Phase equivalent chains of Darboux transformations in scattering theory

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(April 22, 2002)

Abstract

We propose a procedure based on phase equivalent chains of Darboux transformations to generate local potentials satisfying the radial Schrödinger equation and sharing the same scattering data. For potentials related by a chain of transformations an analytic expression is derived for the Jost function. It is shown how the same system of S-matrix poles can be differently distributed between poles and zeros of a Jost function which corresponds to different potentials with equal phase shifts. The concept of shallow and deep phase equivalent potentials is analyzed in connection with distinct distributions of poles. It is shown that phase equivalent chains do not violate the Levinson theorem. The method is applied to derive a shallow and a family of deep phase equivalent potentials describing the \( ^1S_0 \) partial wave of the nucleon-nucleon scattering.

I. INTRODUCTION

The construction of potentials sharing the same scattering data, called in the literature phase equivalent potentials, is one of the most interesting applications of supersymmetric

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(SUSY) quantum mechanics [1] to the inverse scattering problem (see e.g. [2,3]). As it is known, the SUSY approach, when restricted to the derivation of new exactly solvable quantum problems, is basically equivalent to the Darboux transformations method (see e.g. [4]). Therefore we use SUSY and Darboux transformations as synonyms.

To our knowledge, the simplest way of constructing phase equivalent potentials by using Darboux transformations, is due to Sukumar [5]. It consists in a successive application of two transformations with equal factorization constants. The method has been considerably developed, up to an arbitrary modification of the discrete spectrum [6]. It has been extended to coupled channels [7] and used to derive deep and shallow phase equivalent potentials describing nucleon-nucleon [8] and nucleus-nucleus [9,10] collisions. In these applications both deep and shallow potentials have been constructed such as to share the same scattering data at all energies. Shallow potentials possess only ”physical” states while the deep potentials, in addition to the physical states also have ”unphysical” bound states which simulate the effect of the Pauli principle (for details see e.g. [11–13]).

Based on Darboux transformations, here we propose a new way to derive deep and shallow phase equivalent potentials. Our aim is to improve over the practical procedures of Refs. [5] and [6–10]. By replacing a chain of Darboux transformations by an equivalent $N$th order transformation and using a Wronskian formula recently obtained in [14] (see also [15]) we use a determinant approach to construct phase equivalent potentials. When the transformation functions are sufficiently simple (e.g. they are solutions of the free particle Schrödinger equation) this approach happens to be more efficient than the one of previous works [5–10]. The advantages of our method will be clearly seen in its application to the nucleon-nucleon scattering potentials considered here. In particular the potential tail will be cured from the unwanted oscillations obtained in Ref. [6].

As it is known (see e.g. [8]) there are both deep and shallow potentials corresponding to the same value of the partial wave $l$. The value of $l$ defines the long distance behaviour of the potential [2]. The short distance behaviour

$$V(x) \to \frac{\nu(\nu+1)}{x^2}, \quad x \to 0$$

(1.1)

depends on the parameter $\nu$ which we shall call the strength of the singularity at short distances or simply the singularity strength. Following Ref. [8] we generally assume that $\nu$ is different from $l$. In this work we consider only $l$-preserving transformations and restrict our discussion to the $l = 0$ case, although most of the derived formulae are valid in the general case. In a subsequent paper [16] we shall consider $l$-changing transformations which will allow to introduce higher partial waves.

We assume that no Coulomb interaction is present in the potential i.e. we are dealing with the Schrödinger equation on the half axis, $x \in [0, \infty)$, with the potential $V(x)$ satisfying the condition (1.1) and at long distances going to zero faster than $x^{-2}$. For every eigenvalue $E$ the Schrödinger equation with such potentials has two linearly independent solutions with different behavior at short distances, one which behaves as $\sim x^\nu$, called regular solution, and the other as $\sim x^{-\nu-1}$, called singular (or irregular) solution. In general Darboux transformations, which use irregular solutions as transformation functions, do not preserve the property of a solution to remain regular at the origin. As a consequence, the change in the Jost function and the phase shift becomes difficult to control since one has
to know both the Jost solution of the initial equation and the derivative of this solution at
the origin. As far as we know this problem has not been yet solved in the general context.
But we have found that there exist particular chains of transformations where the derivative
of the Jost solution of the initial equation does not appear in the expression of the Jost
function for the transformed Hamiltonian. In this case one can get a closed expression for
the transformed Jost function in terms of the initial Jost function and a rational function
of the momentum $k$ only. Potentials having Jost functions related in such a way are known
in the literature as potentials of the Bargmann class (see e.g. [2]).

Once we know how the Jost function is changed by a chain of Darboux transformations
we can select chains that produce potentials with the same $S$-matrix and hence with the same
phase shift. Such potentials are called in the literature phase equivalent although a more
appropriate term could be isophase potentials since they give exactly the same phase shift at
all energies. Here we shall use both terms. So, to get phase equivalent potentials we propose
to use phase equivalent chains instead of an integral form of Darboux transformations used
in [5–10]. We start from the same exactly solvable reference potential and the action of each
chain will result in potentials always having the same $S$-matrix. Finally we note that this
approach is basically equivalent to the inverse scattering method where the kernel of the
Fredholm operator involved in the Gelfand-Levitan-Marchenko equation is degenerate [2] or
to a supersymmetric transformation [5–10], but it is more direct and has more computational
advantages.

The paper is organized as follows. In the next section we recall the Darboux transfor-
mation method in the context of scattering theory. In Sec. III we introduce special chains for
which a closed expression for the transformed Jost function is derived. Then we select chains
which lead to families of phase equivalent (or isophase) potentials. Sec. IV is devoted to an
application of our method to the derivation of deep and shallow phase equivalent potentials
for the nucleon-nucleon scattering. Conclusions are drawn in the last section. Details of
important analytic calculations are gathered in the Appendix.

II. DARBOUX TRANSFORMATION METHOD

The method of Darboux transformations, well known in soliton theory [17], finds now
more and more applications in other fields of theoretical and mathematical physics [18].
In this section we recall the definition of Darboux transformations and give their main
properties necessary in subsequent sections.

