NUCLEAR MATTER MEAN FIELD WITH EXTENDED NJL MODEL

Steven A. Moszkowski

UCLA, Los Angeles, CA 90095, USA

Constança da Providência\textsuperscript{1}, and João da Providência\textsuperscript{1}

University of Coimbra, P-3000 Coimbra, Portugal

April 19, 2002

Abstract

In this paper we briefly touch on an important aspect of nuclear theory for which
the NJL model seems to be useful: The near semi-binding of the free nucleon-nucleon
system. In the nuclear medium, the same underlying hamiltonian leads to slightly
different masses than in free space. We point out that the NJL model gives a coupling
for scalar meson exchange which is in quite good agreement with what is needed to
fit the properties of nuclear matter. We use the NJL model to obtain an expression
for the energy of nuclear matter as function of density. In two space dimensions, we
obtain a very simple analytic expression for the energy per particle. In fact, it is simply
proportional to the density, so that nuclear matter does not saturate until the density
has reached the critical value at which the effective mass vanishes, i.e. chiral symmetry is
restored. In three space dimensions, the same qualitative behaviour prevails. However,
with an extended NJL model, where the scalar coupling constant is allowed to increase
with density, then it is found that quark matter may saturate at a density where the
effective mass is still finite. A very simple approximate result is obtained for the energy
of nuclear matter,

$$W = \frac{W_0}{M} \left( \frac{N}{M} \right),$$

where $N_c$ is the number of quark colors. The
resulting equation of state can be put into a form somewhat like the one in the derivative
coupling model, \cite{8, 20}. Our model leads to a phase transition from a gas of nucleons
to a gas of quarks at a density about three to four times the equilibrium nuclear matter
density.

1 INTRODUCTION

Strong interaction dynamics of mesons and baryons is believed to be described by QCD,
which exhibits a non perturbative behaviour at low energies. This circumstance renders
the analytic study of the theory rather difficult. The NJL model is a popular substitute
which has in common with QCD important symmetries of the quark-flavour dynamics.
This model has been very successful in the description of the meson sector. The question
arises: does the model allow also for soliton solutions (non-homogeneous solutions) which
are of interest for the description of the baryon sector and of hadronic matter?

\textsuperscript{1}E-mail: stevemos@ucla.edu
\textsuperscript{1}E-mail: cp@teor.fis.uc.pt
\textsuperscript{1}E-mail: providencia@fteor5.fis.uc.pt
The Nambu-Jona-Lasinio (NJL) model[1] was originally developed for the purpose of understanding hadron physics. This was even before quarks were discovered. For reviews of this model see [2], [3], [4] and [5].

In this model, hadron masses are generated by spontaneous symmetry breaking of the vacuum. In the modern form of the NJL model, we start with essentially massless quarks interacting via zero range interactions, but with a cutoff in momentum space. Thus it is a low energy approximation to QCD in which form factor and other finite range effects have been ignored. NJL is thus only an effective theory, and it does not take into account quark confinement. However, it may work quite well in the region of interest in nuclear physics, i.e. for excitations less than the scalar meson mass. We use the NJL model to obtain an expression for the NN interaction at low momentum transfer.

In this paper we briefly touch on an important aspect of nuclear theory for which the NJL model seems to be useful: The near semi-binding of the free nucleon-nucleon system.

The Nambu-Jona-Lasinio Model (NJL) model is based on the following assumptions:

a. The light quarks (u and d) are the basic constituents of nuclei.
b. In the low energy, long wavelength limit, the gluon degrees of freedom are frozen.
c. QCD leads to a local effective interaction between quarks.
d. The interaction between quarks is constructed in accordance with the symmetries of QCD.

The price of making these assumptions is that:

i. The model does not have confinement.
ii. The model is not renormalizable.
iii. The model gives only \textit{effective} interactions valid at low energies (momenta small compared to cutoff momentum.) Thus the model cannot give the correct high momentum (short distance) behaviour of form factors.

The paper has been organized as follows:
In the next section, we discuss the NJL model for a sigma exchange potential. The model gives both the range and coupling strength, and leads to a potential which is close to what is necessary to lead to a zero energy bound state. Also, it is shown that the sigma exchange potential is well approximated by a sum of attractive Yukawa and contact interactions. We then apply the NJL model to calculate the energy density of nuclear matter, and obtain saturation, but only at a density large enough that chiral symmetry has been restored. This is contrary to what is found in nuclei, where the nucleon effective mass is about 0.7 of the free nucleon mass. This problem can be resolved by adding a Lorentz invariant and chirally symmetric term, a product of a scalar and a vector, to the Lagrangian. With this generalization which we refer to as the Extended NJL (ENJL) model, it is possible to get a reasonable nuclear equation of state, and behaviour of the effective mass. There is a brief discussion concerning the role of the kinetic energy, and also of the two dimensional version of the NJL model. Then we list some numerical results, concerning both nuclear matter and...
quark matter, and the hadron to quark phase transition at zero temperature. Finally, there are some remarks concerning the connection of the NJL model to other models, in particular Guichon’s quark meson coupling model, relativistic mean field models, and more traditional nuclear G-matrix models. We discuss, in particular, the interplay between effective mass and short range NN correlations. Finally, some brief conclusions are drawn.

2 NJL MODEL FOR SIGMA EXCHANGE POTENTIAL

Sigma meson exchange leads to an intermediate range Yukawa attraction:

\[ V(r) = -\frac{g_s^2}{4\pi} \frac{e^{-\mu r}}{r}. \]  

(1)

We are using units in which \( \hbar \) and \( c \) are set equal to 1. The NJL model leads to the Goldberger-Treiman relation for the coupling constant \([9]\):

\[ g_s = \frac{M}{F_\pi} = \frac{938}{93} = 10.1. \]  

(2)

Here, \( M \) denotes the nucleon mass. Thus \( g_s^2 \frac{2\pi}{3} = 8.12 \), which is in quite good agreement with the strength deduced from phenomenological OBEP fits \([10]\). This is a quite remarkable fact.

Let us try to generalize this result to \( N_c \) colors. Here the scalar (\( \sigma \)) meson mass is:

\[ \mu = m_\sigma = \frac{2M}{N_c}. \]  

(3)

In our non-covariant treatment\([3]\), we have:

\[ F_\pi^2 = \frac{M^2}{N_c} \int_0^\Lambda \frac{d^3p}{(2\pi)^3(m_q^2 + p^2)^{3/2}}. \]  

(4)

\( m_q \) denotes the constituent quark mass. We can write:

\[ F_\pi^2 = \frac{M^2}{(2\pi)^2 N_c} I_2(\frac{\Lambda}{m_q}), \]  

(5)

where:

\[ I_n(s) = \int_0^s \frac{2p^2}{(1 + p^2)^{n+\frac{1}{2}}} dp. \]  

(6)

In particular:

\[ I_2(s) = \int_0^s \frac{2p^2}{(1 + p^2)^{\frac{3}{2}}} dp = 2 \arctanh \left( \frac{s}{\sqrt{1 + s^2}} \right) - \frac{2s}{\sqrt{1 + s^2}}. \]  

(7)

If the cutoff is chosen to equal the scalar meson mass:

\( \Lambda = m_\sigma = 2m = \frac{2M}{N_c} = 625 \text{ MeV} \), then \( s = 2 \), and \( I_2 = 1.098 \).

