)}(q, \bar{q}) \overline{\Omega}(\bar{q}).$$  \hspace{1cm} (5.3)

Obviously, $f(q)$ is a positive definite function on the $(N - 1)-$dimensional complex moduli space $(q_2, q_3, ..., q_N)$. The solutions to the quantization conditions (3.28) correspond to points on this space, in which the test function vanishes. Since the dimension of the moduli space increases with $N$, the problem of finding zeros of $f(q)$ becomes very nontrivial at higher $N$. To solve it, we applied the algorithm developed in [30].\footnote{This method was implemented in the form of the Fortran-90 code, which is available from the authors upon request.} It allowed us to identify the zeros of the test function (5.3) by the “steepest descent” method starting from some reference point on the moduli space. As yet another test of our approach, we verified that the obtained expressions for the integrals of motion satisfy the conditions (4.30) and (4.31).
5.1. Fine structure of the spectrum

Before summarizing the results of our calculations for $N \geq 3$, let us describe a general structure of the spectrum. We find that for given total $SL(2, \mathbb{C})$ spin of the system, $h = (1 + n_h)/2 + i\nu_h$, the quantization conditions (3.28) provide us with an infinite number of discrete quantized values of the integrals of motion, which can be parameterized by the set of integers

$$q_k = q_k(\nu_h; n_h, \{\ell\}), \quad E_N = E_N(\nu_h; n_h, \{\ell\}), \quad \{\ell\} = (\ell_1, \ell_2, \ldots, \ell_{2(N-2)}) \quad (5.4)$$

with $k = 3, \ldots, N$. The explicit form of this dependence and interpretation of integers $\{\ell\}$ will be given below. Eqs. (5.1) and (5.4) imply that, as a function of the total spin $h$, the quantized values of $q_k$ form the family of one-dimensional continuous nonintersecting trajectories in the $(N-1)$-dimensional space of $q = (q_2, q_3, \ldots, q_N)$. The “proper time” along each trajectory, $\nu_h$, is defined by the imaginary part of the total spin, $\text{Im} \ h = \nu_h$, whereas the integers $n_h, \ell_1, \ldots, \ell_{2(N-2)}$ specify different members of the family. Each trajectory in the $q-$space induces the corresponding trajectory for the energy $E_N = E_N(q, \bar{q})$. We recall that, in contrast with $q$, the energy takes only real values.

The ground state energy of the system

$$E_N^\text{ground} = \min E_N(\nu_h; n_h, \{\ell\}) = E_N(0; 0, \{\ell_\text{ground}\}) \quad (5.5)$$

occurs at $\nu_h = n_h = 0$, or equivalently $h = 1/2$. It belongs to the special trajectory $\{\ell_\text{ground}\}$, to which we shall refer as to the ground state trajectory. The energy along this trajectory is a continuous function of $\nu_h$ and it approaches its minimal value, $E_N^\text{ground}$, at $\nu_h = 0$. At the vicinity of $\nu_h = 0$, one finds an accumulation of the energy levels

$$E_N(\nu_h; 0, \{\ell_\text{ground}\}) - E_N^\text{ground} = \sigma_N \nu_h^2 + \mathcal{O}(\nu_h^4), \quad (5.6)$$

with $\sigma_N$ being the diffusion coefficients.

The spectrum of quantized $q_2, \ldots, q_N$ possesses the following symmetry

$$q_k \rightarrow (-1)^k q_k \rightarrow q_k^* \quad (5.7)$$

with $k = 2, \ldots, N$. Here, the first relation follows from invariance of the Hamiltonian under mirror permutations of particles, Eq. (2.14). The second relation is a consequence of the symmetry of the model at $s = 0$ and $s = 1$ under interchange of the $z-$ and $\bar{z}-$sectors, or equivalently $q_k \leftrightarrow q_k^*$ and $s(s-1) \leftrightarrow \bar{s}(\bar{s}-1)$. The relation (5.7) implies that if the quantization conditions (3.28) are satisfied at some point $\{q_k\}$ on the moduli space, then three other points $\{(-1)^k q_k\}, \{q_k^*\}$ and $\{(-1)^k q_k^*\}$ also belong to the spectrum of the model.

As we will see in the next Section, the spectrum of quantized charges $q$ has a hidden structure, which can be revealed by examining the distribution of the quantized values of the “highest” charge $q_N^{1/N}$. Since $q_N^{1/N}$ is a multi-valued function of complex $q_N$, each eigenstate of the model, specified by a definite value of $q_N$, will be represented on the complex $q_N^{1/N}$-plane by $N$ different points. Together with the symmetry property of the spectrum, Eq. (5.7), this leads to the following transformation on the moduli space

$$q_N^{1/N} \rightarrow \exp(\pi ik/N) q_N^{1/N}, \quad [0 < k < 2N] \quad (5.8)$$

where $k$ is integer for odd $N$ and even for even $N$. It maps one of the eigenstates into itself or into another one with the same energy.

In the rest of this Section we shall present the results of our calculations of the spectrum of the model for the different number of particles $3 \leq N \leq 8$. 

26
5.2. Quantum numbers of the $N = 3$ states

At $N = 3$ the eigenstates depend on the quantum number $q_3$, which is an eigenvalue of the operator $q_3$ defined in (1.6). Some of the eigenvalues of this operator have been already found in [14, 15, 16] using different methods. The eigenstate found in [14] have pure imaginary values of quantized $q_3$ and their quasimomentum is equal to $\theta_3 = 0$. We will demonstrate in this Section, that contrary to the statements made in [16] the spectrum of the operator $q_3$ is not exhausted by pure imaginary values, even in the sector with $\theta_3 = 0$.

Solving the quantization conditions (3.28) at $N = 3$, we reconstructed the full spectrum of quantized $q_3$. We found that apart from pure imaginary $q_3$ calculated in [14, 15], the spectrum also contains (an infinite number of) complex values of $q_3$, including pure real ones. Enumerating the quantized $q_3$ according to their absolute value starting from the smallest one, we notice that $|q_3|$ grows cubically with its number. This suggests to describe the spectrum in terms of $q_3^{1/3}$ rather than $q_3$. To illustrate this, we present in Figure 1 the results of our calculations of quantized $q_3^{1/3}$ for the total $SL(2, \mathbb{C})$ spin of the system $h = 1/2$, or equivalently $n_h = \nu_h = 0$. Similar picture holds for arbitrary $h$.

![Figure 1: The spectrum of quantized $q_3^{1/3}$ for the system of $N = 3$ particles. The total $SL(2, \mathbb{C})$ spin of the system is equal to $h = 1/2$.](image)

The spectrum of quantized $q_3^{1/3}$, shown in Figure 1, is in agreement with the symmetry properties (5.7) and (5.8). Defining the fundamental domain as $0 \leq \arg(q_3^{1/3}) < \pi/3$, we find that the whole spectrum of quantized $q_3^{1/3}$ can be obtained from the points belonging to this domain through the transformations (5.7) and (5.8).

It is difficult do not notice a remarkable regularity in the distribution of quantized charges in Figure 1. Apart from a few points close to the origin, the quantized values of $q_3^{1/3}$ appear to be located at the vertices of the lattice built from equilateral triangles. As a consequence, they can
be parameterized as follows

$$\left[q_{3}^\text{WKB}(\ell_1, \ell_2)\right]^{1/3} = \Delta_{N=3} \cdot \left(\frac{1}{2} \ell_1 + i \frac{\sqrt{3}}{2} \ell_2\right),$$

(5.9)

where $\ell_1$ and $\ell_2$ are integers, such that their sum $\ell_1 + \ell_1$ is even. Here, $\Delta_3$ denotes the lattice spacing. Its value can be calculated from the leading-order WKB solution of the Baxter equation [31]

$$\Delta_3 = \left[\frac{3}{4^{1/3} \pi} \int_{-\infty}^{1} \frac{dx}{\sqrt{1-x^3}}\right]^{-1} = \frac{\Gamma^3(2/3)}{2\pi} = .395175...$$

(5.10)

Quantized $q_3^{1/3}$ occupy the whole complex plane except the interior of the disk of the radius $\Delta_3$

$$|q_{3}^{1/3}| > \Delta_3.$$  

(5.11)

The comparison of (5.9) with the exact expressions for $q_3$ at $h = 1/2$ is shown in Figure 2 and Table 1. We find that the expression (5.9) describes the excited eigenstates with high accuracy. The agreement becomes less impressive for the eigenstates with smaller $q_3$. For the ground state with $iq_3 = 0.20526...$ the accuracy of (5.9) is $\sim 20\%$. Obviously, Eq. (5.9) can be systematically improved by including subleading WKB corrections. Notice that the same WKB formula (5.9) is valid not only at $h = 1/2$ but also for arbitrary spin $h$. In the latter case, it describes correctly the excited states with $|q_3^{1/3}| \gg |q_2^{1/2}|$.

