Solitons on Noncommutative Torus as Elliptic Calogero Gaudin Models, Branes and Laughling wave functions

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Abstract

For the noncommutative torus $T$, in case of the N.C. parameter $\theta = \frac{Z_n}{n}$, we construct the basis of Hilbert space $H$ in terms of $\theta$ functions of the positions $z_i$ of $n$ solitons. The wrapping around the torus generates the algebra $A_n$, which is the $Z_n \times Z_n$ Heisenberg group on $\theta$ functions. We find the generators $g$ of an local elliptic $su(n)$, which transform covariantly by the global gauge transformation of $A_n$. By acting on $H$ we establish the isomorphism of $A_n$ and $g$. We embed this $g$ into the $L$-matrix of the elliptic Gaudin and C.M. models to give the dynamics. The moment map of this twisted cotangent $su_n(T)$ bundle is matched to the $D$-equation with Fayet-Illiopoulos source term, so the dynamics of the N.C. solitons becomes that of the brane. The geometric configuration $(k, u)$ of the spectral curve $\det |L(u) - k| = 0$ describes the brane configuration, with the dynamical variables $z_i$ of N.C. solitons as the moduli $T^\otimes n/S_n$. Furthermore, in the N.C. Chern-Simons theory for the quantum Hall effect, the constrain equation with quasiparticle source is identified also with the moment map equation of the N.C. $su_n(T)$ cotangent bundle with marked points. The eigenfunction of the Gaudin differential $L$-operators as the Laughlin wavefunction is solved by Bethe ansatz.

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1 Introduction

The development of soliton theory on noncommutative geometry is rather impressive (details of reference e.g. see review [1, 2]). The paper [3] gives the solitons expressed by the

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Recently, the solitons on the N.C. torus has attracted a lot of interest [6, 7, 8, 9, 10, 11, 12, 13]. On the compactified torus [14], the duality, the Morita equivalence and the orbifolding have been studied [15].

The equivalence class of projection operator on torus is given by [16] in terms of $U_1$ and $U_2$. The noncommutative algebra $\mathcal{A}$ generated by $U_i$ ($U_1U_2 = u_2U_1e^{i\theta}$) wrapping the torus has been given in [14, 15, 17] in terms of the matrix difference operator. The matrix acts on a $U(n)$ bundle $V_n$ with trivial connection while the difference acts on a $U(1)$ bundle $L$. The Hilbert space $H = V_n \otimes L$ acted by $\mathcal{A}$ is given by the vector functions of real variables. The local covariant derivative operator $\nabla_i$ acted on $L$ is given also. Their commutator gives the constant curvature related to $\theta$. Obviously the double periodicity of wave functions on torus can not be given explicitly by real variable functions. The Schwarz space of smooth functions $S(\mathbb{R})$ of rapid decrease real functions provides a bimodule between $\mathcal{A}_\theta$ and $\mathcal{A}_\pi$ under Morita equivalence [8, 12]. Starting from the Gaussian function $e^{i\pi y^2/\theta}$ as the Schwarz function, Boca [9] constructed the projection operators in terms of $\theta$ functions. This projection operator satisfies the BPS like selfduality condition [6, 18], thus gives constant curvature. But Boca obtained the explicit expressions only for the case with modulus equals $\frac{1}{2}$, moreover, it is in terms of the products of two $\theta$ functions depends separately on $U_1$ and $U_2$, so the symplectic structure of noncommutative torus is unclear. Gopakumar et al [10] starting from the same Gaussian function but orbifolding the double periodic multisoliton solution on the entire N.C. $\mathbb{R}^2$ into a single soliton solution on the torus. They employed the $pq$ representation [19, 20] on a dual lattices for conjugate variables $p$ and $q$, which provides a basis of simultaneous eigenstates of commutative $U_1$ and $U_2$ with $\theta = 2\pi A$, where $A$ is an integer. Thus they succeeded in constructing the soliton on the so-called integral torus with double periodic wave functions. The noncommutative symplectic complex structure on torus appears explicitly and the corresponding Weyl-Moyal transformation is realized as double series of $U_1^{\pm \theta}$ and $U_2^{\pm \theta}$. But since the loop $U_1$ and $U_2$ become commutative in this degenerated $\theta = 2\pi A(e^{i\theta} = 1)$ case, they obtained just a unique projection operator, corresponding to only one soliton in each lattice (i.e. in the torus).