Thus

\[ F_\pi = \frac{M}{2\pi\sqrt{N_c}} \times 1.05. \]  

(8)
Using $M = 938$ MeV, and $N_c = 3$, we obtain $F_\pi = 90.5$ MeV, close to the empirical value of 93 MeV.

For arbitrary number of colors, we find:

$$\frac{g_s^2}{4\pi} = \frac{\pi N_c}{I_2(s)} = 2.85N_c.$$  (9)

Thus, according to NJL, the $N_c$ color sigma exchange potential is:

$$V_{NJL} = -2.85N_c \frac{e^{-\mu r}}{r}.$$  (10)

For three colors, there is a single deeply bound state. This must be removed by some kind of repulsion. [11]

Suppose that the NN interaction gives a semibound state, and, for $N_c$ colors, also $\frac{N_c-1}{2}$ deep bound states, which must be removed.

It is convenient to discuss the volume integral $\tilde{V}_s$ of the scalar potential (including powers of $\hbar$ and $c$):

$$\tilde{V}_s = -4\frac{\pi^2}{I_2(\frac{N_c}{M^2})} N_c \mu c^2 \left(\frac{\hbar}{\mu c}\right)^3 = -9.00 N_c^3 M c^2 \left(\frac{\hbar}{M c}\right)^3.$$  (11)

for the NJL Yukawa potential.

2.1 Sigma exchange potential well approximated by Yukawa plus contact interaction

According to the NJL model, the scalar exchange interaction in momentum space is given by [5, 6]:

$$\tilde{V}_s(q) = -\frac{1}{(1 + q^2/4m_q^2) I_2(\frac{N_c}{m_q}, \frac{q}{m_q})} \pi^2 N_c^3 M c^2 \left(\frac{\hbar}{M c}\right)^3,$$  (12)

where

$$I_2(s, \tilde{q}) = \int_0^s \frac{2\tilde{p}^2 dp}{\sqrt{1 + \tilde{p}^2} (1 + \tilde{p}^2 + \frac{1}{4\tilde{q}^2})}.$$  (13)

Here $I_2(s, 0) = I_2(s)$, which is inversely proportional to the volume integral $\tilde{V}_s(0)$ discussed in the last section. Here $\tilde{p} = \frac{p}{m_q}$ and $\tilde{q} = \frac{q}{m_q}$. The integral can be evaluated analytically [5], and we obtain:

$$I_2(s, q) = 2 \text{arctanh} \left( \frac{s}{\sqrt{1 + s^2}} \right) - 2 \left\{ \sqrt{\frac{4m_q^2 + q^2}{q^2}} \text{arctanh} \left( \frac{s}{\sqrt{4m_q^2 + q^2}} \right) \right\}.$$  (14)

A good approximation to $\tilde{V}_s(q)$ is given by:

$$\tilde{V}_s(q) = -\frac{1}{(1 + q^2/4m_q^2)} \left( \frac{1}{I_2(\frac{N_c}{m_q})} + \frac{q^2}{I_1(\frac{N_c}{m_q})} \right) \pi^2 N_c^3 M c^2 \left(\frac{\hbar}{M c}\right)^3.$$  (15)
In coordinate space this is a superposition of a Yukawa and contact interaction:

$$V_s(r) = -\hbar c\pi N_c \left( \frac{e^{-\mu r}}{r} \left( \frac{1}{I_2(\frac{r}{m_q})} - \frac{1}{I_1(\frac{r}{m_q})} \right) + \frac{4\pi}{I_1(\frac{r}{m_q})} \delta(r) \right).$$

(16)

### 2.2 Poeschl-Teller Potential

The Poeschl-Teller (P-T) potential was introduced many years ago for studies in molecular physics [7]. It will be useful here as well. For this potential, which we identify with scalar meson exchange:

$$V_s(r) = -\frac{\hbar^2 \beta^2}{M} N_c(N_c + 1) \text{sech}^2(\beta r).$$

(17)

We can identify $N_c$ with the number of quark colors. For $N_c$ an odd integer, the Schroedinger equation can be solved analytically. We get a bound state at zero energy, and $\frac{N_c-1}{2}$ deeply bound states in addition. These are at energies $\frac{\hbar^2 \beta^2}{M} \times [2^2, 4^2, \ldots, (N_c-1)^2]$. (To get rid of the latter, we need a repulsion, as mentioned before). For the P-T potentials, this repulsion, which we will later identify with vector meson exchange, equals:

$$V_v(r) = \frac{\hbar^2 \beta^2}{M} N_c(N_c - 1) \text{csch}^2(\beta r).$$

(18)

For large $r$, the asymptotic form of both attraction and repulsion is $e^{-2\beta r}$. Note that at large distances, the ratio of the repulsive to attractive potentials is:

$$\frac{V_v}{V_s} = \frac{N_c - 1}{N_c + 1}.$$  

(19)

Thus, in order to get the same range in the exponent as for scalar meson exchange, let us identify $\beta$ with $\frac{\beta}{2} = \frac{M}{N_c}$.  

### 2.3 Volume Integral

For a Poeschel-Teller potential (Eq. (17)) which has $\frac{N_c-1}{2}$ of bound states, in addition to a semibound state, we get the following. A simple calculation gives for the volume integral of the attraction:

$$\overline{V}_s = -\frac{\pi^3}{3} N_c^2(N_c + 1) M c^2 \left( \frac{\hbar}{M c} \right)^3$$

$$= -10.33 N_c^2(N_c + 1) M c^2 \left( \frac{\hbar}{M c} \right)^3.$$  

(20)

These results are to be compared with those from the NJL model, Eq. (11). As can be seen, the NJL model, which implies the approximation $N_c \gg 1$, is quite good if we use a Poeschel-Teller potential. For the repulsive (vector) potential, the volume integral of $\text{csch}^2(\beta r)$ is twice as large as that of $\text{sech}^2(\beta r)$. Thus the ratio of volume integrals is:

$$\frac{\overline{V}_v}{\overline{V}_s} = -2 \frac{N_c - 1}{N_c + 1}.$$  

(22)
2.4 Nuclear Matter vs Quark Matter Cutoff

It is well known that the NJL model does not contain a mechanism for quark confinement. Thus the clustering of quarks into nucleons is not accounted for in this model. In this paper, quark clustering is put in by hand. The way it is done here is that we treat the nucleons as the elementary particles, but take quark structure into account in the interactions. We will find it convenient to define dimensionless integrals \( I_n \) by:

\[
I_n(x) = \int_0^x \frac{2p^2}{(1 + p^2)^{n - \frac{3}{2}}} dp.
\]