According to (5.9), the quantized values of $q_3$ are parameterized by a pair of integers $\ell_1$ and $\ell_2$ which define the coordinates on the lattice shown in Figures 1 and 2. Eq. (5.9) provides the WKB approximation to the function $q_3(\nu_n; n_h, \ell_1, \ell_2)$ introduced in (5.4). Calculating the
Table 1: Comparison of the exact spectrum of $q_3^{1/3}$ at $\hbar = 1/2$ with the approximate WKB expression (5.9). The last line defines the corresponding energy $E_3(0; \ell_1, \ell_2)$.

<table>
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<td>$0.590i$</td>
<td>$0.358 + 0.621i$</td>
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<tr>
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<td>$-E_3/4$</td>
<td>$-0.2472$</td>
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The quasimomentum of the eigenstate specified by the pair of integers, $\ell_1$ and $\ell_2$, one finds that it is given by the following simple expression

$$\theta_3(\ell_1, \ell_2) = \frac{2\pi}{3} \ell_1 \pmod{2\pi}.$$  (5.12)

In particular, the quasimomentum vanishes for the eigenstates with $\ell_1 = 0 \pmod{3}$. It is easy to see from (5.9) that the corresponding $q_3$ take, in general, complex values. There are, however, special cases, like $\ell_1 = 0$ or $\ell_2 = 0$, when $q_3$ becomes, respectively, pure imaginary or real. The former values have been previously found in [14, 15].

We recall that the spectrum of $q_3$, shown in Figure 1, corresponds to the total spin $\hbar = 1/2$, or equivalently $n_\hbar = \nu_\hbar = 0$ in Eq. (5.4). In general, quantized $q_3$ depend on integer $n_\hbar$ and continuous $\nu_\hbar$. For the sake of simplicity, we present our results only at $n_\hbar = 0$, or equivalently $h = 1/2 + i\nu_\hbar$. We find that quantized $q_3 = q_3(\nu_\hbar, 0, \ell_1, \ell_2)$ are continuous functions of $\nu_\hbar$. For different integers $\ell_1$ and $\ell_2$, the functions $q_3(\nu_\hbar, 0, \ell_1, \ell_2)$ define an infinite set of trajectories in the three-dimensional $(\nu_\hbar, \Re(q_3^{1/3}), \Im(q_3^{1/3}))$ moduli space. The trajectories cross the hyperplane $\nu_\hbar = 0$ at the points shown in Figure 1 and go to infinity for $\nu_\hbar \to \pm\infty$. To illustrate the properties of these trajectories, three representatives, corresponding to $(\ell_1, \ell_2) = (0, 2), (2, 2)$ and $(4, 2)$, are shown in Figure 3. The quasimomentum $\theta_3$ takes constant values along each trajectory, $\theta_3(\ell_1, \ell_2) = 0, 4\pi/3$ and $2\pi/3$, respectively.

Let us now consider the energy spectrum at $N = 3$. Since the energy is a function of the total spin $\hbar$ and the charge $q_3$, each trajectory $q_3 = q_3(\nu_\hbar, \ell_1, \ell_2)$ shown in Figure 3 induces the corresponding trajectory for the energy, $E_3 = E_3(\nu_\hbar, \ell_1, \ell_2)$. Using solutions to the quantization conditions (3.28) and applying (3.14) we obtain the results shown in Figure 4.

We find that the energy is a continuous function of $\nu_\hbar$ along each $(\ell_1, \ell_2)$-trajectory and it approaches its minimal value, $\min_{\nu_\hbar} E_3(\nu_\hbar, \ell_1, \ell_2)$, at $\nu_\hbar = 0$, or equivalently $h = 1/2$. Examining the value of $E_3(0; \ell_1, \ell_2)$ for different sets of integers $(\ell_1, \ell_2)$, we find that $E_3(0; \ell_1, \ell_2)$ increases as one goes towards larger $|q_3^{1/3}(0; \ell_1, \ell_2)|$ (see Table 1). This implies that the ground state corresponds to the point(s) on the plane of quantized $q_3^{1/3}$ (see Figure 1) closest to the origin. It is easy to see from Figure 1 that, in total, there are six such points, $(\ell_1, \ell_2) = (0, \pm 2), (\pm 3, \pm 1)$ and $(\mp 3, \pm 1)$. According to (5.12), their quasimomentum is equal to zero. Going over from $q_3^{1/3}$ to $q_3$, we find that these six points define only two nontrivial eigenstates with opposite values of $q_3$ and the same energy

$$iq_3^{\text{ground}} = \pm 0.20526\ldots, \quad E_3^{\text{ground}} = 0.98868\ldots$$  (5.13)
This implies that the ground state is \textit{double degenerate} at \( N = 3 \). As we will demonstrate below, this property is rather general – the ground state is double degenerate for the systems with \textit{odd} number of particles, \( N \), and it is unique for even \( N \). Eq. (5.13) is in agreement with the results of the previous calculations [14, 15].

At \( N = 3 \) the ground state is located on the \((0, 2)\) – trajectory at \( \nu_h = 0 \) as shown in Figure 4. We notice that close to \( \nu_h = 0 \) there is an accumulation of the energy levels. At small \( \nu_h \) the energy of excited states is described by a general expression (5.6) with the dispersion parameter \( \sigma_3 \) given below in the Table 3.

5.3. Quantum numbers of the \( N = 4 \) states

At \( N = 4 \) the spectrum of the model depends on two complex quantum numbers \( q_3 \) and \( q_4 \). Similar to the previous case, we shall determine their spectrum by solving the quantization conditions (3.28). The only difference with the \( N = 3 \) case is that one has to increase the dimension of the mixing matrices, \( C^{(0)} \) and \( C^{(1)} \), and recalculate the transition matrices, \( \Omega \) and \( \bar{\Omega} \), as functions of \( q_3 \) and \( q_4 \). As before, to understand the structure of the spectrum at \( h = 1/2 \), it becomes convenient to deal with multi-valued complex variables \( q_4^{1/4} \). Then, each value of \( q_4 \) will be represented on the complex \( q_4^{1/4} \) – plane by four different points.

At \( h = 1/2 \) the quantized values of \((q_3, q_4)\) are parameterized by four real numbers and, as a consequence, they do not admit a simple pictorial representation. The results of our calculations are presented in Figure 5. There, each point on the complex \( q_4^{1/4} \) – plane has two additional coordinates corresponding to the quantized values of the charge \( q_3 \). The latter are not displayed for simplicity. It is convenient to separate the eigenstates into two sets according to their quasi-momentum \( \theta_4 = (2\pi k)/4, \ k = 0, 2 \) and \( k = 1, 3 \) as shown in Figure 5. We notice that some points have very close values of \( q_4 \). Nevertheless, the charge \( q_3 \) and the energy \( E_4 \) corresponding to these points are different. Aside from this spurious degeneracy, the spectrum of quantized \( q_4 \) has many features in common with the spectrum of \( q_3 \) at \( N = 3 \) shown in Figure 1.

We notice that quantized \( q_4 \) form the vertices of a square-like lattice. To verify this property
we selected among all eigenstates of the $N = 4$ system only those with $q_3 = 0$ and nonzero values of $q_4$ (see Figure 6). These states have the quasimomentum $\theta_4 = 0$ and they play an important role in our discussion as the ground state of the system has the same quantum numbers. The WKB analysis of the Baxter equation at $N = 4$ leads to the following expression for quantized $q_4$ at $q_3 = 0$ (see Eq. (5.19) below)

$$[q_4^{WKB}(\ell_1, \ell_2)]^{1/4} = \Delta_{N=4} \cdot \left(\frac{\ell_1}{\sqrt{2}} + i \frac{\ell_2}{\sqrt{2}}\right),$$

(5.14)

where the integers $\ell_1$ and $\ell_2$ are such that their sum $\ell_1 + \ell_2$ is even and the lattice spacing is

$$\Delta_4 = \left[\frac{4^{3/4}}{\pi} \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^4}}\right]^{-1} = \frac{\Gamma^2(3/4)}{2\sqrt{\pi}} = 0.423606...$$

(5.15)

As before, the leading-order WKB formula (5.14) is valid only for $|q_4^{1/4}| \gg |q_2^{1/2}|$. The comparison of (5.14) with the exact results for $q_4$ at $h = 1/2$ is shown in Figure 6 and Table 2. We find that quantized $q_4^{1/4}$ occupy the whole complex plane except the interior of the disk of radius $\Delta_{N=4}$

$$|q_4^{1/4}| > \Delta_{N=4},$$

(5.16)

and the WKB formula (5.14) describes their spectrum with a good accuracy.

We recall that the points shown in Figure 6 describe the eigenstates at $N = 4$ with $q_3 = 0$ and the quasimomentum $\theta_3 = 0$. Similar to the $N = 3$ case, the spectrum of $q_4$ is parameterized by the pair of integers $(\ell_1, \ell_2)$, Eq. (5.14). Still, as one can see from Figure 5, there exist the eigenstates that have the same quasimomentum, close value of $q_4$ but $q_3 \neq 0$. Similar phenomenon also occurs for other values of the quasimomentum. In order to describe these additional eigenstates, one has to introduce the second pair of integers $(\ell_3, \ell_4)$. The dependence of $q_4$ on $(\ell_3, \ell_4)$ leads to the “fine splitting” of quantized $q_4$ in Figure 5. It also allows us to describe a nontrivial spectrum
Figure 5: The spectrum of the integrals of motion $q_4$ at $N = 4$ and the total spin $h = 1/2$. The left and the right panels correspond to the eigenstates with the different quasimomentum $e^{i\theta_4} = \pm 1$ and $\pm i$, respectively.