Essentially, the method consists in getting solutions $\varphi$ of one Schrödinger equation

$$ h_1 \varphi = E \varphi, \quad h_1 = -\frac{d^2}{dx^2} + V_1(x), $$

(2.1)

when solutions $\psi$ of another equation

$$ h_0 \psi = E \psi, \quad h_0 = -\frac{d^2}{dx^2} + V_0(x), $$

(2.2)

are known. This is achieved by acting with a differential operator $L$ of the form

$$ \varphi = L \psi, \quad L = -d/dx + w(x), $$

(2.3)
where the real function \( w(x) \), called superpotential, is defined as the logarithmic derivative of a known solution of (2.2) denoted by \( u \) in the following. One has

\[
 w = u'(x)/u(x), \quad h_0 u = \alpha u,
\]

with \( \alpha \leq E_0 \), where \( E_0 \) is the ground state energy of \( h_0 \) if it has a discrete spectrum or the lower bound of the continuum spectrum otherwise. The function \( u \) is called transformation or factorization function and \( \alpha \) its factorization constant or factorization energy. The potential \( V_1 \) is defined in terms of the superpotential \( w \) as

\[
 V_1(x) = V_0(x) - 2w'(x).
\]

The operators \( L \) and \( L^+ = d/dx + w(x) \) establish a one-to-one correspondence between two dimensional subspaces of solutions of Eqs. (2.1) and (2.2) with a given \( E \neq \alpha \). When \( E = \alpha \) the transformed equation (2.1) has \( v = 1/u \) and \( \tilde{v} \) as linearly independent solutions. The latter can be found by using the Wronskian property \( W(v, \tilde{v}) = 1 \) which gives

\[
 \tilde{v} = v \int_{x_0}^{x} v^{-2}(x) dx,
\]

where \( x_0 \) is an arbitrary constant. Hence, the knowledge of all solutions of the initial equation provides the knowledge of all solutions of the transformed equation and in particular of all physical solutions in the usual quantum mechanical sense.

Since the above procedure does not depend on a particular choice of the potential \( V_0 \) the transformed Hamiltonian \( h_1 \) can play the role of the initial Hamiltonian for the next transformation step. In such a way one gets a chain of exactly solvable Hamiltonians \( h_0, h_1, \ldots h_N \) with the potentials \( V_0, V_1, \ldots V_N \). To avoid any confusion we mention that everywhere, except for especially mentioned cases, we shall use subscripts to distinguish between quantities related to different Hamiltonians, \( h_0, h_1, \ldots \) and shall omit them when discussing general properties regarding all Hamiltonians.

A first order Darboux transformation operator \( L_{j,j+1} \) as defined by (2.3) relates two Hamiltonians \( h_j \) and \( h_{j+1} \). If one is not interested in the intermediate Hamiltonians \( h_1, \ldots, h_{N-1} \) and all factorization energies are chosen to be different from each other the whole chain may be replaced by a single transformation given by an \( N \)th order transformation operator denoted by \( L^{(N)} \), defined as a succession of \( N \) first order transformations. A compact representation of this operator is given by [19]

\[
 \psi_N(x, k) = L^{(N)} \psi_0(x, k) = W(u_1, \ldots, u_N, \psi_0(x, k)) W^{-1}(u_1, \ldots, u_N),
\]

where \( \psi_0(x, k) \) is a solution of Eq. (2.2) corresponding to the energy \( E = k^2 \) and \( \psi_N(x, k) \) satisfies

\[
 h_N \psi_N(x, k) = E \psi_N(x, k), \quad E = k^2.
\]

The transformation functions, although labeled by a subscript, are eigenfunctions of the initial Hamiltonian.
\[ h_0u_j(x) = \alpha_j u_j(x), \quad \alpha_j = -a_j^2. \]  

They should be chosen such as the Wronskian \( W(u_1, \ldots, u_N) \) is nodeless in the interval \((0, \infty)\). This condition guarantees the absence of singularities in the potential

\[ V_N = V_0 - 2 \frac{d^2}{dx^2} \log W(u_1, \ldots, u_N), \]  

defining the Hamiltonian \( h_N \) of (2.8), inside this interval. The formula (2.7) is valid for any \( E = k^2 \) except for \( k = ia_j \) \((j = 1, \ldots, N)\). For these values of \( k \) the corresponding solutions are

\[ \psi_N(x, ia_j) = W^{(j)}(u_1, \ldots, u_N)W^{-1}(u_1, \ldots, u_N), \quad j = 1, \ldots, N, \]  

where \( W^{(j)}(u_1, \ldots, u_N) \) is the \((N-1)\)th order Wronskian constructed from \( u_1, \ldots, u_N \) except for \( u_j \).

In general, a chain of \( N \) transformations can have a number of elements corresponding to transformation functions with identical factorization constants. In the simplest \( N = 2 \) case where the first transformation is implemented by the function \( u_1 = u \) and the second by \( v = 1/u \) one gets back the initial potential \( V_0 \). This means that the operator \( L^+ \) realizes the transformation in the opposite direction. More generally, the second transformation can be made with the linear combination \( v_1 = cv + \tilde{v} \) where \( \tilde{v} \) is given by (2.6). The action of such a chain leads to an eigenfunction of \( h_2 \) of the form

\[ \psi_2(x, k) = \psi_0(x, k) - \frac{u(x)}{W_2(x)} \int_{x_0}^x u(t)\psi_0(t, k)dt, \]  

where

\[ W_2(x) = c + \int_{x_0}^x u^2(t)dt. \]  

Choosing \( x_0 = 0 \) (or \( x_0 = \infty \)) together with the condition \( c \geq 0 \) (or \( c \leq 0 \)) guarantees a nodeless \( W_2(x) \) inside the interval \((0, \infty)\). The integral representation (2.12) based on two subsequent transformations with the same factorization constants was used in Refs. [5–10] for getting phase equivalent potentials. We now describe an alternative approach which has the advantage that the above \( N = 2 \) chain of transformations can be included in a larger chain as a subchain.

The potential \( V_2(x) \) is a particular case of (2.10) where \( W(u_1, \ldots, u_N) \) is given by \( W_2(x) \) of (2.13). It has been shown [14] that the function \( W_2 \) is nothing else but the limiting value of the ratio \( W(u_1, u_2)/\alpha_1 - \alpha_2 \) when \( u_2 \to u_1 \). This fact, together with a property of the Wronskians established in [14,15], which allows to reduce the order of the determinants involved in calculations, implies that one can use the same formulae (2.7) and (2.10) also in the case of pairwise identical factorization constants but the Wronskians should be calculated with the help of another representation. If the Wronskian in (2.7) is of even order, \( N = 2m \), it becomes proportional to an \( m \times m \) determinant.
For an odd order Wronskian, \( N = 2m + 1 \), one has

\[
W(\psi, u_1, \ldots, u_{2m}) = \prod_{l \neq j} (\alpha_l - \alpha_j) \prod_{k=1}^{2m} (\alpha_{2k} - E) 
\]