According to the NJL model, the volume integral of the \( NN \) interaction is given by Eq.(11): Let us now try to get the same \( NN \) volume integral with a single color. This can be accomplished by using a lower value of the cutoff, denoted here by \( \Lambda' \). Further, from here on, we will express all masses in units of the free nucleon mass \( M \). Also, we will set \( \hbar \) and \( c \) equal to 1. We find that:

\[
I_2(\Lambda') = \frac{I_2(\Lambda N_c)}{N_c^3}.
\]

3 Nuclear Matter Energy Density According to NJL Model

3.1 Formulation

Next is the calculation of the energy of nuclear matter in the NJL model with nucleons. We start with Eq.(3.3) of Klevansky’s review article on the NJL model, (and follow also an article by Fiolhais et al. on solitons in the NJL model) [3], [12]. In this paper, we restrict our consideration to zero temperature. There have been several papers in which the NJL model was applied to dense matter at finite temperature, see, for example [13, 14].

\[
E = -2 N_c N_f \sum_{p=p_f}^{N_c} \frac{p^2}{\sqrt{m^2 + p^2}} - m^2 G_s (N_c N_f)^2 \left[ \sum_{p=p_f}^{N_c} \frac{1}{\sqrt{m^2 + p^2}} \right]^2.
\]

The only modification we have made is to use the scalar coupling constant \( G_s \) which is \( \frac{G}{4} \) in terms of the \( G \) defined in [3]. If now the number of colors \( N_c \) is set equal to 1, we can write for the energy density:

\[
e = \frac{E}{2N_f} = - \sum_{p=p_f}^{N_c} \frac{p^2}{\sqrt{m^2 + p^2}} - \frac{1}{2} m^2 G_s N_f \left[ \sum_{p=p_f}^{N_c} \frac{1}{\sqrt{m^2 + p^2}} \right]^2.
\]

Here, \( m \) denotes the effective nucleon mass in units of the free mass. \( N_f \) is the number of flavors. For symmetric nuclear matter \( 2N_f = 4 \). Eq. (26) gives the energy expectation value of quark matter in the Hartree approximation. The only modification we have made here is to replace the integrals over momenta by sums over momenta. The sums over spins,
and flavors give the additional factor 4. Minimizing Eq. (26) with respect to the effective quark mass \( m \), we obtain the gap equation:

\[
\sum_{p=p_f}^{\Lambda'} \frac{1}{\sqrt{m^2 + p^2}} = \frac{1}{2G_s}.
\]  

(27)

In free space, the sum over momenta starts at 0 rather than at \( p_f \). We can write the gap equation in the form:

\[
\sum_{p=p_f}^{\Lambda'} \frac{1}{\sqrt{m^2 + p^2}} = \sum_{p=0}^{\Lambda'} \frac{1}{\sqrt{1 + p^2}}
\]  

(28)

Thus the energy density can be written as:

\[
e = -\sum_{p=p_f}^{\Lambda'} \sqrt{m^2 + p^2} + \sum_{p=0}^{\Lambda'} \frac{m^2}{2} \sum_{p=0}^{\Lambda'} \frac{1}{\sqrt{1 + p^2}}.
\]  

(29)

Relative to the value in free space, the energy density is given by:

\[
e(m) - e(1) = -\sum_{p=p_f}^{\Lambda'} \sqrt{m^2 + p^2} + \sum_{p=0}^{\Lambda'} \sqrt{1 + p^2} + \frac{m^2}{2} \sum_{p=0}^{\Lambda'} \frac{1}{\sqrt{1 + p^2}}.
\]  

(30)

Eq. (30), which can also be derived from a paper by Fiolhais et al. [12], is the starting point for our energy calculations. We first replace the sum over states by an integral over momenta. This is:

\[
\sum_{p=p_1}^{p_2} f(p) = \frac{1}{\pi^2} \int_{p_1}^{p_2} f(p) \, 2p^2 \, dp.
\]  

(31)

In particular, the nucleon density \( \rho_N \) is given by:

\[
\rho_N = \sum_{p=0}^{p_f} 1 = \frac{1}{\pi^2} \int_{p_1}^{p_f} 2p^2 \, dp = \frac{2}{3\pi^2} p_f^3.
\]  

(32)

We will find it convenient to define the quantity:

\[
\rho_f = \int_{p_1}^{p_f} 2p^2 \, dp = \frac{2}{3} p_f^3 = \frac{\rho_N}{q},
\]  

(33)

where

\[
q = \frac{N_c N_f}{2\pi^2} = \frac{1}{\pi^2}.
\]  

(34)

The gap equation reads:

\[
\int_{p_f}^{\Lambda'} \frac{2p^2}{\sqrt{m^2 + p^2}} \, dp = \int_{p=0}^{\Lambda'} \frac{2p^2}{\sqrt{1 + p^2}} \, dp.
\]  

(35)

The energy per particle \( W \), in units of \( Mc^2 \) not including the rest energy, is:

\[
W = \frac{de}{\rho_f} - 1,
\]  

(36)
where
\[ de = - \int_{p_f}^{\Lambda} \sqrt{m^2 + p^2} 2p^2 dp + \int_0^{\Lambda} \sqrt{1 + p^2} 2p^2 dp + \frac{m^2 - 1}{2} \int_0^{\Lambda} \frac{2p^2 dp}{\sqrt{1 + p^2}}. \] (37)

The gap equation can be written as:
\[ m^2[I_1(\frac{\Lambda'}{m}) - I_1(\frac{p_f}{m})] = I_1(\Lambda') \] (38)

and the energy density is given by:
\[ de = -m^2[I_0(\frac{\Lambda'}{m}) - I_0(\frac{p_f}{m})] + m^2 - 1 \frac{1}{2} I_1(\Lambda') \] (39)

from which we can obtain the energy per nucleon.

For densities where the nucleon mass vanishes, the energy per particle is given by:
\[ W = c_1 \rho^\frac{3}{2} - c_2 + c_3 \rho^{-1}, \] (40)

where \( c_1, c_2 \) and \( c_3 \) are constants.

We will find that for quark matter \( W \) does not have a minimum except for unreasonably small values of the cutoff momentum \( \Lambda \). On the other hand, for nuclear matter, \( W \) does not have a minimum except at a rather high density where the quark mass is small or vanishing. Thus the pure NJL model does not give a proper equation of state. We may summarise the situation as follows: without clustering of quarks into nucleons there is no binding, while with clustering there is overbinding and collapse. But, as we will show, with a modification of the NJL model we can obtain saturation at a density where the effective mass is finite.