<table>
<thead>
<tr>
<th>$(\ell_1, \ell_2)$</th>
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<th>(2,2)</th>
<th>(3,1)</th>
<th>(4,0)</th>
<th>(3,3)</th>
<th>(4,2)</th>
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<tbody>
<tr>
<td>$q_4^{\text{exact}}$</td>
<td>.626</td>
<td>.520 + .520 $i$</td>
<td>.847 + .268 $i$</td>
<td>1.158</td>
<td>.860 + .860 $i$</td>
<td>1.159 + .574 $i$</td>
</tr>
<tr>
<td>$q_4^{\text{WKB}}$</td>
<td>.599</td>
<td>.599 + .599 $i$</td>
<td>.899 + .299 $i$</td>
<td>1.198</td>
<td>.899 + .899 $i$</td>
<td>1.198 + .599 $i$</td>
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<tr>
<td>$-E_4/4$</td>
<td>0.6742</td>
<td>−1.3783</td>
<td>−1.7919</td>
<td>−2.8356</td>
<td>−3.1410</td>
<td>−3.3487</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the exact spectrum of $q_4^{1/4}$ at $q_3 = 0$ and $h = 1/2$ with the approximate WKB expression (5.14). The last line defines the exact energy $E_4$.

of $q_3$ at $N = 4$. As we will argue below, the complete description of the spectrum of $q_3$ and $q_4$ can be obtained within the WKB approach.

Let us now examine the energy spectrum at $N = 4$. Calculating the energy of the eigenstates with the quantum numbers shown in Figure 5, we find that $E_4 > 0$ for all points on the $q_4^{1/4}$-plane except of four points with the coordinates $(\ell_1, \ell_2) = (\pm 2, 0)$ and $(0, \pm 2)$. Due to a residual symmetry $q_4^{1/4} \rightarrow \exp(ik\pi/2)q_4^{1/4}$, Eq. (5.8), they describe a single eigenstate, which can be identified as the ground state of the $N = 4$ system

$$q_3^{\text{ground}} = 0, \quad q_4^{\text{ground}} = 0.153589..., \quad E_4^{\text{ground}} = -2.696640...$$

with $h = 1/2$. It has the quasimomentum $\theta_4 = 0$ and, in contrast with the $N = 3$ case, it is unique.

Going from $\nu_h = 0$ to nonzero values of $\nu_h$, one finds that, similar to the $N = 3$ case (see Figure 3), the eigenvalues of the integrals of motion flow along the trajectories on the moduli space $(\nu_h, q_3, q_4)$. Namely, each point shown in Figure 5 creates its own trajectory parameterized by the set of integers $\{\ell\}$. Among all trajectories the one with $(\ell_1, \ell_2) = (2, 0)$ plays the special role as it contains the ground state of the model. We find that $q_3 = \Im q_4 = 0$ for arbitrary $\nu_h$ on the ground state trajectory, whereas $\Re(q_4)$ and $E_4$ vary with $\nu_h$ as shown in Figure 7. Accumulation of the energy levels next to the ground state at $\nu_h = 0$ is described by Eq. (5.6) with the dispersion parameter $\sigma_4$ given below in the Table 3.
As we have seen before, the spectrum of quantized charges at $N = 3$ and $N = 4$ is described by the simple formulae, Eqs. (5.9) and (5.14). Their derivation is based on the WKB approach to the eigenproblem for the Baxter operator [31]. In this approach, $Q_{q,q}(x, \bar{x})$ is constructed as a quasiclassical wave function in the separated coordinates $\vec{x} = (x, \bar{x})$. The underlying classical dynamics is described by the spectral curve (“equal energy level” equation)

$$\Gamma_N : \quad y^2 = t_N^2(x) - 4x^{2N} = (4x^N + q_2x^{N-2} + ... + q_N)(q_2x^{N-2} + ... + q_N),$$

(5.18)

where $t_N(x)$ was defined in (2.8) and $y(x) = 2x^N \sinh p_x$ is related to the holomorphic part of the momentum of a particle in the separated coordinates, $p_x$. For arbitrary complex $x$, the equation (5.18) has two solutions for $y(x)$. As a consequence, $y(x)$ becomes a single-valued function on the Riemann surface corresponding to the complex curve $\Gamma_N$. This surface is constructed by gluing together two copies of the complex $x$–plane along the cuts $[\sigma_1, \sigma_2], ..., [\sigma_{2N-3}, \sigma_{2N-2}]$ running between the branching points $\sigma_j$ of the curve (5.18). The latter are defined as simple roots of the equation $t_N^2(\sigma_j) = 4\sigma_j^{2N}$. Their positions on the complex plane depend on the values of the integrals of motion $q_2, q_3, ..., q_N$. In general, the Riemann surface defined in this way has a genus $g = N - 2$ which depends on the number of reggeons, $N$. It is a sphere at $N = 2$, a torus at $N = 3$ and so on.

Let us define on $\Gamma_N$ the set of oriented closed $\alpha$– and $\beta$–cycles. The cycles $\alpha_k$ encircle the cuts $[\sigma_{2k-1}, \sigma_{2k}]$ with $k = 1, ..., N - 2$ and belong to the both sheets of $\Gamma_N$. The cycles $\beta_k$ run from the cut $[\sigma_{2N-3}, \sigma_{2N-2}]$ to $[\sigma_{2k-1}, \sigma_{2k}]$ on the upper sheet and, then, back on the lower sheet. Then, classical trajectories of a particle in the separated coordinates correspond to wrapping around $\alpha$– and $\beta$–cycles on the Riemann surface (5.18). Requiring $Q_{q,q}^{WKB}(x, \bar{x} = x^*)$ to be single-valued on the complex $x$–plane, one obtains the following WKB quantization conditions\(^{12}\)

$$\text{Re} \oint_{\alpha_k} dS_0 = \pi \ell_{2k-1}, \quad \text{Re} \oint_{\beta_k} dS_0 = \pi \ell_{2k}, \quad (k = 1, ..., N - 2)$$

(5.19)

\(^{12}\)We are most grateful to A. Gorsky for collaboration on this point.
Figure 7: The dependence of the energy, $-E_{4/4}$, and the quantum number, $q_4/q_2$, with $q_2 = 1/4 + \nu_h^2$, on the total spin $h = 1/2 + i\nu_h$ along the ground state trajectory at $N = 4$.

where $\ell_k$ are integer and $dS_0$ is the “action” differential on the curve (5.18) [31]

$$dS_0 = dx p_x \approx \frac{N t_N(x) - xt'_N(x)}{\sqrt{t''_N(x) - 4x^2}} dx. \tag{5.20}$$

Solving (5.19), one can find the explicit expressions for the integrals of motion, Eq. (5.4). At $N = 3$ and $N = 4$, for $|q_{1/N}^1| \gg q_2^{1/2}$, one arrives at (5.9) and (5.14). The general analysis of (5.19) turns out to be rather involved and it will be presented in the forthcoming publication.

5.4. Quantum numbers of the states with higher $N$

In this Section we will describe the results of our calculations of the spectrum of the model for higher $N$. As we have seen in the previous Sections, the structure of the spectrum gets more complicated as one increases $N$. Therefore, instead of presenting a detailed description of the whole spectrum, as was done before for $N = 3$ and $N = 4$, we will restrict our analysis to the properties of the ground state trajectory only.

Solving the quantization conditions (3.28) for $N \geq 5$, we found that the ground state trajectories have different properties for even and odd number of particles. Namely, for even $N$ the integrals of motion $q_k$ with odd indices $k$ vanish and for even $k$ they take pure real values

$$q_3 = q_5 = ... = 0, \quad \text{Im } q_4 = \text{Im } q_6 = ... = 0, \quad [N = \text{even}] \tag{5.21}$$

For odd $N$, the integrals of motion $q_k$ take nonvanishing values. They are pure imaginary for odd $k$ and real otherwise

$$\text{Re } q_3 = \text{Re } q_5 = ... = 0, \quad \text{Im } q_4 = \text{Im } q_6 = ... = 0, \quad [N = \text{odd}] \tag{5.22}$$

We recall that the total spin of the system on the ground state trajectory is equal to $h = 1/2 + i\nu_h$, so that the quantized charges $q_k$ and the energy $E_N$ are functions of $\nu_h$. At $N = 3$ and $N = 4$ the $\nu_h$–dependence of $q_N$ and $E_N$ is shown in Figures 3, 4 and 7, respectively.
Table 3: The exact quantum numbers, \( q \), and the energy, \( E_N \), of the ground state of \( N \) reggeized gluons in multi-colour QCD. The dispersion parameter, \( \sigma_N \), defines the energy of the lowest excited states, Eq. (5.6).