Now in this paper, by $n$ times orbifolding such integral torus $T$ into a $T_n$, we construct the $n$-th multi-soliton solution with $n$ zeros $z_i$ in $T$, such that the moduli space of this kind of multiple soliton becomes $T^\otimes n/S_n$. Meanwhile, the fundamental cycles $U_i$ wrapping around $T$, will be subdivided $n$ times into corresponding $W_i$ upon $T_n$, and $W_1W_2W_1^{-1}W_2^{-1} = e^{i\frac{2}{n} \theta}$. The torus $T$ now covers $T_n$ $n$ times, i.e. the original 2-brane on $T$ turns to be $n$ 2-branes with a $U(n)$ symmetry $G \sim U(n)$. The global $W_i$ generates an $\mathcal{A}_n$ algebra as the $Z_n \times Z_n$ Heisenberg group $G_H$. Moreover, using the modules $z_i$, $\tilde{z}_i(\sim \partial_i)$, we construct the local operators $E_a$ which generates the algebra $g \sim u_n(T)$ of $G$.

Then for the cotangent bundle of this $su_n(T)$ bundle on torus $u$ twisted by $U_i$, by using a moment map with sources at the marked points, we obtain the quantum elliptic Gaudin Lax differential operator $L(u)$ as its $n \times n$ covariant Hermitean section. In case of only one marked
point we gauge transform this $L(u)$ into the elliptic Calogero-Moser Lax operator. Obviously the moduli space becomes symplectic $(z_i, \bar{z}_i \to q_i, p_i)$ with Hamiltonian flows generated by C.M. Hamiltonian. The spectral curve determines the ”shape” in target space of the brane configuration in DHWW construction. The moment map equation serves as the D-equation. The dynamics of brane is given by the evolution of the $n$ solutions $z_i(t)$ of the C.M. equation.

Furthermore, we consider this moment map equation as the Gaussian constrain equation in QHE. The quantum eigenfunctions of Hamiltonian operators of Gaudin models becomes the Langhlin wave functions on torus.

In the next section, after review the construction by Gopakumar et. al [10] for the soliton on the integral torus $T$, we show by $n \times n$ times orbifolding this $T$ into $T_n$, that the one dimensional trivial Heisenberg group i.e. the double periodicity under wrapping by $U_i$ on $T$ in paper [10] will be refined into a $Z_n \times Z_n$ $G\mathcal{H}(n)$ generated by loops $W_i(= U_i^{\pi})$ around $T_n$.

In section 3, we use the $symplectic$ of the N.C. $Z_n \times Z_n$ $su(n)$ bundle. Then we gauge transform the Gaudin Lax into the Lax of the elliptic Calogero-Moser Lax operator. Obviously the $L(u)$ is obtained from the classical Yang-Baxter $r$-matrix $r(u) = \sum_\alpha w_\alpha(u)I_\alpha \otimes I_\alpha$ acted on $V_n \otimes V_n$ space by replacing one $V_n$ into a quantum space, i.e. replacing the Heisenberg matrix $I_\alpha$ on it by the $su_n(T)$ operator $E_\alpha$. The $r$-matrix is nondynamic and depends only on the spectral (evaluation) parameter $u$ through the meromorphic sections $w_\alpha(u)$. The double Heisenberg properties of $w_\alpha(u)$ and hence of $r(u)$ ensures $L(u)$ to be a section on a twisted $su(n)$ bundle. Then we gauge transform the Gaudin Lax into the Lax of the elliptic Calogero-Moser models. This Lax is equivalent to the elliptic Dunkle operators representing Weyl reflections. The trace of the quadratic of $L$ gives the Hamiltonian of the C.M. which is assumed to give dynamics of the N.C. solitons on the brane. As in section 3, we have shown that the isomorphism of the N.C. $z_i$ and $\bar{z}_i$ to the $z_i$ and local translation $\partial_i$ in N.C. $R^2$ case; under orbifold $\frac{k^2}{Z \otimes Z}$, becomes the isomorphism of the $Z_n \times Z_n$ Heisenberg $W_n$ in $A_n$ to the $E_\alpha$ in $su_n(T)$. So the correspondence of noncommutative $\partial_i(\sim \bar{z}_i)$ and $z_i$ in $E_\alpha$ to dynamical $p_i$ and $q_i$ in $L(u)$, naturally endows the N.C. torus with symplectic structure. Thus naturally the solitons on brane satisfy the dynamics of Calogero Gaudin systems.

In section 5 we describe the brane picture of the N.C. solitons. The moment map for reducing this twisted cotangent $su_n(T)$ bundle is matched to the $D$-equations with Fayet-Illiopoulos source term, so the dynamics of the N.C. solitons becomes that of the brane with impurities. The geometric configuration of the spectral curve $\det[L(u) - k] = 0$ describes the brane configuration $K(u) = 0$, with the dynamical variables $z_i$ of N.C. solitons as the modules describing the brane.
Furthermore, in section 6 we find the Laughlin wavefunction for quantum Hall effect (QHE) on N.C. torus. In the N.C. Chern-Simons theory for the QHE, the constrain equation with quasiparticle source is identified also with the moment map equation of the N.C. $su_n(T)$ cotangent bundle with marked points. In the following we show that the Laughlin wave function is given as a special case of the Bethe ansatz solution of Gaudin differential $L$ oprators. In the B.A. given by Felder and Varchenko [21], the so called "spectral parameter $\zeta$ is generic. We specialize it to be matched with the physical quantities in QHE etc. in the following way: we analyze succeedingly how it is related to the twisting phase of the B.A. function, to the twisted boundary condition and boundary Hamiltonian. Then we consider the algebraic geometrical form of the eigenfunction of C.M., so we find that this phase shift is determined by the filling fraction in the Gaussian constrain in quantum Hall fluid equation. Thus we confirm that this specialized B.A. eigenfunction is really the Laughlin wave function on torus. We give the solution for the case of only one quasiparticle, i.e. the Gaudin bundle with one marked point, which is equivalent to the C.M. bundle. It is easy to generalize it to the multi-quasiparticle case, i.e. Gaudin bundle with more marked points.