\[
\begin{vmatrix}
\psi & W_{E, \alpha_2} & W_{E, \alpha_4} & W_{E, \alpha_6} & \cdots & W_{E, \alpha_N} \\
\psi & W_{\alpha_1, \alpha_2} & W_{\alpha_1, \alpha_4} & W_{\alpha_1, \alpha_6} & \cdots & W_{\alpha_1, \alpha_N} \\
u_1 & W_{\alpha_3, \alpha_2} & W_{\alpha_3, \alpha_4} & W_{\alpha_3, \alpha_6} & \cdots & W_{\alpha_3, \alpha_N} \\
u_3 & W_{\alpha_5, \alpha_2} & W_{\alpha_5, \alpha_4} & W_{\alpha_5, \alpha_6} & \cdots & W_{\alpha_5, \alpha_N} \\
u_5 & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{N-1} & W_{\alpha_{N-1}, \alpha_2} & W_{\alpha_{N-1}, \alpha_4} & W_{\alpha_{N-1}, \alpha_6} & \cdots & W_{\alpha_{N-1}, \alpha_N} \\
\end{vmatrix}.
\tag{2.15}
\]

where

\[ W_{\alpha_k, \alpha_l} = \frac{W(u_k, u_l)}{\alpha_k - \alpha_l}, \quad W_{E, \alpha_l} = \frac{W(\psi, u_l)}{E - \alpha_l}. \tag{2.16} \]

The product \( \prod_{l \neq j} (\alpha_l - \alpha_j) \) introduced above contains only the factors \((\alpha_l - \alpha_j)\) appearing in \( W_{\alpha_k, \alpha_l} \). For identical factorization constants the function \( W_{\alpha_k, \alpha_l} \) has to be calculated from the formula (2.13).

### III. PHASE EQUIVALENT CHAINS OF TRANSFORMATIONS

Our method is based on a choice of transformation functions which allows to transform the Jost function through a special chain of Darboux transformations. We follow the usual definitions of the scattering theory which can be found for example in Refs. [2,3].

**A. Jost function for a special chain of transformations**

Let us choose the following set of eigenfunctions of the Hamiltonian \( h_0 \)

\[ v_1(x), \ldots, v_\nu(x), u_{\nu+1}(x), v_{\nu+1}(x), \ldots, u_n(x), v_n(x), \quad \nu \geq 0, \tag{3.1} \]

\[ h_0 u_j(x) = -a_j^2 v_j(x), \quad h_0 v_j(x) = -b_j^2 v_j(x), \quad \text{Im } a_j = \text{Im } b_j = 0, \tag{3.2} \]

as transformation functions for the Darboux transformation of order \( N = 2n + \nu \). The role of the first \( \nu \) functions \( v_j(x) \) which are absent in (3.1) for \( \nu = 0 \) will be clear below. Let
the a’s and the b’s be different from each other and choose them such as the corresponding factorization energies be smaller than the ground state energy of \( h_0 \) when it has a discrete spectrum or less than the lower bound of the continuous spectrum otherwise.

We distinguish between the functions \( u \) and \( v \) from their behaviour at the origin. The functions \( v \) are regular, \( v_j(0) = 0 \), and hence are uniquely defined up to a constant factor which is not essential for our purpose. They form a regular family and are exponentially increasing at large \( x \) and therefore it is natural to restrict all b's to be positive. The functions \( u_j \) are irregular at the origin, \( u_j(0) \neq 0 \), and from a singular family. They are defined as linear combinations of two Jost solutions \( f(x, \pm k) \):

\[
 u_j(x) = A_j f(x, -ia_j) + B_j f(x, ia_j). \tag{3.3}
\]

When \( A_j \neq 0 \) the functions \( u_j \) increase exponentially at large \( x \) provided \( a_j > 0 \). At fixed \( a_j \) they form a one-parameter \( (A_j/B_j) \) family. The case \( A_j = 0, B_j \neq 0 \), with \( a_j > 0 \) is equivalent to the case \( A_j \neq 0, B_j = 0 \), with \( a_j < 0 \). They correspond to a decreasing exponential function (3.3), uniquely defined (up to a constant factor). In this case in order to stress the asymptotical decrease of this function we allow \( a_j \) to be negative and set, \( B_j = 0, A_j \neq 0 \). So, for \( a_j > 0 \) the functions (3.3) have an increasing asymptotic behavior and for \( a_j < 0 \) their asymptotic behavior is opposite. It is useful to note that if \( A_j/B_j = F_0(ia_j)/F_0(-ia_j) \) the function (3.3) becomes regular. In the chain of functions (3.1) we shall allow to move some functions from the singular family to the regular family. This preserves the number of functions \( N \) but changes the value of \( \nu \). The effect of such a modification of a chain will be discussed below. The particular case where the initial potential is zero and \( A_j = B_j \) has been considered earlier in Ref. [20].

In the Appendix we show that the chain of transformations with the transformation functions (3.1) increase the singularity parameter of the initial potential by \( \nu \) units, \( \nu_0 \to \nu_N = \nu_0 + \nu \), and the transformed Jost function \( F_N(k) \) is related to the initial function \( F_0(k) \) by

\[
 F_N(k) = F_0(k) \prod_{j=1}^{\nu} \frac{k}{k + i b_j} \prod_{j=\nu+1}^{n} \frac{k - i a_j}{k + i b_j}. \tag{3.4}
\]

Since a Jost function is analytic in the upper half of the complex \( k \)-plane (see [2,3]) all b’s must be positive whereas the a’s can have any sign, which agrees with our sign convention for \( a_j \) and \( b_j \) and every positive \( a_j \) corresponds to a discrete level \( E_j = -a_j^2 \) of \( V_N \). A remarkable property of the relation (3.4) is that it is independent of the ratio \( A_j/B_j \) with \( A_j \) and \( B_j \) from the definition (3.3). But the potential \( V_N \) does depend on this ratio. Hence, keeping \( F_N(k) \) fixed but varying \( A_j/B_j \) such as \( u_j \) remains irregular we can obtain a family of isospectral potentials. Nevertheless the potential \( V_N \) belongs to this isospectral family only if \( A_j \neq 0 \) and the function \( u_j(x) \) increases at large \( x \). The choice \( A_j = 0 \) is equivalent to changing the sign of \( a_j \) and leads to losing the level \( E = -a_j^2 \) in the potential \( V_N \). This property does not have an analogue in the usual scheme of SUSY transformations [18].

## B. Phase equivalent chains

Now we turn to the construction of phase equivalent or isophasic potentials. Such potentials have identical S-matrices which are defined in terms of the Jost function by
\[ S(k) = e^{2i\delta(k)} = \frac{F(-k)}{F(k)}. \] (3.5)

Therefore if for a real \( k \) two Jost functions differ from each other by a real factor they correspond to isophase potentials. In particular, these potentials can have equal Jost functions. Taking into consideration the discussion at the end of the previous subsection we see that by changing the ratio \( A_j/B_j \), which corresponds to a discrete level of the potential \( V_N \), we can get not only isospectral but also isophase potentials. As it is clear from the Appendix, the potentials are isospectral provided the function \( u_j \) remains singular. If it moves from the singular to the regular family, the transformed potential does not belong to the family of isospectral potentials any more but as we show below it remains in the family of isophase potentials.