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</tbody>
</table>

It follows from Eqs. (5.21) and (5.22) that for even \( N \) the ground state trajectory is invariant under the symmetry transformations (5.7), whereas for odd \( N \) it is mapped into another trajectory that has different quantum numbers \((-1)^k q_k\) and the same energy \( E_N \). This implies, that the ground state of the system is double degenerate for odd number of particles \( N \) and it is unique for even \( N \). The degeneracy is related to the properties of the system under mirror permutations of particles [6, 5].

The results of our calculations of the ground state for the system with the number of particles \( N \leq 8 \) are summarized in the Table 3 and Figure 8. We recall that for odd \( N \) there exists the second eigenstate with the same energy \( E_N \) and the charges \((-1)^k q_k\).

The energy of the ground state \( E_N \), as a function of the number of particles \( N \), has a number of interesting properties. \( E_N \) changes a sign as one increases the number of particles – it is negative for even \( N \) and positive for odd \( N \). Our results also indicate that the absolute value of the energy decreases with \( N \), \( |E_N| \sim 1/N \) for \( N \to \infty \) (see Figure 8). For large even and odd number of particles \( N \) it approaches the same asymptotic value \( E_{2\infty} = E_{2\infty+1} = 0 \).

These properties are unique (and quite unexpected) features of noncompact \( SL(2, \mathbb{C}) \) Heisenberg spin magnets. It is instructive to compare them with similar properties of the \( SL(2, \mathbb{R}) \) Heisenberg magnets studied earlier in [32, 33, 34, 35] in the relation with the Evolution Equations for high-twist operators in high-energy QCD. There, the number of sites of the spin magnet is equal to the twist of the operators and the ground state energy defines the minimal anomalous dimension of these operators. Roughly speaking, the Hamiltonian of the \( SL(2, \mathbb{R}) \) magnet is given by the holomorphic part of the reggeon Hamiltonian (1.5) with the only difference that the holomorphic \( z \)-coordinates take real values and the eigenstates are polynomials in \( z \). One finds [33, 32, 34, 35], that for the \( SL(2, \mathbb{R}) \) spin magnet all charges \( q_3, ..., q_N \) take only real quantized values, the ground state energy \( E_N \) is positive and it monotonically increases with \( N \). Thus, the properties of the \( SL(2, \mathbb{C}) \) and \( SL(2, \mathbb{R}) \) Heisenberg spin magnets turn out to be completely different.

The spectrum of the \( SL(2, \mathbb{C}) \) magnet is gapless for arbitrary \( N \). Accumulation of the energy levels next to the ground state is described by (5.6) with the dispersion parameter \( \sigma_N \) given in the Table 3. The states belonging to the ground state trajectory have the total \( SL(2, \mathbb{C}) \) spin \( h = 1/2 + i \nu_h \) and their quantum numbers \( q_3, ..., q_N \) satisfy (5.21) and (5.22) for even and odd \( N \), respectively. The explicit expressions for \( q_k \) can be obtained in the WKB approach from Eq. (5.19).
Figure 8: The dependence of the ground state energy, $-E_N/4$, on the number of particles $N$. The exact values of the energy are denoted by crosses. The upper and the lower dashed curves stand for the functions $1.8402/(N - 1.3143)$ and $-2.0594/(N - 1.0877)$, respectively.

Another unusual feature of the $SL(2, \mathbb{C})$ magnet can be revealed by examining the dependence $E_N = E_N(\nu_h)$ along the ground state trajectory for different $N$. For $2 \leq N \leq 8$ this dependence is shown in Figure 8. At $\nu_h = 0$ the values of $E_N(0)$ coincide with those depicted by crosses in Figure 8. We notice that the flow of the energy $E_N$ with $\nu_h$ is such that the hierarchy of the energy levels at $\nu_h = 0$ and large $\nu_h$ is completely different. The reason for this is that for the total $SL(2, \mathbb{C})$ spin $h = 1/2 + i\nu_h$ with $\nu_h$ large the system approaches a quasiclassical regime [36, 31], in which the energy $E_N(\nu_h)$ and the quantum numbers $q_N$ have a universal scaling behaviour

$$E_N(\nu_h) \sim 4 \ln |q_N|, \quad |q_N| \sim C^N \nu_h^2,$$

with $C \approx 0.52$. As can be seen from the right panels in Figures 7 and 9, this regime starts already at $\nu_h \approx 2$.

6. Summary

In this paper, we have continued the study of noncompact Heisenberg $SL(2, \mathbb{C})$ spin magnets initiated in [5]. Having solved this model, we obtained for the first time a complete description of the spectrum of the multi-reggeon compound states in QCD at large $N_c$.

From point of view of integrable models, the results presented in this paper provide an exact solution of the spectral problem for completely integrable quantum mechanical model of $N$ interacting spinning particles in two-dimensional space. A unique feature of this model, leading to many unusual properties of its spectrum, is that the quantum space is infinite-dimensional for finite $N$ and conventional methods, like the Algebraic Bethe Ansatz, are not applicable. To overcome this problem, we applied the method of the Baxter $Q-$operator developed in application to the $SL(2, \mathbb{C})$ spin magnets in [5]. Solving the Baxter equations, we were able to find the exact expressions for the eigenvalues of the $Q-$operator. They allowed us to establish the quantization conditions for the integrals of motion and, finally, reconstruct the spectrum of the model.
Figure 9: The dependence of the energy $-E_N(\nu h)/4$ and the “highest” integral of motion $|q_N|/q_2$ with $q_2 = (1/4 + \nu h^2)$ on the total spin $h = 1/2 + i\nu h$ along the ground state trajectory for different number of particles $2 \leq N \leq 8$. At large $\nu h$, $-E_8 > ... > -E_3 > -E_2$ on the left panel and $|q_8/q_2| < ... < |q_3/q_2|$ on the right panel.

From point of view of high-energy QCD, we calculated the spectrum of the colour-singlet compound states built from $N$ reggeized gluons for $N \leq 8$ in the multi-colour limit, $N_c \rightarrow \infty$. The obtained expressions allowed us to reveal the general properties of the spectrum for arbitrary $N$. Our analysis was based on the identification of the $N$–reggeized gluon states in multi-colour QCD as the ground states for the noncompact Heisenberg magnet of the length $N$ and the $SL(2, \mathbb{C})$ spins $(s = 0, \bar{s} = 1)$.

The identification is not straightforward, however, since one has to verify that the obtained multi-reggeon states have correct Bose properties, that is they have to be symmetric under interchange of any pair of reggeon coordinates and their colour indices. In the multi-colour limit, the Bose symmetry is reduced to the invariance of the wave function under the cyclic and mirror permutations. Since the wave function of the $N$–reggeon state is factorized, as $N_c \rightarrow \infty$, into the product of the colour tensor and the scalar function $\Psi(\vec{z}_1, ..., \vec{z}_N)$, the both factors have to possess the same parity under the cyclic and mirror permutations. Since the operators of the corresponding transformations, $P$ and $M$, do not commute with each other, this leads to the following selection rules on the eigenstates of the noncompact Heisenberg magnet.

By the construction, the eigenstates $\Psi_{\vec{p}(q, \bar{q})}(\vec{z}_1, ..., \vec{z}_N)$ diagonalize the operator of cyclic permutations, Eq. (2.5). Then, making use of (2.14), the eigenstates of the operator of mirror permutations, $M \Psi_{\vec{p}(q, \bar{q})}^{(\pm)} = \pm \Psi_{\vec{p}(q, \bar{q})}^{(\mp)}$, can be defined as

$$\Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) = \frac{1 \pm M}{2} \Psi_{\vec{p}(q, \bar{q})}(\vec{z}) = \frac{1}{2} \left[ \Psi_{\vec{p}(q, \bar{q})}(\vec{z}) \pm \Psi_{\vec{p}(-q, -\bar{q})}(\vec{z}) \right].$$

(6.1)

where $q = \{q_k\}$ and $-q = \{(-1)^k q_k\}$. Although these states do not diagonalize the integrals of motion, $\{q, \bar{q}\}$, they are the eigenstates of the Hamiltonian with the same energy $E_N = E_N(q, \bar{q}) = E_N(-q, -\bar{q})$. Using $\theta_N(-q, -\bar{q}) = -\theta_N(q, \bar{q})$, we find from (2.5) that the operator of cyclic permutations acts on them as

$$P \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) = \cos(\theta_N(q, \bar{q})) \cdot \Psi_{\vec{p}(q, \bar{q})}^{(\pm)}(\vec{z}) + i \sin(\theta_N(q, \bar{q})) \cdot \Psi_{\vec{p}(\bar{q}, q)}^{(\mp)}(\vec{z}).$$

(6.2)
The eigenstates $\Psi_{\tilde{p}(q, \bar{q})}^{(\pm)}(\vec{z})$ diagonalize the operators $P$ and $M$ simultaneously only for $\sin(\theta_N(q, \bar{q})) = 0$. Together with (2.5), this condition select among all eigenstates of the magnet only those with the quasimomentum $e^{i\theta_N(q, \bar{q})} = \pm 1$ for even $N$ and $e^{i\theta_N(q, \bar{q})} = 1$ for odd $N$. Obviously, it is satisfied for the ground state that has $\theta_N(q, \bar{q}) = 0$.