4 GENERAL TREATMENT OF THE ENJL MODEL

4.1 Lagrangian and Hamiltonian

In this section we give a more detailed discussion of the Extended NJL (ENJL) model. The NJL model [3] is defined by the Lagrangian density
\[ \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu)\psi + G_s[(\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\tilde{T}\psi)^2]. \] (41)

A regularizing momentum cut-off \( \Lambda \) is part of the model. The Lagrangian is equivalent to the Hamiltonian
\[ \mathcal{H}_{NJL} = \sum_{k=1}^{N} \vec{p}_k \cdot \vec{\alpha}_k + G_s \sum_{k,j=1}^{N} \delta(\vec{r}_k - \vec{r}_j)\beta_k\beta_j(1 - \gamma_k\gamma_j \vec{\tau}_k \cdot \vec{\tau}_j). \] (42)

The vacuum is described by a Slater Determinant \( |\Phi_0\rangle \) constructed from plane waves which are negative energy eigenfunctions of the single particle Hamiltonian \( \mathcal{H} = \vec{p} \cdot \vec{\alpha} + \beta m \). The “constituent mass” \( m \) is a variational parameter.
If moreover positive energy eigenfunctions with momentum $\vec{p}$ satisfying $|\vec{p}| < p_F$ are occupied, so that $p_F$ is the Fermi momentum, the energy expectation value $E = \langle \Phi_0 | H_{NJL} | \Phi_0 \rangle$ is given by Eq. (25).

For quark matter, the degeneracy is $\nu = 2N_cN_f$ and $\Lambda$ is such that $m = 313$ MeV is the quark constituent mass in the vacuum. For nuclear matter the degeneracy is $\nu = 2N_f$ and $\Lambda$ is such that $m = 939$ MeV is the nucleon mass in the vacuum.

We assume that $G_s$ for nuclear matter is either determined by its value for quark matter according to the previously defined prescription, Eq. (24), or is nine times bigger than the quark matter value. The philosophy behind this assumption is that the NN interaction is a manifestation of the instanton interaction between quarks predicted by QCD in the weak coupling regime. Following Guichon [15], we assume that the field emanating from the quark scalar density in moving nucleons, due to the instanton force, acts on the quarks of another nucleon, and produces in this way the nucleon-nucleon force.

The NJL model can be extended [16] to yield more reasonable saturation of nuclear matter by means of a modified gap equation which will be presented later, (Eqs. (52), (53)). An effective density dependent coupling constant is indeed obtained if the following extended NJL Lagrangian density, which actually pushes chiral symmetry restoration to higher densities, is considered,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu)\psi + G_s[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \vec{r}\psi)^2] - G_v(\bar{\psi}\gamma^\mu \psi)^2 - G_{sv}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5 \vec{r}\psi)^2](\bar{\psi}\gamma^\mu \psi)^2.$$ (43)

The resulting thermodynamical potential per volume is

$$\omega(p_F, \mu) = \langle \bar{\psi}(-\vec{p})\psi \rangle - G_s\langle \bar{\psi}\psi \rangle^2 + G_v\langle \psi^\dagger \psi \rangle^2 + G_{sv}\langle \bar{\psi}\psi \rangle^2\langle \psi^\dagger \psi \rangle^2 - \mu\langle \bar{\psi}\psi \rangle$$ (44)

where exchange terms have been neglected. By $\langle \bar{\psi}\Gamma\psi \rangle$ we denote the following expectation value per volume

$$\langle \bar{\psi}\Gamma\psi \rangle = \frac{1}{V} \langle \Phi_0 | \sum_k \beta_k \Gamma_k | \Phi_0 \rangle.$$ (45)

We find

$$\langle \bar{\psi}(-\vec{p})\psi \rangle = -\nu \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E_p} (\theta_\Lambda - \theta_F)$$ (46)

$$\langle \bar{\psi}\psi \rangle = -\nu \int \frac{d^3p}{(2\pi)^3} \frac{m}{E_p} (\theta_\Lambda - \theta_F)$$ (47)

$$\langle \psi^\dagger \psi \rangle = \nu \int \frac{d^3p}{(2\pi)^3} \theta_F$$ (48)

with $E_p = \sqrt{m^2 + p^2}$, $\theta_\Lambda = \theta(\Lambda^2 - p^2)$, $\theta_F = \theta(p_F^2 - p^2)$. The condition $\partial \omega / \partial m = 0$ implies

$$m = -2G_s\langle \bar{\psi}\psi \rangle + 2G_{sv}\langle \bar{\psi}\psi \rangle\langle \psi^\dagger \psi \rangle^2.$$ 

The condition $\partial \omega / \partial p_F = 0$ implies

$$E_{p_F} = \mu - 2G_v\langle \psi^\dagger \psi \rangle - 2G_{sv}\langle \bar{\psi}\psi \rangle\langle \psi^\dagger \psi \rangle\langle \bar{\psi}\psi \rangle^2.$$
These conditions fix the values of $p_F$, $m$ for given $\mu$.

In terms of the following functions $F_0(m, p) = \frac{m^2}{2} I_1(p/m)$, $F_2(m, p) = \frac{m^4}{2} (I_1(p/m) - I_1(p/m))$ we have

$$
\langle \bar{\psi} \gamma \cdot \vec{p} \psi \rangle = -\nu \frac{4\pi}{(2\pi)^3} (F_2(m, \Lambda) - F_2(m, p_F)) \tag{49}
$$

$$
\langle \bar{\psi} \psi \rangle = -\nu \frac{4\pi m}{(2\pi)^3} (F_0(m, \Lambda) - F_0(m, p_F)) \tag{50}
$$

$$
\langle \bar{\psi} \gamma^\dagger \psi \rangle = \nu \frac{4\pi}{(2\pi)^3} \frac{p_F^3}{3}. \tag{51}
$$

The properties of the extended NJL model are now easily computed. The terms in $G_v, G_{sv}$ are supposed to simulate a chiral invariant short range repulsion between nucleons which, due to the overlap of the bags, is increasingly felt as the density increases. Thus, for quark matter we set $G_v = G_{sv} = 0$.

### 4.2 Calculation of Effective Mass and Energy

Here is how the NJL model can be modified to yield more reasonable saturation of nuclear matter.

Let us revisit the gap equation:

$$
\sum_{p=p_f}^{\Lambda'} \frac{1}{\sqrt{m^2 + p^2}} = \frac{1}{2G_s(\rho)}. \tag{52}
$$

As indicated here, we allow the coupling constant $G_s$ to depend on $\rho$.

$$
G_s(\rho_f) = \frac{G_0}{F(\rho_f)}. \tag{53}
$$

This is equivalent to having an additional chirally invariant coupling term involving both scalar and vector densities [16].