Let us suppose for example that the function \( u_1(x) \) becomes regular from being singular. Then for a regular \( u_1 \) the Jost function of the potential \( V_N(x) \) corresponds to (3.4) with \( \nu = 0 \) and for a singular \( u_1 \) it should be calculated by the same formula where \( \nu = 2 \) and \( b_1 \) is replaced by \( a_1 \). From here it follows that the ratio of these two Jost functions is equal to \( k^2/(k^2 + a_1^2) \) which is real for a real \( k \) and hence these two chains of transformations are isophase. It is also clear that if \( \nu \) functions from the singular family move to the regular family (i.e. \( \nu \) zeros of the Jost function are transformed into its poles) the potential \( V_N \) loses \( \nu \) discrete levels, and acquires an additional singularity strength equal to \( 2\nu \), but remains in the family of isophase potentials. Furthermore, this procedure is completely reversible. If one has a singular potential with \( \nu \geq 2 \) then one of the poles of its Jost function can be transformed into a zero in the upper half of the complex \( k \)-plane which leads to the creation of a discrete level and a decrease of the singularity strength \( \nu \rightarrow \nu - 2 \). This procedure can be repeated either up to the point where the potential loses all discrete levels and all free parameters, becoming uniquely defined by its phase shift (see e.g. [2,3]) or up to the point where \( \nu \) becomes equal to zero or one. To avoid additional singularities inside the interval \((0, \infty)\) one has to take these poles such as the additional level be located below the existing discrete levels. The latter restriction is well known in the SUSY approach [18].

The above method of deriving isophase potentials is not unique. As it is known one can create a new energy level independent of the existence of poles of the Jost function, at any desirable position even in the middle of the continuous spectrum, and preserving the \( S \)-matrix. For this purpose one can use a two-step Darboux transformation with the same factorization energy [5–8]. After a new level has been created the singularity of the potential loses two units \( \nu \rightarrow \nu - 2 \) [6]. This procedure can be repeated up to a limit where the singularity parameter becomes equal to either zero or one. For a regular potential or a potential with \( \nu = 1 \) no possibility exists to create a level by a phase preserving procedure. In this context the smallest possible value is \( \nu = 2 \) when the potential with an additional discrete level becomes regular after transformation. According to Ref. [14] a chain of two transformations with coinciding factorization constants is equivalent to considering the limit \( a_{j+1} \rightarrow -a_j \) and using the Wronskian formulae (2.14) and (2.15). Without losing generality let us consider \( a_2 \rightarrow -a_1 \) for our system of transformation functions (3.1) with \( \nu = 0 \). In this limit the Jost function (3.4) where we set \( \nu = 0 \)

\[ F_N(k) = F_0(k) \frac{k^2 + a_1^2}{(k + ib_1)(k + ib_2)} \prod_{j=3}^n \frac{k - ia_j}{k + ib_j}. \] (3.6)
has no singularity. The corresponding potential $V_N$ can be considered as a result of the application of the above two-step phase preserving SUSY transformation on a singular potential $V_{N-2}$ generated by the system of transformation functions $v_1, v_2, v_3, u_3, \ldots, u_n, v_n$ whose Jost function $F_{N-2}$ is given by the same formula (3.4) with $\nu = 2$. Finally we conclude that $V_{N-2}$ is isophase with $V_N$ since the ratio $F_N/F_{N-2} = (k^2 + a_1^2)/k^2$ is real.

C. The $S$-matrix poles

Once the Jost function $F(k)$ is known one can calculate the phase shift $\delta(k)$ by using the definition (3.5). The asymptotic behavior of the scattering wave function of the potential $V_N$ with $l=0$ and singularity strength $\nu$ at long distances is

$$\psi_N(x, k) \sim \sin(kx - \frac{1}{2} \nu \pi + \delta_N(k)), \quad x \to \infty.$$  \hspace{1cm} \text{(3.7)}

In our approach $F_N(k)$ differs from $F_0(k)$ by a rational function of momentum $k$ and therefore the expression for $\delta_N(k)$ becomes rather complicated when the number of transformation functions is sufficiently large. An equivalent expression which is more convenient for practical calculations is

$$\delta_N(k) = \delta_0(k) - \sum_{j=\nu+1}^{n} \arctan(k/a_j) - \sum_{j=1}^{n} \arctan(k/b_j),$$  \hspace{1cm} \text{(3.8)}

in agreement with the result obtained in [6]. From (3.8) it follows that the $S$-matrix acquires additional poles at $a_j$ and $b_j$ through the application of the transformation $L^{(N)}$ consistent with the discussion of the previous section.

For $l=0$ the effective range expansion is (see e.g. [21])

$$\lim_{k \to 0} k \cot \delta(k) = -\frac{1}{a} + \frac{1}{2} r_0^2 - P r_0^3 k^4 + \ldots,$$  \hspace{1cm} \text{(3.9)}

where $a$ is the scattering length and $r_0$ effective range. Expanding the phase shift (3.8) in a power series one obtains

$$\frac{1}{a^{(N)}} = \frac{1}{a^{(0)}} + \left[ \sum_{j=\nu+1}^{n} a_j^{-1} + \sum_{j=1}^{n} b_j^{-1} \right]^{-1},$$  \hspace{1cm} \text{(3.10)}

and

$$r^{(N)} = r^{(0)} + \frac{2a^{(N)}}{3} \left[ 1 - a^{(N)-3} \left( \sum_{j=\nu+1}^{n} a_j^{-3} + \sum_{j=1}^{n} b_j^{-3} \right) \right].$$  \hspace{1cm} \text{(3.11)}

These formulae can be used to calculate theoretical values to be compared with experimental results, as for example those obtained in the neutron-proton scattering experiments. But first one has to fit the parameters $a_j$ and $b_j$ in (3.8) such as to reproduce the phase shift data as close as possible. As a result of the fit one gets a sequence of $n_+$ positive and $n_-$ negative poles of the $S$-matrix all having absolute values below the pole associated with the
ground state of $h_0$ if this has a discrete spectrum or below the lower bound of its continuum spectrum otherwise. (We shall comment below on the cases when this is not so.) Every negative pole is associated with a singular transformation function decreasing at infinity and belonging to the singular family of (3.1). This is just the Jost solution of the initial equation at $k_j = ia_j$ which is uniquely defined. A positive pole corresponds to an increasing transformation function (3.3), which can be either regular or singular at $x = 0$. When regular it is uniquely defined, whereas singular functions form a one-parameter family.