We conclude that the ground state of the magnet has a definite parity with respect to the cyclic and mirror permutations simultaneously and, therefore, its wave function $\Psi_{\tilde{p}(q, \bar{q})}^{(\pm)}(\vec{z})$ can be identified as the $\vec{z}$–dependent part of the full wave function of the compound states of $N$ reggeized gluons as $N_c \to \infty$. The ground state of the magnet has different properties for even and odd number of particles. For even $N$, its quantum numbers satisfy $q_{2k+1} = 0$, Eq (5.21), and, as a consequence, two sets of the quantum numbers, $(q, \bar{q})$ and $(-q, -\bar{q})$ coincide leading to $\Psi_{\tilde{p}(q, \bar{q})}^{(+)}(\vec{z}) = \Psi_{\tilde{p}(q, \bar{q})}^{(-)}(\vec{z})$ and $\Psi_{\tilde{p}(q, \bar{q})}^{(-)}(\vec{z}) = 0$. For odd $N$, the ground state is double degenerate. The degeneracy occurs due to the symmetry of the energy $E_N$ under $q_{2k+1} \to -q_{2k+1}$, Eq. (2.18). This allows us to construct two mutually orthogonal ground states, $\Psi_{\tilde{p}(q, \bar{q})}^{(\pm)}(\vec{z})$, which are invariant under the cyclic permutations, $\theta_N = 0$, and possess a definite parity under the mirror permutations, $M \Psi_{\tilde{p}(q, \bar{q})}^{(\pm)}(\vec{z}) = \pm \Psi_{\tilde{p}(q, \bar{q})}^{(\pm)}(\vec{z})$. In virtue of the Bose symmetry, the colour part of the wave function of the $N$ reggeized gluon state should have the same parity under the charge conjugation, $t^a \to -(t^a)^T$, with $t^a$ being the $SU(N_c)$ generators in the quark representation. This allows us to distinguish the ground states according to their $C$–parity. For even $N$ the ground states with the parity $M = 1$ have the same $C$–parity as the Pomeron, $C = 1$. For odd $N$, the ground states with the parity $M = 1$ and $M = -1$ have the $C$–parity of the Odderon [13], $C = -1$, and the Pomeron, $C = 1$, respectively.

Our results indicate that, in the multi-colour limit, in the Pomeron sector, only compound states built from even number of reggeized gluons $N$ provide the contribution to (1.1) rising with the energy $s$. Their intercept $\alpha_N \equiv 1 - \bar{\alpha}_s E_N/4$ is bigger than one, but it decreases at large $N$ as $\alpha_N - 1 \sim 1/N$. In the Odderon sector, the situation is different. The intercept of the compound states is smaller than one, and it increases with $N$ as $1 - \alpha_N \sim 1/N$. As a consequence, the contribution to the scattering amplitude (1.1) from the $N = 3$ state (“bare Odderon”) is subdominant at high-energy with respect to the contribution of the $N = 5$ state and so on. This means that the high-energy asymptotics of the scattering amplitude (1.1) in the Odderon sector is governed, as $N_c \to \infty$, by the contribution of the states with an arbitrary large odd number of reggeized gluons, which increase the effective value of the Odderon intercept and lead to $\alpha_{\text{Odderon}} = \alpha_{2N+1} = 1$. Finally, in the both sectors, the intercept of the $N$–reggeon states approaches the same value $\alpha_\infty = 1$ as $N \to \infty$ and their contribution to the scattering amplitude, $A(s, t)/s$, ceases to depend on the energy $s$ as $N \to \infty$. It is interesting to notice that this result has been anticipated a long time ago within the bootstrap approach [9]. It is also in agreement with the upper bound on the energy of the compound reggeized gluon states established in [12].

We would like to stress, however, that these results were obtained in the multi-colour limit and the important question remains: may the nonplanar corrections change the $N$–dependence of the energy $E_N$ of the compound reggeized gluon states? One expects that the nonplanar $1/N_c^2$ corrections to the reggeon Hamiltonian will break the integrability of the Schrödinger equation (1.2) and calculations will be more involved. This problem deserves additional studies.

Analyzing the high-energy asymptotics of the scattering amplitudes, one is trying to identify the effective theory, which describes the QCD dynamics in the Regge limit. The main objects of this effective theory are the $N = 2, 3, \ldots$ reggeon compound states constructed in this paper. In the generalized leading logarithmic approximation, these states propagate between the scattered
hadrons and do not interact with each other. In the topological $1/N_c^2$-expansion [9], these states emerge from the summation of Feynman diagrams having the form of a cylinder, whose walls are built from the reggeized gluons. These diagrams can be interpreted as describing the propagation a closed string between two scattered hadrons. This suggests that there should exist the stringy representation for the effective dynamics of the multi-reggeon compound states in multi-colour QCD [37].

**Acknowledgements**

We would like to thank A. Gorsky, I. Kogan, N. Nekrasov, F. Smirnov, A. Turbiner, G. Veneziano and J. Wosiek for illuminating discussions. This work was supported in part by the Blaise Pascal Research Chair awarded to G. Veneziano, by the grant KBN-PB-2-P03B-19-17 (J.K.), by the grant of Spanish Ministry of Science (A.M.) and by the grant 00-01-005-00 of the Russian Foundation for Fundamental Research (S.D. and A.M.).

**A Appendix: Solution to the Baxter equation at $N = 2$**

In this Appendix we summarize the properties of the eigenvalues of the Baxter operator $Q(u, \bar{u})$ at $N = 2$. As was shown in the Section 3, $Q(u, \bar{u})$ is equal to the integral (3.1) of the function $Q(z, \bar{z})$ defined in (3.32) over the two-dimensional plane.

The function $Q_s(z; h)$ entering (3.32) is expressed in terms of the Legendre function on the second-kind, Eq. (3.31). Using the properties of the Legendre functions [24], one finds the behaviour of $Q_s(z; h)$ around singular points $z = 0$ and $z = 1$ as

$$Q_s(z; h) \overset{z \to 0}{\sim} z^{1-s} \left[ -\frac{1}{2} \ln z - \psi(1-h) + \psi(1) + O(z) \right], \quad (A.1)$$

$$Q_s(z; h) \overset{z \to 1}{\sim} (1-z)^{2s-h-1} \left[ \frac{\Gamma^2(1-h)}{2\Gamma(2-2h)} + O(1-z) \right], \quad (A.2)$$

Comparing (A.1) and (A.2) with the asymptotic behaviour of the functions $Q_n^{(0)}(z)$ and $Q_m^{(1)}(z)$ defined in (3.7) and (3.16), one finds that $Q_s(z; h)$ can be decomposed over the fundamental basis of solutions around $z = 0$ as

$$Q_s(z; h) = - \left[ \psi(1-h) - \psi(1) - \frac{1}{2} \right] Q_1^{(0)}(z) - \frac{1}{2} Q_2^{(0)}(z),$$

$$Q_s(z; 1-h) = - \left[ \psi(h) - \psi(1) - \frac{1}{2} \right] Q_1^{(0)}(z) - \frac{1}{2} Q_2^{(0)}(z), \quad (A.3)$$

and over the fundamental basis of solutions around $z = 1$ as

$$Q_s(z; h) = \frac{\Gamma^2(1-h)}{2\Gamma(2-2h)} Q_1^{(1)}(z), \quad Q_s(z; 1-h) = \frac{\Gamma^2(h)}{2\Gamma(2h)} Q_2^{(1)}(z). \quad (A.4)$$

The functions $Q_n^{(0)}(z)$ and $Q_m^{(1)}(z)$ are related to each other through the transition matrix $Q_n^{(0)}(z) = \sum_m \Omega_{nm} Q_m^{(1)}(z)$ defined in (3.26). Comparison of the the r.h.s. of (A.3) and (A.3) yields

$$\Omega(h) = \left( \begin{array}{cc} \Delta(h) & \Delta(1-h) \\ \delta(h) & \delta(1-h) \end{array} \right), \quad (A.5)$$
where $\Delta(h) = \Gamma(2h - 1)/\Gamma^2(h)$ and $\delta(h) = -2 \left[ \psi(h) - \psi(1) - \frac{1}{2} \right] \Delta(h)$. Similar matrix in the antiholomorphic sector is equal to $\text{Im} = \Omega(\bar{h})$.

The general expression for the function $Q(z, \bar{z})$ is given by (3.32) with the expansion coefficients $c_h$ defined in (3.33). Substituting (A.3) and (A.4) as well as analogous relations in the antiholomorphic sector into (3.32), one arrives at Eqs. (3.9) and (3.20) with the mixing matrices given by

$$C^{(0)} = \begin{pmatrix} \alpha_1(h) - 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad C^{(1)} = \begin{pmatrix} \beta_h & 0 \\ 0 & \beta_{1-h} \end{pmatrix}.$$  \hspace{1cm} (A.6)

with $\alpha_1(h)$ defined in (3.35) and

$$\beta_h = (-1)^{n_h+1} \frac{\Gamma^2(1 - \bar{h}) \Gamma(2h - 1)}{\Gamma^2(h) \Gamma(2 - 2h)}.$$  \hspace{1cm} (A.7)

It is straightforward to verify the matrices $C^{(0)}$ and $C^{(1)}$ satisfy the quantization conditions (3.28).