In the 3D case, we still have $m$ as the solution of:

$$
m^2 [I_1(\frac{\Lambda'}{m}) - I_1(\frac{p_F}{m})] = I_1(\Lambda')F(\rho_f). \tag{54}
$$

We will suppose that:

$$
F(\rho) = \frac{1}{1 + \frac{\rho_{sv}^2}{I_1(\Lambda')I_2(\Lambda')\rho_f^2}}. \tag{55}
$$

We see that the coupling constant actually increases with increasing density. Chiral symmetry breaking is enhanced, i.e. the mass is closer to the free space value of $m_0$. However, the extended NJL hamiltonian (with linear dependence on $\rho_{sv}^2$) still preserves chiral symmetry. A linear dependence of $F$ on density itself would not accomplish this. Then we obtain for the energy density:

$$
de = -m^4 [I_0(\frac{\Lambda'}{m}) - I_0(\frac{p_F}{m})] + I_0(\Lambda') + \frac{m^2 F(\rho) - 1}{2} I_1(\Lambda'). \tag{56}
$$
When we calculate the energy per particle, \( W = \frac{d e}{\rho_f} - 1 \), the density dependence of the coupling constant also leads to an attractive term linear in density, which has to be cancelled by a vector coupling contribution.

4.3 Strong Coupling Approximation to ENJL Model

Let us now go to the strong coupling limit, in which we neglect all momenta. We have:

\[
\int_{p_f}^{\Lambda'} \sqrt{m^2 + p^2} dp = m \int_{p_f}^{\Lambda'} 2p dp = m (\Omega \lambda - \Omega_{p_f}). \tag{57}
\]

The two omega terms can be expressed in terms of the NJL scalar coupling, \( \bar{V}_s \) (11), and density:

\[
\Omega \lambda = \frac{\pi^2}{\bar{V}_s}, \tag{58}
\]

\[
\Omega_{p_f} = \pi^2 \rho_N. \tag{59}
\]

We then get a simple expression for the energy per particle:

\[
W = m - 1 + \frac{(m - 1)^2}{2 \bar{V}_s \rho_N} - \frac{m^2 [1 - F(\rho)]}{2 \bar{V}_s \rho_N}. \tag{60}
\]

Here \( \bar{V}_s \) is the volume integral of the attractive potential, and \( \bar{V}_s \rho_N \) are expressed in units of \( Mc^2 \). In the strong coupling approximation, we obtain:

\[
F(\rho) = \frac{1}{1 + c_{sv} \bar{V}_s^2 \rho_N^2}. \tag{61}
\]

Minimizing \( W \) w.r.t. the effective mass, we get the following simple expression for \( m \) (We set the free nucleon mass \( M = 1 \)):

\[
m = \frac{1 - \bar{V}_s \rho_N}{F(\rho)} = (1 - \bar{V}_s \rho_N)(1 + c_{sv} \bar{V}_s^2 \rho_N^2), \tag{62}
\]

and the energy per particle is:

\[
W = -\frac{1}{2} \bar{V}_s \rho_N [1 + c_{sv} (1 - \bar{V}_s \rho_N)^2]. \tag{63}
\]

Now we need to add a vector repulsion in order to cancel the extra attraction due to the \( c_{sv} \) term. Actually, we will choose the vector repulsion to be twice as large as this scalar-vector term. It may be convenient to define \( c_v \) here:

\[
\bar{V}_v = c_v \bar{V}_s = 2 c_{sv} \bar{V}_s. \tag{64}
\]

Thus we have:

\[
W = -\frac{1}{2} (1 - c_{sv}) \bar{V}_s \rho_N + c_{sv} \bar{V}_s^2 \rho_N^2 - \frac{1}{2} c_{sv} \bar{V}_s^3 \rho_N^3. \tag{65}
\]
Our choice for the extra coupling terms is, admittedly somewhat arbitrarily:

\[ c_{sv} = \frac{N_c - 1}{N_c + 1}, \quad c_v = 2c_{sv}. \]  

(66)  

(67)

It might be not coincidental that the quantity \( c_v \) is, in fact, the ratio of the volume integrals of the repulsive and attractive terms in the Poeschl-Teller potential necessary to lead to a single semibound two-body state.

With our choice for the extra coupling terms, the low density equation of state is:

\[ W = -\frac{1}{N_c + 1} \bar{V}_s \rho_N + \frac{N_s - 1}{N_c + 1} \bar{V}_s^2 \rho_N^2. \]  

(68)

The minimum energy is:

\[ W_0 = -\frac{1}{4(N_c^2 - 1)}. \]  

(69)

For three colors, this means, putting in the rest mass: \( W_0 = -\frac{M c^2}{32} = -29MeV \). This is not quite correct, but at least within a factor of two of the correct value. In any case, when we correct for the kinetic energy, we get a much better result. The effective mass at saturation is well approximated by:

\[ m = 1 - \frac{1}{2(N_c - 1)}, \]  

(70)

which equals 0.75 for 3 colors. The equilibrium density is given by:

\[ \rho_0 = \frac{1}{2(N_c - 1) \bar{V}_s}. \]  

(71)

Now putting in the value previously obtained for \( \bar{V}_s \), in the NJL model we obtain:

\[ \rho_0 = \frac{1}{18 N_c^3(N_c - 1)} \left( \frac{Mc}{\hbar} \right)^3. \]  

(72)

For three colors, we obtain an equilibrium density of 0.11 fm\(^{-3}\), which is 2/3 to 3/4 of the generally accepted value, for example, \( k_f = 1.30 fm^{-1} \) corresponding to a density \( \rho_N = 0.148 fm^{-3} \), in any case in qualitative agreement with empirical results. It should be emphasized that there are no arbitrary parameters in this model, except, perhaps the \( c_{sv} \) and \( c_v \) terms.

5 RESULTS INCLUDING KINETIC ENERGY

5.1 Low Density Expansion of ENJL

We find the following expressions for the effective mass (the free mass is set equal to 1) and the energy per particle, up to order \( \rho^2 \):

\[ m = 1 - (1 + T) \bar{V}_s \rho_N + (c_{sv} - \frac{3}{2} \xi) \bar{V}_s^2 \rho_N^2 + ... \]  

(73)

\[ W = T + [T - \frac{1}{2}(1 - c_{sv})] \bar{V}_s \rho_N + (c_{sv} - \frac{1}{2} \xi) \bar{V}_s^2 \rho_N^2 + ... \]  

(74)
Here
\[ \xi = 1 - \frac{I_3(N')}{I_2(N')} \]  
and \( T \) is the kinetic energy.

### 5.2 Original NJL Model in Two Dimensions

For this case, we get simple analytic results. The average kinetic energy \( T \) is given by:
\[ T = \frac{1}{2} \rho_f - \frac{1}{6} \rho_f^2 + \ldots, \]
where \( \rho_f = 2\pi \rho_N \), and
\[ \xi = \frac{1}{g} - \frac{1}{3g^2}, \]
where \( g = \frac{\sqrt{v}}{2\pi} \). Thus we get, (\( e_{sw} = 0 \) for pure NJL):
\[ m = 1 - 2 \rho_f - \frac{1}{2} \rho_f^2 + \ldots, \quad (76) \]
\[ W = -\frac{1}{2} \rho_f + 0 \rho_f^2 + \ldots. \quad (77) \]

For the two dimensional case, we can obtain exact results for these two quantities, not just for low densities:
\[ m^2 = 1 - 2g \rho_f + (g - 1)^2 \rho_f^2, \quad (78) \]
\[ W = -\frac{1}{2} \rho_f. \quad (79) \]

It is remarkable that higher order terms in \( W \), cancel exactly. The term of order \( \rho_f^2 \) in Eq. (77) has three components:
1. Relativistic correction to energy momentum relation.
2. Increased kinetic energy due to reduced effective nucleon mass.
3. Increased attraction due to reduction in effective mass of scalar meson (similar to that of the nucleon).