According to the choice of transformation functions (3.1) our method is restricted by the condition $n_- \leq n_+$ and one has to distribute the functions related to positive poles among regular and singular families such as the number of singular transformation functions does not exceed the number of regular ones. This means that at least $n_-$ functions related with positive poles should be regular. For the remaining positive poles there are different possibilities either related to the regular $\{v_j\}$ family or to the singular $\{u_j\}$ family. All these possibilities correspond to the same phase shift and hence will give different phase equivalent chains of transformations. When all the remaining positive poles are associated with the regular family the potential will acquire the largest singularity strength $\nu = n_+ - n_-$ and according to the formula (3.4) will have no additional discrete levels with respect to $V_0$. In this case every function from the system (3.1) is uniquely defined and the potential $V_N$ has no free parameters. This agrees with the well-known statement of the inverse scattering method (see e.g. [2,3]) that in the absence of discrete spectrum a local potential is uniquely defined by its phase shift. Such potentials are called in the literature shallow potentials.

When a maximal number of positive poles (equal to $\gamma = (n_+ - n_-)/2$ for $N$ even and $\gamma = (n_+ - n_- - 1)/2$ for $N$ odd) is included in the singular family every such pole corresponds to an additional discrete level of $V_N$ and this potential will acquire the smallest singularity strength $\nu = 0$ (regular potential) for $N$ even and $\nu = 1$ for $N$ odd and $\gamma$ additional discrete levels. Every positive pole $a_j$ associated with a singular function $u_j$ will correspond to a discrete level $E_j = -a_j^2$ of $V_N$. According to (3.3) the function $u_j$ has a free parameter $A_j/B_j$. Thus we obtain a $\gamma$-parameter family of isophase and isospectral potentials. Such potentials are called in the literature deep potentials. Of course there are also intermediate possibilities. When $n_+ = n_-$ or $n_+ = n_- + 1$ one can construct only a deep (regular if $n_+ = n_-)$ potential without additional discrete levels. For this potential phase preserving transformations which at the same time do not change the angular momentum $l$ are impossible.

When $\nu = n_+ - n_- \geq 2$ one can construct a $2\gamma$-parameter family of isophase potentials by introducing $\gamma$ energy levels into the shallow potential. For this purpose one has to use a chain of $N + 2\gamma$ transformations, phase equivalent to the initial chain of $N$ transformations, to which $2\gamma$ transformation functions with pairwise equal factorization constants $a_j^2 = b_j^2$ should be added. To calculate the Wronskian of such a system one has to use either the formula (2.14) or (2.15). The potentials $V_N$ form a $2\gamma$-parameter family where $\gamma$ energy levels play the role of parameters. The other $\gamma$ parameters correspond to the constant $c$ of (2.13).

It may happen that the absolute values of some poles is above the position defined by an excited state energy of $h_0$. Then using a phase preserving procedure, we first have to delete all such levels including ground state level from the spectrum of $h_0$ and then use the above described technique. This means to include additional pairs of functions with equal factorization constants in the system of transformation functions (3.1). The
corresponding energies have to coincide with the positions of the discrete levels to be deleted, which evidently does not change the fitted phase shift. The Wronskian of the entire system of transformation functions has to be calculated by using either (2.14) or (2.15).

When the number of negative poles exceeds the number of positive ones, the problem becomes more sophisticated. In some cases it can be solved by introducing transformations changing the orbital angular momentum \( l \). We are planning to discuss these problems in a subsequent paper [16].

In this subsection all potentials have an exponential decrease at infinity, provided the initial potential corresponds to \( l = 0 \). Therefore all transformed potentials also correspond to \( l = 0 \). An useful remark is that when fitting an experimental phase shift one can consider either real poles of the \( S \)-matrix or pairwise mutually conjugated poles. In either case they should be associated with the regular family of the system (3.1). The corresponding Wronskian is either real or purely imaginary, since the transformation functions are mutually conjugated. The potential difference \( V_N - V_0 \) resulting from (2.10) is real in either case.

D. The Levinson theorem

Here we show that the formula (3.8) is in agreement with the Levinson theorem in its generalized form [22] (see also [8]). For a potential satisfying the condition (1.1) and decreasing faster than \( x^{-2} \) at long distances this theorem states that

\[
\delta(0) - \delta(\infty) = [N + \frac{1}{2}(\nu - l)]\pi ,
\]

(3.12)

where \( N \) is the number of bound states \( l \) the partial wave and \( \nu \) the singularity strength. In our case we have to set \( l = 0 \). Suppose that this theorem holds for the initial potential \( V_0 \). We shall show that in this case it also holds for the transformed potential \( V_N \). Since the right hand side of (3.12) is a linear form in the variables \( N \) and \( \nu \) it remains to show that this formula is valid for the case where \( \delta(0), \delta(\infty), N \) and \( \nu \) represent differences between the same quantities associated with the potentials \( V_0 \) and \( V_N \). In what follows in this subsection, by these quantities we understand the corresponding increments.

The potential \( V_N \) is supposed to be obtained from \( V_0 \) by an \( l \)-preserving chain of transformations with transformation functions (3.1). When all additional positive poles of the \( S \)-matrix are related to the regular family, the potential \( V_N \) has the same number of bound states as \( V_0 \) but its singularity strength is \( \nu_N = \nu = n_+ - n_- \). Using (3.8) one finds that \( \delta(0) = 0, \delta(\infty) = -\pi(n_+ - n_-)/2 \) which is in the agreement with (3.12) at \( N = 0 \). If now we transform \( N \leq (n_+ - n_-)/2 \) regular functions into singular ones the potential \( V_N \) acquires \( N \) additional discrete levels and loses \( 2N \) units of singularity strength which also is in agreement with the relation (3.12). Hence, we conclude that \( l \)-preserving chains of transformations with transformation functions (3.1) do not violate the Levinson theorem.