To obtain the $Q$-block at $N = 2$ one inserts (3.31) into (4.12)

$$Q(u; h) = \frac{1}{\Gamma(2s - h)} \int_0^1 dz \, z^{iu - s}(1 - z)^{2(s-1)} Q_{-h} \left( \frac{1 + z}{1 - z} \right).$$  \hspace{1cm} (A.8)

Integration can be performed by replacing the Legendre function by its integral representation leading to (4.20). The Baxter function, $Q(u, \bar{u})$, with required analytical properties is given by the bilinear combination of the holomorphic and antiholomorphic $Q$-blocks, Eq. (4.21).

As was explained in Section 4.4, to calculate the energy and quasimomentum it is convenient to introduce the blocks, $Q_0(u)$ and $Q_1(u)$ defined in Eqs. (4.32) and (4.34), respectively. The block $Q_0(u)$ is fixed (up to an overall normalization) by the requirement to have poles of the order not higher than $N - 1$ at the points $u = -i(s - m)$ with $m > 0$. At $N = 2$ one has [2, 6]

$$Q_0(u) = \frac{1}{\Gamma(2s - 1 + h) \Gamma(2s - h)} \int_0^1 dz \, z^{iu - s}(1 - z)^{2(s-1)} P_{-h} \left( \frac{1 + z}{1 - z} \right) \quad = \quad \frac{1}{\Gamma^2(2s)} F_2^3 \left( \begin{array}{c} s - iu, 2s - h, 2s - 1 + h \\ 2s, 2s \end{array} \mid 1 \right).$$  \hspace{1cm} (A.9)

The antiholomorphic block $\overline{Q_0}(\bar{u})$ can be obtained from (A.9) by replacing $u \rightarrow -\bar{u}$, $s \rightarrow \bar{s}$ and $h \rightarrow \bar{h}$. The block $Q_1(u)$ is defined according to (4.34) as

$$Q_1(u) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^2 \left( \overline{Q_0}(u^*) \right)^* = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^2 Q_0(u; 1 - s),$$  \hspace{1cm} (A.10)

where in the last relation we indicated explicitly the dependence of the block $Q_0$ on the spin $s$.

The two sets of the blocks are linearly dependent

$$\frac{1}{\Gamma(2s - 1 + h)} Q(u; h) - \frac{1}{\Gamma(2s - h)} Q(u; 1 - h) = -\pi \cot(\pi h) Q_0(u),$$

$$\frac{1}{\Gamma(1 - 2s + h)} Q(u; h) - \frac{1}{\Gamma(2 - 2s - h)} Q(u; 1 - h) = -\pi \cot(\pi h) Q_1(u).$$  \hspace{1cm} (A.11)
The inverse relation reads

\[
Q(u, h) = \rho(s, h) \left[ \frac{1}{\Gamma(2 - 2s - h)} Q_0(u) - \frac{1}{\Gamma(2s - h)} Q_1(u) \right],
\]

\[
Q(u, 1 - h) = \rho(s, h) \left[ \frac{1}{\Gamma(1 - 2s + h)} Q_0(u) - \frac{1}{\Gamma(2s - 1 + h)} Q_1(u) \right], \tag{A.12}
\]

where \(\rho(s, h) = \pi^2 / (2 \sin(2\pi s) \sin(\pi h))\). Substituting (A.12) into (4.21) one finds that the resulting expression matches (4.36).

Using the properties of the \( _3F_2 \)–series [38], one can show that the block \(Q_0(u)\) defined in (A.9) satisfies the following relation

\[
Q_0(-u; s) - \frac{\sin(\pi h)}{\sin(2\pi s)} Q_0(u; s) = -\Gamma \left[ \frac{2s, 1 - 2s, 1 - s - iu, 1 - s + iu}{2s - h, 2s - 1 + h, s - iu, s + iu} \right] Q_0(u; 1 - s), \tag{A.13}
\]

where we indicated explicitly the dependence on the spin \(s\). Applying (A.10), (A.12) and (A.13) one can verify that the eigenvalue of the Baxter operator at \(N = 2\), Eq. (4.21), can be rewritten (up to an overall normalization) as

\[
Q(u, \bar{u}) \simeq \Gamma \left[ \frac{1 - \bar{s} - i\bar{u}, 1 - \bar{s} + i\bar{u}}{\bar{s} - i\bar{u}, \bar{s} + i\bar{u}} \right] \left\{ Q_0(u) (Q_0(-\bar{u}^*))^* + (-1)^{n_h} Q_0(-u) (Q_0(\bar{u}^*))^* \right\}. \tag{A.14}
\]

Finally, two equivalent expressions for the eigenvalues of the Baxter \(Q\)–operator at \(N = 2\), Eqs. (A.14) and (4.21), admit an elegant representation in terms of two-dimensional Feynman diagrams (see Figures 10a and b in Ref. [5]).

**B Appendix: Properties of the \(Q\)–blocks**

In this Appendix we establish different useful relations between the blocks \(Q(u; h, q)\) and \(\overline{Q}(\bar{u}; \bar{h}, \bar{q})\) defined in Eq. (4.12).

**Intertwining relations**

It is well-known that the \(SL(2, \mathbb{C})\) representations of the principal series of the spins \((s, \bar{s})\) and \((1 - s, 1 - \bar{s})\) are unitary equivalent. At the level of the eigenvalues of the Baxter \(Q\)–operator, this property leads to the following intertwining relation between the blocks

\[
Q_s(u; h, q) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^N Q_{1-s}(u; h, q), \tag{B.1}
\]

where subscript indicates the corresponding value of the holomorphic spin. Indeed, one verifies that the both sides of this relation satisfy the Baxter equation (2.7), have the same analytical properties (4.16) and asymptotic behaviour (4.17) at infinity. In a similar manner, using the identities \(s^* = 1 - \bar{s}\), \(h^* = 1 - \bar{h}\) and \(\bar{q}_k = q_k^*\), one can show that

\[
(Q_{1-s}(\bar{u}^*; h, q))^* = Q_s(-\bar{u}; 1 - \bar{h}, -\bar{q}). \tag{B.2}
\]
Combining together Eqs. (B.1), (B.2) and (4.18), we obtain the relations between the holomorphic and antiholomorphic blocks

\[ Q(u; h, q) = \left[ \frac{\Gamma(1 - s + iu)}{\Gamma(s + iu)} \right]^N \overline{Q}(u^*; 1 - \bar{h}, \bar{q})^*, \]

\[ \overline{Q}(\bar{u}; \bar{h}, \bar{q}) = \left[ \frac{\Gamma(1 - \bar{s} - i\bar{u})}{\Gamma(\bar{s} - i\bar{u})} \right]^N (Q(u^*; 1 - h, q))^*. \] (B.3)

Here, the ratio of \(\Gamma\)–functions in the r.h.s. compensates different analytical properties and asymptotic behaviour of two blocks at infinity.

**Wronskian relations**

The functions \(Q(u; h, q)\) and \(Q(u; 1 - h, q)\) satisfy the same Baxter equation (2.7). This suggests to define the Wronskian for the holomorphic blocks as

\[ W(u) = \left[ \frac{\Gamma(iu + s)}{\Gamma(iu - s)} \right]^N \left[ Q(u + i; h, q) Q(u; 1 - h, q) - Q(u; h, q) Q(u + i; 1 - h, q) \right]. \] (B.4)

It follows from (2.7) that \(W(u)\) is periodic, \(W(u + i) = W(u)\). In addition, taking into account the properties of the blocks, Eqs. (4.16) and (4.17), one finds that \(W(u)\) is analytical in the half-plane \(\text{Re}(iu - s + 1) > 0\) and behaves there at large \(u\) as \(W(u) \sim u^0\). This implies that \(W(u)\) takes constant values for arbitrary \(u\). Then, substituting (4.17) into (B.4), one finds

\[ W(u) = 1 - 2h. \] (B.5)

Similar consideration in the antiholomorphic sector leads to

\[ \overline{Q}(\bar{u} - i; \bar{h}, \bar{q}) \overline{Q}(\bar{u}; 1 - \bar{h}, \bar{q}) - \overline{Q}(\bar{u}; \bar{h}, \bar{q}) \overline{Q}(\bar{u} - i; 1 - \bar{h}, \bar{q}) = (1 - 2\bar{h}) \left[ \frac{\Gamma(-i\bar{u} - \bar{s})}{\Gamma(-i\bar{u} + \bar{s})} \right]^N. \] (B.6)

For \(u = i(s + \epsilon)\) and \(\bar{u} = -i(\bar{s} + \epsilon)\) we find from the Wronskians (B.4) and (B.6) that the functions \(\Phi(\epsilon)\) and \(\bar{\Phi}(\epsilon)\), defined in (4.26), satisfy the relations

\[ \Phi(\epsilon) - \Phi(\epsilon + k) = \mathcal{O}(\epsilon^N), \quad \bar{\Phi}(\epsilon) - \bar{\Phi}(\epsilon + k) = \mathcal{O}(\epsilon^N) \] (B.7)

with \(k\) being positive integer. Here, we used the fact that \(Q(i(s + \epsilon); h, q)\) and \(Q(i(s + \epsilon); 1 - h, q)\) are finite for \(\epsilon \to 0\).