For the more realistic three dimensional case, the physics is very similar, and again there is a near cancellation between these three terms. It cannot, of course, be perfect, since in three dimensions, these terms are proportional to \( \rho^{4/3}, \rho^{5/3}, \) and \( \rho^2 \), respectively.

### 6 NUMERICAL RESULTS

#### 6.1 Nuclear Matter

We give now numerical results for effective mass and energy per nucleon calculated using the various approximations. The equation of state referred as EOS-I is based on the prescription discussed in the text, according to Eqs. (66, 67). Table 1 lists results for Fermi momentum and effective mass. Also the energy per particle, both neglecting and including kinetic energy corrections, are shown.

In Table 2, numerical results pertaining to two different realizations of the model are presented. The set I, as referred before, arises from the the requirements specified by Eqs.
Table 1: Fermi momentum, effective mass and energy per particle both neglecting and including kinetic energy corrections

<table>
<thead>
<tr>
<th>APPROX.</th>
<th>Section</th>
<th>$p_f$ (fm$^{-1}$)</th>
<th>m</th>
<th>$W_0$ (MeV)</th>
<th>K.E.corr. (MeV)</th>
<th>$W_N$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>6</td>
<td>1.14</td>
<td>0.796</td>
<td>-29.65</td>
<td>15.90</td>
<td>-13.75</td>
</tr>
<tr>
<td>Str.Coup.</td>
<td>4.3</td>
<td>1.15</td>
<td>0.779</td>
<td>-32.01</td>
<td>17.96</td>
<td>-14.05</td>
</tr>
<tr>
<td>Low Dens.</td>
<td>5.1</td>
<td>1.10</td>
<td>0.819</td>
<td>-28.23</td>
<td>15.00</td>
<td>-13.23</td>
</tr>
</tbody>
</table>

(24,66,67). In model I no free parameters are present. Here, we assume $\Lambda = 2m_\sigma$ and the values of $F_\pi$ and the saturation properties are not inputs, but outputs. The set II arises when, as inputs, one uses the value of the saturation properties of nuclear matter (binding energy and saturation density), so that, in this case, Eqs. (66,67) are relaxed. In model III the inputs are $F_\pi$, the saturation properties of nuclear matter and we have also relaxed Eq.(24). Instead, we have assumed in model III that $G_s$ for nuclear matter is nine times bigger than the quark matter value determined by the value of $F_\pi$. We have taken, for parameterisation II and III, $E/A = -15.8$ MeV and $\rho_0 = 0.148$ fm$^{-3}$ according to [17, 18]. It is a remarkable fact that model I, which contains no free parameters, reproduces experimental data rather well and does not deviate very much from the other two models (II and III). The compression modulus comes out with very reasonable values in all the different parameterisations.

The crucial role played by clustering must be stressed. It should be pointed out that without clustering there is no real binding and that, moreover, the incompressibility becomes unacceptable. This is well illustrated by the curves in Fig 1. At some density, the nucleon matter curve intercepts the quark matter curve, so it is clear that, at high density, quark matter prevails and may be found, for instance, in the core of neutron stars. Fig. 2 shows that clustering pushes chiral symmetry restoration to higher densities.
Table 2: Numerical results pertaining to two realizations of the model. Sets I and II arise from the the requirements specified by Eqs. (64, 11). The set III arises from the specification of $F_\pi$ and of the saturation energy (-15.9 MeV) and saturation density of nuclear matter, respectively 0.148 fm$^{-3}$ corresponding to a Fermi momentum of 0.13 fm$^{-1}$[17, 18]: $\rho_0 = 0.148$ fm$^{-3}$.

<table>
<thead>
<tr>
<th></th>
<th>quark-matter EOS-I(II)</th>
<th>quark-matter EOS-III</th>
<th>nuclear-matter EOS-I</th>
<th>nuclear-matter EOS-II</th>
<th>nuclear-matter EOS-III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GA^2$</td>
<td>2.172</td>
<td>2.139</td>
<td>18.81</td>
<td>18.81</td>
<td>13.78</td>
</tr>
<tr>
<td>$G$(fm$^2$)</td>
<td>0.215</td>
<td>0.194</td>
<td>4.856</td>
<td>4.856</td>
<td>1.746</td>
</tr>
<tr>
<td>$G_v$(fm$^2$)</td>
<td>0</td>
<td>0</td>
<td>5.339</td>
<td>4.666</td>
<td>3.387</td>
</tr>
<tr>
<td>$G_{sv}$(fm$^8$)</td>
<td>0</td>
<td>0</td>
<td>-11.082</td>
<td>-6.583</td>
<td>-1.839</td>
</tr>
<tr>
<td>$c_v$</td>
<td>0</td>
<td>0</td>
<td>1.00</td>
<td>0.874</td>
<td>1.620</td>
</tr>
<tr>
<td>$c_{sv}$</td>
<td>0</td>
<td>0</td>
<td>0.50</td>
<td>0.297</td>
<td>1.639</td>
</tr>
<tr>
<td>$F_\pi$ (MeV)</td>
<td>90.4</td>
<td>93.0</td>
<td>90.4</td>
<td>90.4</td>
<td>93.0</td>
</tr>
<tr>
<td>$m_0$ (MeV)</td>
<td>313</td>
<td>313</td>
<td>939</td>
<td>939</td>
<td>939</td>
</tr>
<tr>
<td>$\Lambda$ (MeV)</td>
<td>626</td>
<td>654</td>
<td>387.7</td>
<td>387.7</td>
<td>553</td>
</tr>
<tr>
<td>$I_1(\Lambda/m_0)$</td>
<td>3.028</td>
<td>3.359</td>
<td>0.045</td>
<td>0.045</td>
<td>0.124</td>
</tr>
<tr>
<td>$I_2(\Lambda/m_0)$</td>
<td>1.098</td>
<td>1.162</td>
<td>0.0407</td>
<td>0.0407</td>
<td>0.104</td>
</tr>
<tr>
<td>$m^*/m_0$ $(\rho/\rho_0)$</td>
<td>1.59</td>
<td>1.62</td>
<td>2.5</td>
<td>2.4</td>
<td>8.0</td>
</tr>
<tr>
<td>$E_{Bq} = E_{BN}$ (MeV) $(\rho/\rho_0)$</td>
<td>27 (2.0)</td>
<td>72 (3.5)</td>
<td>80 (3.5)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

satisfaction properties

| $E/A - m_0$(MeV) | 0.099                 | 0.149                 | 0.148                 |
| $\rho$ (fm$^{-3}$) | 1.14                  | 1.30                  | 1.30                  |
| $\rho_f$ (fm$^{-1}$) | 0.796                 | 0.68                  | 0.89                  |
| $K$ (MeV)               | 264                   | 280                   | 339                   |