IV. APPLICATION TO NEUTRON-PROTON ELASTIC SCATTERING

Here we apply our method to derive shallow and deep phase equivalent potentials, both reproducing the neutron-proton \((np)\) experimental elastic scattering phase shift for the \(^1S_0\) partial wave in the laboratory energy interval 0-350 MeV. The experimental values
taken from Ref. [23] are displayed in Table I. Our choice \( V_0(x) = 0 \) which allows to use solutions of the free particle Schrödinger equation simplifies the calculations considerably. The theoretical phase shift also displayed in Table I under \( \delta_6 \) is obtained from the formula (3.8) with \( \delta_0(k) = 0 \) and \( \nu = 0 \) where we had to fix \( n \) optimally and search for \( a_j \) and \( b_j \) (\( j = 1,2,\ldots,n \)). By using the standard Mathematica package NonlinearRegress [24] we found that we need six \( S \)-matrix poles in order to fit the experimental phase shift very closely. Thus we have \( N = 2n = 6 \) and the poles are

\[
\begin{align*}
  a_1 &= -0.0401, \quad a_2 = -0.7540, \quad a_3 = 4.1650, \\
  b_1 &= 0.6152, \quad b_2 = 2.0424, \quad b_3 = 4.6000,
\end{align*}
\]

in fm\(^{-1} \) units. In the notation of Subsection III C we have \( n_- = 2 \) and \( n_+ = 4 \). From Eqs. (3.10) and (3.11) with \( 1/a^{(0)} = 0 \) and \( r^{(0)} = 0 \) we obtain \( a^{(6)} = -23.7032 \) fm and \( r^{(6)} = 2.6235 \) fm. The comparison with the recommended values \( a_{\text{exp}} = -23.721 \pm 0.020 \) fm and \( r_{\text{exp}} = 2.658 \pm 0.062 \) fm [25] shows that the phase shift is sufficiently well described by the system of poles (4.1). The negative poles correspond to transformation functions of type \( u_j = \exp(a_j x) \). Positive poles can correspond either to regular solutions \( u_j = \sinh(a_j x) \) or to

\[
  u_j = A_j \exp(a_j x) + B_j \exp(-a_j x),
\]

as implied by (3.3). According to the previous section the poles \( b_j \) correspond to regular solutions \( v_j = \sinh(b_j x) \).

We can now derive isophase potentials associated with \( \delta_6 \) by using (2.10). The system of poles (4.1) can correspond either to a one level potential (deep potential) with \( E_0 = -a_3^2 \) or to a shallow potential which has no discrete levels. The latter is the desired case and can be obtained when the function \( u_3 \) is a sinh function. Fig. 1 shows our shallow potential \( V_6 \) obtained in this way together with the Reid68 potential [26]

\[
V_{\text{Reid68}}(x) = \left[ -10.463 \exp(-0.7x) - 1650.6 \exp(-2.8x) + 6484.2 \exp(-4.9x) \right]/(0.7x),
\]

and the potential of Ref. [6]. The dots represent an updated version [23] of the Reid soft core potential [26]. In contrast with Ref. [6] our potential has a correct short range behaviour \( V_6(x) \to 6x^{-2} \) corresponding to the case \( \nu = n_+ - n_- = 2 \) (the potential of Ref. [6] has \( \nu = 1 \)). In Fig. 2 we present the tail of the absolute value of various potentials in a logarithmic scale. One can see that \( V_6 \) has a smooth behaviour consistent with the one-meson exchange theory as it is the Reid’s potential [26], in contradistinction to the SUSY potential from Fig. 3 of Ref. [8] which displays an undesired oscillation around \( r = 2 \) fm\(^1 \). Interestingly, our potential is very close to the Reid68 potential (4.3) in a large interval up to 5 fm.

Actually the choice (4.2) for \( u_3 \) gives an isospectral and isophase family of \( V_6 \) potentials with the bound state energy \( E_0 \) fixed. In Fig. 3 we display a few deep potentials, members of this family and isophase with the shallow \( V_6 \) potential. These potentials differ from each

\footnote{The potential drawn in Figs. 1 and 2 under Ref. [8] represents the output of Eq. (32) of this reference.}
other only by the value of the coefficient $A'_3 = A_3/B_3$ in (4.2). One can see that by increasing $A'_3$ the potential well shifts to the right without changing its depth. The bound state with $E_0 = -a_3^2$ follows the movement, being localized at larger and larger values of $x$. A similar behavior has been observed by Bargmann [27] already in 1949 (see also [28]). When $A'_3$ approaches $-1$ the well concentrates near the origin and becomes very narrow and deep. The ground state becomes well localized near the origin. At very short distances a barrier appears before the well which grows when $A'_3 \to -1$. In the limit $A'_3 = -1$ the well becomes infinitely narrow and completely collapses, being replaced by an infinitely high barrier which becomes a repulsive core, as for a shallow potential.

One can also construct a deep potential isophase with the shallow potential $V_6$ but having a discrete level at a desired place. First one has to delete the bound state $E_0 = -a_3^2$ of $V_6$ following the procedure described in Sec III. Next one has to use a 2-parameter family of isophase potentials (Subsection III C). This is possible by adding two new transformation functions, which gives a total of 8 functions. The new functions $u_4$ and $v_4$ must have $a_4 = -b_4$ ($a_4 < 0$) in order to give $\delta_8(k) = \delta_6(k)$. It is enough to take $u_4 = \exp(b_4x)$ and calculate the Wronskian of the added pair from (2.13). When $|a_4| = a_3$ the potential behaves like a deep $V_8$ potential with the energy $E = -a_3^2$. By varying $a_4$ one can change the depth of the well. The family of $V_8$ potentials has in fact two parameters: the energy $E = -a_4^2$ and the parameter $c$ of (2.13). One can select a potential from this family by fixing these two parameters. Here we look for a deep potential which describes the nucleon-nucleon interaction according to microscopic theories based on the quark structure of nucleons (see e.g. Refs. [12] or [29]). We wish to obtain a potential close to the deep potential $V_K$ of Ref. [29], given by the formula

$$V_K = -1106.21 \exp(-1.6x^2) - 10.464 \exp(-0.7x)/(0.7x)[1 - \exp(-3x)], \quad (4.4)$$

where $x$ is given in fm. We varied $a_4$ and $c$ such as to be as close as possible to (4.4). From a least square fit we obtained $c = -0.155$ fm and $a_4 = -3.7944$ fm$^{-1}$ which gives $E = 596.42$ MeV. In this way we obtained the potential $V_8$ which is compared to $V_K$ of (4.4) in Fig. 4. One can see that the two potentials are quite close to each other. The bound state of $V_K$ is 442. MeV which is not much different from our value. The form of $V_K$ was chosen on qualitative grounds related to the approximations used in microscopic studies of the nucleon-nucleon interaction [12]. The potential $V_8$ of Fig. 4 is as legitimate as $V_K$ inasmuch as it correctly describes the experimental phase shift. The difference between the two potentials is partly due the difference in the fitted phase shifts and the $E_{lab}$ interval chosen for the fit in each case. If instead of fitting the phase shift to the experimental values we were fitting it to the phase shift corresponding to the potential (4.4), the difference would have been smaller.