**Series representation**

We can obtain a series representation for the block \(Q(u; h, q)\) in the different regions on the complex \(u\)–plane, by replacing the function \(Q_1(z)\) in (4.12) by its expressions in terms of the fundamental solutions, Eqs. (4.6) and (4.7), defined in (3.7) and (3.16). For \(z \to 1\) the function \(Q_1(z)\) is given by

\[ Q_1(z) = z^{1-s}(1 - z)^{Ns - h - 1} \sum_{n=0}^{\infty} v_n (1 - z)^n \] (B.8)
with \( v_0 = 1 \) and \( v_n \equiv v_n(1)(-q) \) defined in (3.16) and (3.17). In this way, one gets from (4.12)

\[
Q(u; h, q) = \frac{\Gamma(1 - s + iu)}{\Gamma(-h + Ns)} \sum_{n=0}^{\infty} v_n \frac{\Gamma(n - h + Ns)}{\Gamma(n + 1 - h - s + iu + Ns)}.
\]

(B.9)

For \( z \to 0 \) the function \( Q_1(z) \) is given by

\[
Q_1(z) = z^{1-s} \sum_{n=0}^{N-1} (\ln z)^k \sum_{n=0}^{\infty} w_n^{(k)} z^n,
\]

(B.10)

with \( w_n^{(k)} = \sum_{b=k+1}^{N} [\Omega^{-1}(q)]_{b-1} c_n^{b-k} (-q) \) and the expansion coefficients \( v_n^{(m)} \) defined in (3.7) and (3.8). This leads to

\[
Q(u; h, q) = \frac{1}{\Gamma(Ns - h)} \sum_{n=0}^{\infty} \sum_{k=0}^{N-1} (-1)^k w_n^{(k)} k! \left( iu - s + n + 1 \right)^{k+1},
\]

(B.11)

where the sum over \( n \) goes over the \( k \)-th order poles located at \( u = i(n + 1 - s) \).

We notice that (B.9) reproduces correctly the asymptotic behaviour of \( Q(u; h, q) \) at large \( u \), Eq. (4.17), and the position of its poles on the \( u \)-plane, Eq. (4.16), but not their order. The reason for this is that the series (B.9) is convergent only for \( \text{Re}(1 - s + iu) \geq 0 \). Indeed, the large-order behaviour of the expansion coefficients \( v_n \) in (B.8) is determined by the asymptotics of the function \( Q_1(z) \) at \( z = 0 \), Eq. (B.10)

\[
v_n = \frac{1}{2\pi i} \int_{|z|<\epsilon} dz \frac{z^{s-1} Q_1(z)}{(1 - z)^{n-h+Ns}} \sim \int_{-\infty}^{0} dz \frac{\ln^{N-1} z}{(1 - z)^{n+1}} \sim \frac{\ln^{N-1} n}{n}.
\]

(B.12)

and the series (B.9) diverges at large \( n \) as \( \sum_n v_n/n^{1-s+iu} \sim \sum_n \ln^{N-1} n/n^{2-s+iu} \).

C Appendix: Contour integral representation

Let us demonstrate that the two-dimensional integral (4.5) can be decomposed into the sum of products of simple contour integrals, Eq. (4.9). The derivation is based on the technique developed in [28] for calculation of the correlation functions in CFT. To simplify notations, we rewrite the integral over (4.5) as

\[
Q = \int d^2z \sum_{n,k=1}^{N} q_n(z) C_{nk} \bar{q}_k(z) \equiv \int d^2z q^T(z) \cdot C \cdot \bar{q}(z)
\]

(C.1)

with \( q_n(z) = z^{-iu-1} Q_n(z) \) and \( \bar{q}_k(z) = z^{-i\bar{u}-1} Q_k(z) \).

The functions \( q_n(z) \) and \( \bar{q}_k(z) \) defined in this way have three isolated singular points located at \( z_i = \bar{z}_i = 0, 1 \) and \( \infty \). Around these points, they have a nontrivial monodromy

\[
q_n(z) \overset{z \to \bar{z}_i}{\sim} [M_i]_{nk} q_k(z), \quad \bar{q}_n(z) \overset{z \to \bar{z}_i}{\sim} [\bar{M}_i]_{nk} \bar{q}_k(z),
\]

(C.2)
Stokes’s theorem cuts running from the singular points 0 and 1 to infinity, as shown in Figure 10, and apply the function on the two-dimensional plane, the mixing matrix clockwise and clockwise directions, respectively. For the integrand in (C.1) to be a single-valued with calculated as follows

\[ M_i^T C \bar{M}_i = C, \quad (i = 0, 1, \infty). \quad (C.3) \]

Let us look at \( q(z) \) and \( \bar{q}(\bar{z}) \) as on analytical functions defined on the complex plane with two cuts running from the singular points 0 and 1 to infinity, as shown in Figure 10, and apply the Stokes’s theorem

\[ Q = \int_\Sigma d^2z \frac{\partial}{\partial \bar{z}} \left[ q^T(z) \cdot C \cdot \int_{\bar{z}_{\text{aux}}}^\bar{z} dz' \bar{q}(\bar{z}') \right] = \frac{1}{2i} \int_{\partial \Sigma} dz q^T(z) \cdot C \cdot \int_{\bar{z}_{\text{aux}}}^\bar{z} dz' \bar{q}(\bar{z}'), \quad (C.4) \]

where \( \bar{z}_{\text{aux}} \) is an arbitrary point and the \( \bar{z}' \)-integration goes along the contour that does not cross the cut. It becomes convenient to choose \( \bar{z}_{\text{aux}} = 0 \) and split the integral over (infinite) contour \( \partial \Sigma \) into four integrals along the different edges of two cuts. Their contribution to (C.4) can be calculated as follows

\[
\begin{align*}
Q_{[b'a']} &= -\frac{1}{2i} \int_{a'}^{b'} dz q^T(z) \cdot C \cdot \left[ \int_{1}^{\bar{z}} dz' \bar{q}(\bar{z}') + \int_{0}^{1} dz' \bar{q}(\bar{z}') \right], \\
Q_{[ab]} &= \frac{1}{2i} \int_{a}^{b} dz q^T(z) \cdot C \cdot \left[ \int_{1}^{\bar{z}} dz' \bar{q}(\bar{z}') + \int_{0}^{1} dz' \bar{q}(\bar{z}') \right]. \quad (C.5)
\end{align*}
\]

In these relations the integration goes along two different edges of the same cut. In spite of the fact that \( q(z) \) and \( \bar{q}(\bar{z}) \) are discontinuous across the cut, their bilinear combination remains continuous due to (C.3)

\[
\int_{a'}^{b'} dz q^T(z) C \int_{1}^{\bar{z}} dz' \bar{q}(\bar{z}') = \int_{a}^{b} dz [M_1 q(z)]^T C \int_{1}^{\bar{z}} dz' \bar{M}_1 \bar{q}(\bar{z}') = \int_{a}^{b} dz q^T(z) C \int_{1}^{\bar{z}} dz' \bar{q}(\bar{z}').
\]

This leads to a partial cancellation of terms in the sum \( Q_{[b'a']} + Q_{[ab]} \). Then, one calculates the contribution of the second cut

\[
\begin{align*}
Q_{[cd]} &= -\frac{1}{2i} \int_{d}^{c} dz q^T(z) \cdot C \cdot \int_{0}^{\bar{z}} dz' \bar{q}(\bar{z}'), \\
Q_{[d'c']} &= \frac{1}{2i} \int_{d'}^{c'} dz q^T(z) \cdot C \cdot \int_{0}^{\bar{z}} dz' \bar{q}(\bar{z}'). \quad (C.6)
\end{align*}
\]
and finds that the same property leads to \( Q_{[cd]} + Q_{[d'c']} = 0 \). Combining together (C.5) and (C.6), we obtain the following expression for the two-dimensional integral (C.4)

\[
Q = \frac{1}{2i} \left[ \int_a^b dz \, q^T(z) - \int_{a'}^{b'} dz \, q^T(z) \right] \cdot C \cdot \int_0^1 dz' \, \tilde{q}(z') \\
= \frac{1}{2i} \int_1^\infty dz \, q^T(z) \cdot (1 - M_1^T) \cdot C \cdot \int_0^1 dz' \, \tilde{q}(z').
\]

(C.7)

Finally, one replaces the functions \( q(z) \) and \( \tilde{q}(z) \) by their actual expressions, Eq. (C.1), and arrives at (4.9).