The relation between the coupling constants $G_s$, $G_v$, $G_{sv}$ and $c_v$, $c_{sv}$ is fixed as follows. From eqs. (27) and (28) we have $G_s = \pi^2/(2I_1(\Lambda'))$. From eqs. (64) and (11) we have $G_v = 1/4 c_v \tilde{V}_s = c_v \pi^2/(2 I_2(\Lambda'))$. From eqs.(53,55) we have $G_{sv} = -c_{sv}\pi^4 G_s/(I_2(\Lambda')I_1(\Lambda')) = c_{sv}\pi^6/(2I_2 I_1^2)$. 


Figure 1: Energy vs density in NJL Model

Figure 2: Effective Mass vs density in NJL Model
6.2 Quark Matter

As discussed in the last section, at some density, depending on the parameterisation used, the nucleon matter curve intercepts the quark matter curve. At high density the quark matter prevails. In order to study the confinement-deconfinement phase transition we make a Maxwell construction: at the phase transition the temperature, pressure and chemical potential are equal in both phases:

\[ H = Q; \quad T_H = T_Q; \quad p_H = p_Q. \]

In the present simplified approach we only require barionic charge conservation. For a single conserved charge the transition occurs at constant pressure [19]. In fig. 3 we plot the pressure versus the barionic density. The letters \( H \) and \( Q \) denote the end of the confined (hadronic) and beginning of the deconfined (quark) phases. We conclude that the density at which the confinement-deconfinement phase transition occurs is quite sensitive to the parameterisations used. We will not discuss the properties of the family of neutron stars predicted by the models discussed above. In order to do this study, it is important to implement both barionic and charge conservation in order not to get a region of instability between the minimum and maximum mass configurations. The kink in EOSII corresponds to the chiral symmetry restoration which occurs at a lower density than the confinement-deconfinement phase transition.

7 CONNECTION WITH OTHER MODELS

We would like to account for the value of the coefficient \( c_{sv} \), which provides a crucial contribution to the energy of nuclear matter: \( c_{sv} V_s^2 \rho^2 \). We will consider several models each of which has a scalar potential with volume integral \( V_s \). In the ENJL model, with a scalar-vector coupling \( c_{sv} = 0.297 \), we can fit the saturation density and energy of nuclear matter. However, this constant and also \( c_v \) are purely phenomenological, unlike the other constants used in the model. If, as was done in an earlier section, it is supposed that the constants are associated with the number of colors: \( c_{sv} = \frac{N_c - 1}{N_c + 1} = 0.5 \), then we get a somewhat smaller equilibrium density and slightly less binding. However, it would be desirable to find some kind of microscopic basis for the scalar-vector coupling constant.
7.1 Guichon quark meson coupling model

One promising possibility is the quark-meson coupling model proposed by Guichon [15]. In this model, the nucleon mass is generated dynamically using a MIT bag model. Let us define the dimensional coupling constant:

\[ B_s = \frac{g_s^2 \rho_0}{m_s^2} = \frac{\rho_0 V_s}{M} \quad (80) \]

Then in the Guichon model, we automatically obtain a saturating term which can be put into our form:

\[ W_{\rho^2} = c_{sv} B_s^2 \rho^2 M c^2 \quad (81) \]

where \( c_{sv} = 0.340 \frac{m_s c}{\hbar} r_{bag} \), and \( \hat{\rho} = \frac{\hat{\rho}}{\rho_0} \).

The coefficient 0.340 is a dimensionless constant which can be derived from the MIT bag model. [15] In that work, nuclear matter saturation and some properties of finite nuclei are fitted with a bag radius of \( r_{bag} = 0.6 fm \). This means that:

\[ c_{sv} = 0.340 \cdot \frac{313}{197} \cdot 0.6 = 0.324, \]

slightly larger than the value 0.297 obtained using the ENJL model. Their value of the scalar coupling \( B_s = 0.298 \) is slightly smaller than what is used in our NJL model: \( B_s = 0.332 \).

7.2 Relativistic Mean Field Models

In the non-linear and derivative coupling mean field models, we also get a term quadratic in the density. In the strong coupling limit, neglecting kinetic energy, we can write for the energy per particle:

\[ W = m - 1 + \frac{1}{2} B_v \hat{\rho} + \frac{(m - 1)^2}{2 B_s \hat{\rho} m^\alpha} \quad (82) \]

The effective mass is obtained by setting \( \frac{dW}{dm} = 0 \).

In the low density limit, the effective mass is given by:

\[ m = 1 - B_s \hat{\rho} + \frac{3}{2} \alpha B_s^2 \hat{\rho}^2 + \ldots \quad (83) \]

and

\[ W = \frac{1}{2} (B_v - B_s) \hat{\rho} + \frac{1}{2} \alpha B_s^2 \hat{\rho}^2 + \ldots \quad (84) \]

Making a comparison with Eq. 81, we obtain \( c_{sv} = \frac{1}{4} \alpha \). In the original derivative coupling model [8], we get \( \alpha = 2 \), and \( c_{sv} = 1 \), which seems too large. However, in the hybrid model developed by Glendenning et al [20], we have \( \alpha = 1 \), and \( c_{sv} = 0.5 \), and happens to agree (coincidence?) with our theoretical value based on the number of colors. Notice that, nevertheless, in parametrization III we find \( c_{sv} = 1.6 \).
7.3 Non-Relativistic Models

7.3.1 Need for Reduced Effective Mass

As has been pointed out, for nuclear matter, the core volume is only about 1% of the total volume, and here there have to be other ingredients for saturation in addition to the core and quantum kinetic energy. A typical repulsive core radius for the NN interaction is 0.4 fm, while the nuclear matter density is, \(0.15 \text{ fm}^{-3}\). Thus the fraction of the volume occupied by the core is roughly \(0.15 \times 0.4^3 = 0.01\). Thus while the core plays a role for nuclear saturation, it cannot be decisive. (By contrast, for liquid helium, the core volume is more than 10% of the total volume. This can be accounted for by the repulsion, together with quantum kinetic energy effects.) Thus there must be additional contribution for getting the right nuclear saturation properties. We will see that a reduced effective mass provides a simple mechanism for this. (However, there are other ways as well to accomplish this, such as many body interactions, exchange interactions, relativistic effects or higher order effects of the tensor forces.)