**V. CONCLUSIONS**

In this study we propose an alternative method to derive families of phase equivalent (isophase) local potentials based on the application of an $N$th order Darboux transformation or an equivalent chain of $N$ first order transformations. In practical applications this method is simpler and more efficient than the conventional SUSY approach of Refs. [5–10]. First
one has to look for imaginary poles of the S-matrix, i.e. one has to find the parameters \( a_j \) and \( b_j \) of the phase shift \( \delta_N(k) \) of (3.8) which fit a particular experimental phase shift. One can start with the lowest order \( N = 2 \) in (3.8) and increase it until a good fit of the phase shift is achieved. The second step consists in finding an appropriate mapping between the S-matrix poles found in this way and the system of functions (3.1) which give a nodeless Wronskian. In terms of these functions the formula (2.10) gives phase equivalent potentials, among which one is shallow and the others form families of deep potentials from which one can select a particular one for a specific physical problem. We have applied our method to derive potentials describing the neutron-proton experimental phase shift for the \( ^1S_0 \) partial wave. We obtained a shallow potential very close to the Reid’s soft core potential and its deep phase equivalent partner is close to potentials resulting from microscopic derivations of the nucleon-nucleon interaction. In order to obtain phase equivalent potentials describing higher partial waves \( l \)-changing Darboux chains of transformations are necessary. This is the scope of a subsequent study.

APPENDIX A

Here we prove the formula (3.4). The proof is split into two steps: 1) Taking an initial potential with \( \nu = \nu_0 = 0 \) we prove (3.4) for this case. 2) We introduce a number of regular solutions to the previous case which corresponds to \( \nu > 0 \). Finally we comment on singular initial potentials.

Let us consider a chain of \( N = 2n \) Darboux transformations for a potential \( V_0(x) \) with \( \nu = \nu_0 = 0 \) and transformation functions given by (3.1) with \( \nu = 0 \). Since all \( a \)'s and \( b \)'s are different from each other this chain is equivalent to an \( N \)th order transformation. The transformed potential \( V_N \) is given by (2.10) and the eigenfunctions \( \psi_N \) by (2.7) or (2.11). One can also find the Jost solution of \( h_N \) by applying the same operator \( L^{(N)} \) to the Jost solution of \( h_0 \). The result is the function \( \tilde{f}_N(x, k) = L^{(N)} f_0(x, k) \), which like \( \psi_N \) from (2.7), is proportional to the ratio of two determinants. The long-distance asymptotic value of \( \tilde{f}_N(x, k) \) is proportional to the asymptotic value of the Jost solution. Therefore using the usual definition of the Jost function as the value of the Jost solution at the origin \([2,3]\) one has

\[
F_N(k) = \frac{\tilde{f}_N(0, k)}{[e^{-ikx}\tilde{f}_N(x, k)]_{x\to\infty}} \quad (A1)
\]

The definition (2.7) and our special choice of transformation functions (3.1) implies that

\[
\tilde{f}_N(x, k) = W(u_1, v_1, \ldots, u_n, v_n, f_0(x, k)) W^{-1}(u_1, v_1, \ldots, u_n, v_n) \equiv W f W_N^{-1} \quad (A2)
\]

Therefore both for \( x = 0 \) and \( x \to \infty \) we have to calculate the ratio of two determinants of orders \( N + 1 \) and \( N \) respectively, but of a similar structure.

First let us consider the function \( \tilde{f}_N(x, k) \) at \( x = 0 \). Using the definition \( f_0(0, k) = F_0(k) \)
the determinant appearing in the numerator of (A2) becomes

\[
W_f = \begin{vmatrix}
C_1 & 0 & C_2 & 0 & \ldots & C_n & 0 & F_0(k) \\
c_1 & D_1 & c_2 & D_2 & \ldots & c_n & D_n & \tilde{F}(k) \\
C_{1+1} & 0 & C_{2+1} & 0 & \ldots & C_{n+1} & 0 & F_0(k)(ik)^2 \\
c_{1+1} & D_{1+1} & c_{2+1} & D_{2+1} & \ldots & c_{n+1} & D_{n+1} & \tilde{F}(k)(ik)^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
C_{1+2n-2} & 0 & C_{2+2n-2} & 0 & \ldots & C_{n+2n-2} & 0 & F_0(k)(ik)^{2n-2} \\
c_{1+2n-2} & D_{1+2n-2} & c_{2+2n-2} & D_{2+2n-2} & \ldots & c_{n+2n-2} & D_{n+2n-2} & \tilde{F}(k)(ik)^{2n-2} \\
C_{1+2n} & 0 & C_{2+2n} & 0 & \ldots & C_{n+2n} & 0 & F_0(k)(ik)^{2n} \\
\end{vmatrix}
\]

\[
= C_1 \ldots C_n D_1 \ldots D_n F_0(k) \\
= -F_0(k) \prod_{j=1}^n (C_j D_j) \prod_{m>j} (a_j^2 - a_m^2)(b_j^2 - b_m^2) \prod_{j=1}^n (a_j^2 + k^2)
\]  

(A3)

Here \( C_j = u_j(0), \) \( c_j = u'_j(0), \) \( D_j = v'_j(0), \) \( \tilde{F}(k) = f'_0(0,k). \) Note that the derivative of the Jost function, \( \tilde{F}(k), \) does not appear in the final expression for \( W_f. \) The above result has been obtained by first using the following determinant property

\[
\begin{vmatrix}
M_{p \times p} & 0 & \ldots & \tilde{M}_{q \times q} \\
\vdots & \vdots & \ddots & \vdots \\
\end{vmatrix} = \det M_{p \times p} \det \tilde{M}_{q \times q}
\]  

(A4)

and next by expanding the two Vandermonde determinants (see e.g. [30]) from the middle line.

The determinant \( W_N \) which appears in the denominator of (A2) is of even order. It is obtained from the above determinant by the removal of the last row and column. Its final form is similar to the one above. It has opposite sign and the factors depending on \( k \) are absent. Finally one finds

\[
\tilde{f}_N(0,k) = (-1)^n F_0(k) \prod_{j=1}^n (k^2 + a_j^2)
\]  

(A5)

Let us now consider the denominator of (A1), i.e., \( \tilde{f}_N(x,k) \) for \( x \to \infty. \) In this limit all functions entering the Wronskians of (A2) tend to exponentials which cancel out with each other except for \( \exp(ikx) \) which cancels out with the exponential present in (A1). The remaining factor is a ratio of two Vandermonde determinants of orders \( N + 1 \) and \( N \) respectively containing powers of \( a_j \) and \( b_j. \) The determinant in the numerator differs from that in the denominator by the presence of a column containing powers of \( ik \) and an additional row composed of \((N + 1)\)th power of \( a_1, b_1, \ldots a_n, b_n, ik. \) Using once more the expression of Vandermonde’s determinant one obtains
\[ \bar{f}_N(x, k)|_{x \to \infty} \longrightarrow \imath^N \prod_{j=1}^{n} (k + ia_j)(k + ib_j) \] (A6)

Introducing (A5) and (A6) in (A1) one gets the following relation between the initial and final Jost functions

\[ F_N(k) = F_0(k) \prod_{j=1}^{n} \frac{k - ia_j}{k + ib_j}. \] (A7)

Moreover, since the Wronskian \( W_N \) is different from zero at \( x = 0 \) the singularity strength parameter of \( V_N \) is zero and it is regular at the origin.