One can obtain another but equivalent representation for \( Q \) by starting instead of (C.4) with

\[
Q = \int_\Sigma d^2z \frac{\partial}{\partial z} \left[ \int_0^z d'z' \, q^T(z') \cdot C \cdot \tilde{q}(z') \right] = -\frac{1}{2i} \int_{\partial\Sigma} dz \int_0^z d'z' q^T(z') \cdot C \cdot \tilde{q}(z).
\]

(C.8)

Repeating the same analysis one gets

\[
Q = -\frac{1}{2i} \int_0^1 d'z' q^T(z') \cdot C(1 - M_1) \cdot \int_1^\infty dz \, \tilde{q}(z).
\]

(C.9)

Let us now take into account that \( q(z) \) is analytical inside \( \Sigma \) and, therefore, \( \int_{\partial\Sigma} dz \, q(z) = 0 \). As before, splitting the integration contour into four pieces and taking into account that \( M_1 \int_a^b dz \, q(z) = \int_{a'}^{b'} dz \, q(z) \) and \( \int_c^d dz \, q(z) = \int_{c'}^{d'} dz \, q(z) \) one obtains

\[
-(1 - M_0^{-1}) \int_0^\infty dz q(z) + (1 - M_1) \int_1^\infty dz q(z) = 0.
\]

(C.10)

Here, two terms in the l.h.s. correspond to the contribution of two cuts. We conclude from (C.10) that

\[
-(1 - M_0) \int_0^1 dz q(z) = (1 - M_0 M_1) \int_1^\infty dz q(z).
\]

(C.11)

Obviously, similar property holds for the function \( \tilde{q}(z) \) in the antiholomorphic sector. Recalling the definition of the function \( q(z) \), Eq. (C.1), one obtains

\[
\int_0^1 dz \, q_n(z) = \int_0^1 dz \, z^{-iu-1} Q_n(z) , \quad \int_1^\infty dz \, q_n(z) = \int_0^1 dz \, z^{iu-1} Q_n(1/z).
\]

(C.12)

Eq. (C.11) allows us to rewrite (C.7) and (C.9) in the form

\[
Q = \frac{1}{2i} \int_0^1 dz \, q^T(z) \cdot \left( \left( 1 - (1 - M_0^T)^{-1} \right) \left( 1 - M_1^T \right)^{-1} \right)^{-1} C \cdot \int_0^1 dz \, \tilde{q}(z),
\]

(C.13)

in which the symmetry between \( z- \) and \( \bar{z} - \) sectors becomes manifest.

### D Appendix: Degenerate \( Q \)-blocks

In our analysis of the quantization conditions, performed in the previous section, we have tacitly assumed that the blocks \( Q(u; h, q) \) are well-defined for arbitrary spins \( h = (1 + n_h)/2 + iv_h \) and,
in addition, \( Q(u; h, q) \) is finite at \( u = i(s + n - 1) \) with \( n \geq 1 \). As was already mentioned in Section 4.2, the first condition is not satisfied at \( \nu_h = 0 \). The second condition does not hold for (half)integer spins \( s \) since, by the definition, the block \( Q(u; h, q) \) has poles at \( u_m^- = -i(s - m) \) with \( m \geq 1 \) and for positive integer \( n \) and \( m \), such that \( 2s - 1 = m - n \), the point \( u = i(s + n - 1) \) coincides with the pole \( u_m^- \). In this Section, we will work out the quantization conditions for (half)integer spins \( h = (1 + n_h)/2 \) and \( s \). It worth mentioning that one has to deal with these two cases calculating the ground state of the Schrödinger equation (1.2) at \( \nu_h \) to \( \partial \nu_h \) are valid only for \( \nu_h \) entering (B.8) are valid only for \( \nu_h \) entering (B.8) are valid only for \( \nu_h \) entering (B.8) if h = (1 + n_h)/2 and \( s \). It worth mentioning that one has to deal with these two cases calculating the ground state of the Schrödinger equation (1.2) at \( h = 1/2 \) and \( s = 0 \).

To start with, let us examine the series representation (B.8) for the block \( Q(u; h, q) \) at \( h = (1 + n_h)/2 + i\nu_h \) in the limit \( \nu_h \to 0 \) and \( n_h > 0 \). The expansion coefficients \( \nu_h \) entering (B.8) satisfy the \( N \)-term recurrence relations, which lead to \( \nu_n \sim 1/\nu_h \) as \( \nu_h \to 0 \) for \( n \geq n_h \). The resulting expression for \( Q(u; h, q) \) can be written as

\[
Q(u; h, q) = \frac{A_{nh}(q)}{\nu_h} Q(u; 1 - h, q) + \tilde{Q}(u; n_h, q) + \mathcal{O}(\nu_h), \tag{D.1}
\]

where \( \tilde{Q}(u; h, q) \) and \( Q(u; 1 - h, q) \) are finite for \( \nu_h \to 0 \) and \( A_{nh}(q) = \lim_{\nu_h \to 0} [v_{nh}(q)\nu_h] \). Taking into account (4.17), we find that the function \( \tilde{Q}(u; n_h, q) \) defined in this way has the following asymptotic behaviour at large \( u \)

\[
\tilde{Q}(u; n_h, q) \sim (iu)^{-N_s+(1+n_h)/2} \left[ 1 + \mathcal{O}(1/u) \right] \\
+ A_{nh}(q) (iu)^{-N_s+(1-n_h)/2} \ln u \left[ 1 + \mathcal{O}(1/u) \right]. \tag{D.2}
\]

According to the definition (D.1), the function \( \tilde{Q}(u; n_h, q) \) is a linear combination of two degenerate blocks and, therefore, it satisfies the holomorphic Baxter equation (2.7). Eqs. (D.1) and (D.2) are valid only for \( n_h > 0 \). At \( n_h = 0 \), or equivalently \( h = 1/2 \), the function \( Q(u; 0, q) \) is defined as

\[
\tilde{Q}(u; 0, q) = \partial_{\nu_h} Q(u; 1/2 + i\nu_h, q) \bigg|_{\nu_h=0}^{u \to \infty} \sim (iu)^{-N_s+1/2} \ln u \left[ 1 + \mathcal{O}(1/u) \right]. \tag{D.3}
\]

It is straightforward to verify that for \( n_h \geq 0 \) the function \( \tilde{Q}(u; n_h, q) \) satisfies the chiral Baxter equation (2.7).

Let us now insert (D.1) into the quantization conditions (4.28) and examine the limit \( \nu_h \to 0 \). It follows from (4.26) that

\[
\Phi(\epsilon) = \frac{A_{nh}(q)}{\nu_h} + \frac{\tilde{Q}(i(s+\epsilon); n_h, q)}{Q(i(s+\epsilon); (1-n_h)/2, q)} + \mathcal{O}(\nu_h). \tag{D.4}
\]

Then, comparing the coefficients in front of different powers of \( \nu_h \) we find from (4.28) the set of \( N - 1 \) quantization conditions on the charges \( q_3, ..., q_N \)

\[
\frac{\partial^n}{\partial \epsilon^n} \text{Im} \left[ (A_{nh}(q))^* \frac{\tilde{Q}(i(s+\epsilon); n_h, q)}{Q(i(s+\epsilon); (1-n_h)/2, q)} \right] \bigg|_{\epsilon=0}^{\nu_h=0} = 0, \tag{D.5}
\]

with \( n = 1, ..., N - 1 \) and \( h = (1 + n_h)/2 \). We recall that the additional set of \( N \) quantization conditions follows from (4.31) in the antiholomorphic sector. At \( h = 1/2 \) the relation (D.5) leads to

\[
\frac{\partial^n}{\partial \epsilon^n} \frac{\partial}{\partial \nu_h} \text{Im} \ln Q\left( i(s+\epsilon); 1/2 + i\nu_h, q \right) \bigg|_{\epsilon=\nu_h=0} = 0. \tag{D.6}
\]

46
Let us now consider the quantization conditions for (half) integer spins \( s \). In distinction with the previous case, the blocks \( Q_s(u; h, q) \) as well as the eigenvalues of the Baxter operator \( Q(u, \bar{u}) \) remain finite in the limit \( s = (1 + n_s) / 2 \) and possess a correct analytical properties as functions of \( u \) and \( \bar{u} \). Nevertheless, the quantization conditions have to be modified because the two sets of points (4.25) overlap. Repeating the analysis of Section 4.4, we find that the function \( Q(u, \bar{u}) \) has correct analytical properties for (half)integer \( s \) provided that the following conditions are satisfied

\[
\arg \left[ \frac{Q(i(1 + |n_s|)/2 + \epsilon; h, q)}{Q(i(1 + |n_s|)/2 + \epsilon; 1 - h, q)} \right] = \pi \left( \frac{n_h}{2} + \ell \right) - \Theta_{q,\bar{q}} + O(\epsilon^{2N}).
\]  

(D.7)

Here, in distinction with (4.30), the \( Q \)-blocks are calculated at the vicinity of the point \( u = i(1 + |n_s|)/2 \) that belongs to the both sets (4.25) simultaneously. In addition, the small \( \epsilon \)-expansion in the r.h.s. starts with the terms \( \sim \epsilon^{2N} \).

References


47