The average energy per nucleon can be written as follows, in the Hartree approximation:

\[
W = \frac{\rho}{2} \frac{\hbar^2}{m*} \int (\nabla \psi)^2 d^3 r + \frac{\rho}{2} \int V \psi^2 d^3 r \tag{85}
\]

Here, the Fermi kinetic energy has been neglected and the normalization of the wave function for the relative motion \(\psi\) is such that \(\psi(r) = 1\) for \(r > d\). The first term is, of course, the kinetic energy, which will be increased if the effective mass \(m^*\) is reduced. This expression uses the effective mass approximation, which implies that the single particle energy varies quadratically with the mass, even up to high momenta. More detailed many body G-matrix calculations seem to bear out the validity of this approximation. [21].

Now let us separate the interaction into short and long range parts. For the long range part, we can use perturbation theory, i.e. \(\psi = 1\). For the short range part, it is readily shown that, as a consequence of the Schroedinger equation, we have:

\[
\frac{\hbar^2}{m} \int_0^d (\nabla \psi)^2 d^3 r = - \int_0^d V \psi^2 d^3 r. \tag{86}
\]

This quantity is a kinetic energy of correlations, which we will denote by \(T_{\text{corr}}\). Thus a part of the integrals in Eq. (86) cancel. We are left with:

\[
W = \frac{\rho}{2} \frac{\hbar^2}{m} \left(\frac{m}{m^*} - 1\right) \int_0^d (\nabla \psi)^2 d^3 r + \frac{\rho}{2} \int_d^{\infty} V_L d^3 r. \tag{87}
\]

Here \(V_L\) denotes the long range potential, i.e. that part at distances larger than the separation distance \(d\). The effective mass is related to the scalar potential by:

\[
\frac{m}{m^*} - 1 = \frac{\rho V_s}{mc^2} \tag{88}
\]
at least for small density. Here, as before, $\bar{V}_s$ denotes the volume integral of the scalar potential. Then the term quadratic in the density can be written as:

$$W_{\rho^2} = \frac{1}{2}\rho^2 T_{\text{corr}} \frac{\bar{V}_s}{mc^2}$$

(89)

Let us make a connection with the ENJL. In the strong coupling low density approximation, we have:

$$W = \frac{1}{2}(-1 - c_{sv} + c_v) \bar{V}_s \rho + c_{sv} \frac{\bar{V}_s^2 \rho^2}{mc^2}$$

(90)

Equating the coefficients of the $\rho^2$ terms, we obtain:

$$c_{sv} = \frac{T_{\text{corr}}}{2 \bar{V}_s}$$

(91)

Thus our scalar-vector coupling constant can be calculated from a non-relativistic G-matrix model. We need only the volume integral of the attractive scalar potential, and the correlation kinetic energy.

### 7.3.2 Poeschl-Teller Potential

Here we have (expressing all energies in units of $\frac{4\pi\hbar^2}{m}$, and the radial coordinate in units of $\frac{N\hbar}{mc}$):

$$\bar{V}_s = N(N + 1) \int \text{sech}^2(r) r^2 dr = \frac{\pi^3}{12} N(N + 1) = \pi^2$$

for 3 colors.

For a local PT attraction and repulsion, i.e. for a PT potential of the form:

$$V(r) = \frac{\hbar^2 \beta^2}{M} [6 \text{csch}^2(\beta r) - 12 \text{sech}^2(\beta r)]$$

(92)

The ground state radial wavefunction, $u(r) = r \psi(r)$, which has zero energy, is given by:

$$u(r) = (\tanh r)^3$$

(93)

For this wavefunction, the separation distance is $d = 1.4 \text{fm}$, and the correlation kinetic energy, in units of $\frac{4\pi\hbar^2}{m}$, is $T_{\text{corr}} = 0.354$. This leads to a value of the scalar-vector coupling constant,

$$c_{sv} = \frac{T_{\text{corr}}}{2 \bar{V}_s} = \frac{0.354}{2 \times 9.87} = 0.0180$$

(94)

which is an order of magnitude smaller than needed. Thus, a repulsive core alone, even with a reduced effective nucleon mass, does not saturate nuclear matter at a low enough density. Something else is needed. Let us suppose now that we have only the scalar part of the PT potential for the local part of $V_{NN}$. Then there would be a deeply bound state, and it is the first excited state which is at zero energy. This state has a node. Let us remove this state by adding a repulsive separable (of course non-local) potential whose kernel is just the same as the wavefunction of the deeply bound state. If the repulsion is strong
enough, this deeply bound state will move up to be above zero energy, so the nodal state will become the ground state. The wavefunction of this state is:

\[ u(r) = \frac{3}{2} \tanh r + \frac{5}{2} \tanh^3 r \]  \hspace{1cm} (95)

For this case, the separation distance is slightly larger than for the local case \( d = 1.8 \text{ fm} \), and the correlation kinetic energy is much larger. That is since the wavefunction deviates much more from the free particle one \( u(r) = r \) than for the local case, even having the opposite sign for small \( r \). If the strength of the repulsion is taken to infinity, it is equivalent to imposing the orthogonality condition that any wavefunction, including the one at zero energy must be orthogonal to the deeply bound state. We obtain:

\[ T_{\text{corr}} = 8.62 \text{ and } c_{sv} = \frac{8.62}{2 \times 0.87} = 0.437 \]

This is even larger than our fit with the ENJL model. Probably this is because degree of non-locality imposed by the orthogonality condition is excessive. There is likely some non-locality in the NN interaction, but not as much as implied by orthogonality. Models with some non-locality have been investigated by Doleschall and Borbely [22].

8 Results and Conclusions

Three variants of the proposed model, defined by appropriate sets of the parameters, \( G_s, G_v, G_{sv} \), and \( \Lambda \), have been studied numerically. The properties of the model in these variants are exhibited at the Table 2. By horizontal lines we separate the values of the defining parameters, the vacuum properties, the properties at saturation and the properties at restoration of chiral symmetry. For the vacuum we show the values of the pion decay constant \( F_\pi \), the constituent quark mass \( m_0 \), the order parameter \( \langle \bar{u}u \rangle \). At saturation we show the nucleon density \( (\rho_N)_S \), Fermi momentum \( (p_F)_S \), where \( \rho_0 \) denotes the nuclear matter density, the constituent quark mass \( m^* \), the incompressibility of hadronic matter per nucleon \( K \), and the energy per nucleon \( (E/N)_S \). At restoration of chiral symmetry we show the nucleon density \( (E/N)_c \) and the Fermi momentum \( (p_F)_c \). The term in \( G_{sv} \), responsible for the density dependence of the effective coupling constant, plays an important role in pushing to higher energies the restoration of chiral symmetry and in lowering the incompressibility.

Acknowledgements

The authors are grateful to P. Alberto, M. Ericson, M. Malheiros, C. Shakin, C. Sousa, and Fl. Stancu for useful discussions and correspondence. This work was partially supported by FCT and FEDER under the projects POCTI/FIS/451/94 and POCTI/35308/FIS/2000.
References