Consider now the case where the first \( \nu \) functions of (3.1) belong to the regular family and calculate the singularity strength of \( V_N \). Note that the system (3.1) can be divided into two subsystems, one containing an equal number of singular and regular functions, and the other composed of regular functions only. Accordingly the \( N \)th order transformation can be decomposed into a superposition of two transformations. The first transformation of order \( N - \nu \) which is based on functions paired by their regular property will not affect the regularity of the potential at the origin, whereas the second one of order \( \nu \) composed of regular functions only will increase the singularity parameter of the potential by one unit for each transformation function (see e.g. [6]). So, the singularity strength parameter of \( V_N(x) \) is equal to \( \nu \).

In the following we have to use the definition of the Jost function \([2,3]\)

\[ F(k) = \lim_{x \to 0} \frac{(kx)^\nu}{i^\nu (2\nu - 1)!} f(x, k) \] (A8)

We give detailed calculations for the particular case \( \nu = 2 \) from which a general formula can easily be derived. In this case both \( W_f = W(v_1, v_2, v_3, \ldots, u_n, v_n, f_0) \) and \( W_N = W(v_1, v_2, v_3, \ldots, u_n, v_n) \) are equal to zero at \( x = 0 \). Since \( f_0(0, k) \neq 0 \) only one function (either \( v_1 \) or \( v_2 \)) in \( W_f \) is an unpaired regular solution and hence the zero of \( W_f(0) \) is of the first degree. For \( \nu = 2 \) the second logarithmic derivative of \( W_N(x) \) at \( x = 0 \) is of order \( \nu(\nu + 1) = 6 \). We conclude that the zero of \( W_N(0) \) is of the third degree. To evaluate the indeterminate form defining the Jost function (see (A8) for \( \nu = 2 \) and \( f(x, k) = f_N(x, k) \)) one can use L'Hôpital’s rule to get \( (kx)^2 W_f/W_N = k^2 W_f/x^3/W_N \to 3! k^2 W_f(0)/W_N(0) \) when \( x \to 0 \), because the first two derivatives of \( W_N(0) \) are zero. Since the derivative of an \( N \)th order Wronskian \( W(f_1, \ldots, f_N) \) is a determinant obtained from the same Wronskian by the replacement of the last row composed of the derivatives \( f_j^{(N-1)} \) by the derivatives \( f_j^{(N)} \) one can use the property (A4) to factorize \( W_f(0) \) into a product of two Vandermonde determinants \( W_{V1} \) and \( W_{V2} \) of orders \( n \) and \( n + 1 \) respectively. When calculating the third derivative of \( W_N \) at \( x = 0 \) one can see that it is equal to a sum of two equal determinants which can be factorized according to (A4) in a product of two Vandermonde determinants \( W_{V3} \) and \( W_{V4} \) of orders \( n - 1 \) and \( n + 1 \) respectively. We do not give here the explicit form of these determinants. The factor 2 coming from two equal Vandermonde determinants will cancel out with the corresponding factor coming from 3! and the factor 3 from this factorial will cancel out with \( (2\nu - 1)! \) for \( \nu = 2 \), the latter being present in the definition of the Jost function (A8). Hence, \( (kx)^2 W_f/W_N \to k^2 W_{V1} W_{V2}/(W_{V3} W_{V4}) \). When evaluating the
corresponding Vandermonde determinants one can see that the factors independent of \( k \) are exactly the same both in the numerator and the denominator. So, one gets the following Jost function for \( V_N(x) \)

\[
F_N(k) = F_0(k) \frac{k^2}{(k + ib_1)(k + ib_2)} \prod_{j=3}^{n} \frac{k - ia_j}{k + ib_j}
\]  

(A9)

It is clear from here that every additional regular function in the system (3.1) with the parameter \( b_j \) produces a factor \( k/(k + ib_j) \) in the Jost function which corresponds exactly to the formula (3.4). Moreover, a regular at the origin potential admits two kinds of transformations, the ones which do not affect its singularity (\( \nu = 0 \) in (3.1)) and others which increase its singularity (\( \nu > 0 \)). Actually if the initial potential has the singularity parameter \( \nu_0 \neq 0 \), three types of transformations are possible. In addition to two the previous types of transformations existing for the case \( \nu_0 = 0 \) transformations which decrease the singularity parameter are also possible. All these possibilities become clear if one takes into consideration the fact that all transformations are invertible (this means that from an intermediate element of a chain one can move in both directions) and an intermediate member of a previously considered chain is just a potential with a singularity strength different from zero.

ACKNOWLEDGMENTS.

We are most grateful to Mart Rentmeester for providing details about the Reid93 potential. Illuminating discussions with Daniel Baye and Francis Michel are gratefully acknowledged. One of us (B.F.S) is grateful to the Theoretical Fundamental Physics Laboratory of the University of Liège for warm hospitality during autumn 2001 when this work has been started. He also acknowledges a partial support from RFBR.
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TABLE I. The $^1S_0$ experimental phase shift from Ref. [23] together with the theoretical value $\delta_6$. $E_{lab}$ is related to the momentum $k$ ($fm^{-1}$) by $k = \sqrt{\frac{m_n E_{lab}}{2\hbar^2 c^2}}$ where we took $m_n = 940$ MeV and $\hbar c = 197.33$ MeV fm.

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FIG. 1. Comparison between the shallow potential $V_6$ (solid line), the Reid68 potential [26] (dotted line), the potential of Ref. [8] (dashed line) and the Reid93 potential [23] (dots) represented as a function of $x \equiv r$.

FIG. 2. The asymptotic behaviour of the absolute value of $V_6$ (solid line), the Reid68 potential [26] (dotted line) and the potential of Ref. [8] (dashed line) in (natural) logarithmic scale.
FIG. 3. Family of isospectral potentials isophase with $V_6$. Solid line - $A'_3 = 0$; Dashed line - $A'_3 = 10^6$; dash-dotted line — $A'_3 = -0.95$. For the latter case appearance of a barrier is clearly visible.

FIG. 4. Comparison between two deep potentials: $V_8(x)$ (solid line) and the potential of Ref. [29] (dashed line).