In the last section, we shortly describe the subjects which will be investigated later.

2 Solitons on the "integral torus" [10] and its further orbifolding

In this section we at first shortly review the result of paper [10] for solitons on noncommutative space $\mathbb{R}^2$: $[\hat{x}_1, \hat{x}_2] = i\Theta$. This $\mathbb{R}^2$ has been orbifolded to a torus $T$ with periodicities $L$ and $\tau L$.

The generators of the fundamental group of $T$ are:

$$ U_1 = e^{-i\hat{g}_1^2}, \quad U_2 = e^{i(\tau_2\hat{g}_1^3 - \tau_1\hat{g}_2^3)}, $$

where the normalized length $l \equiv L\Theta^{\frac{1}{2}}$, $\hat{g}_i = \frac{1}{\sqrt{\Theta}} \hat{x}_i$, $(i = 1, 2)$ and

$$ U_1U_2 = U_2U_1e^{-i\tau_2l^2} $$

As in [10], let us consider the case $\frac{\tau_2l^2}{2\pi} \in \mathbb{N}$ (or $\mathbb{Z}_+$) i.e. the normalized area $A = \frac{\tau_2l^2}{2\pi}$, is an integer. The projection operators constructed by [10] has its image spanned by the lattice of coherent states

$$ U_1^{j_1}U_2^{j_2}|0\rangle, (j_1, j_2 \in \mathbb{Z}^2) $$

here $|0\rangle$ is the Fock Bargmann vacuum vector: $a|0\rangle = 0$, where $a = \frac{1}{\sqrt{2}}(\hat{g}_1^1 + i\hat{g}_2^2)$, $a^\dagger = \frac{1}{\sqrt{2}}(\hat{g}_1^1 - i\hat{g}_2^2)$, $[a, a^\dagger] = 1$. They found a particular linear combination:

$$ |\psi\rangle = \sum_{j_1, j_2} c_{j_1, j_2} U_1^{j_1}U_2^{j_2}|0\rangle $$

4
that satisfies
\[ \langle \psi | U_1^{j_1} U_2^{j_2} | \psi \rangle = \delta_{j_1,0} \delta_{j_2,0}. \]  
(5)

Then they found the projection operator
\[ P = \sum_{j_1, j_2} U_1^{j_1} U_2^{j_2} \langle \psi | U^{-j_2} U_{-j_1} \rangle. \]
\[ P^2 = P. \]  
(6)

Since the orthonormalities (5), we have \( U_i P U_i^{-1} = P, \) (i = 1, 2), i.e. \( P \) actually acts on \( T = \mathbb{R}^2 / \mathbb{Z} \times \mathbb{Z} \) i.e. with periods \( U_i \) on \( \mathbb{R}^2 \).

To find the solution satisfying the orthonormality condition (5), the wave function in (4) on \( \mathbb{R}^2 \) has been found [10] in terms of the \( pq \) representation [19, 20]:
\[ |pq\rangle = \sqrt{\frac{l}{2\pi}} e^{-ir_1(q^2/2\tau_2)} \sum_j e^{ijlp} |q + jl\rangle. \]
\[ (7) \]

where \( |q\rangle \) is an eigenstate of \( \hat{y}^1: \hat{y}^1 |q\rangle = q|q\rangle \), such that
\[ e^{il\tau_2} |q\rangle = e^{ilq_2} |q\rangle, \quad e^{-il\tau_2} |q\rangle = |q + l\rangle. \]  
(8)

Obviously, in case \( A \) being an integer, \( |pq\rangle \) is the common eigenstate of \( U_1 \) and \( U_2 \):
\[ U_1 |pq\rangle = e^{-ilp} |pq\rangle, \quad U_2 |pq\rangle = e^{ilq_2} |pq\rangle. \]  
(9)

Remarks: From the completeness of the coherent states [22], we know that \( |pq\rangle \) is defined in the double periods: \( 0 \leq p \leq \frac{2\pi}{\tau_1}, 0 \leq q \leq l \), and constitute an orthonormal and complete basis for the Hilbert space \( \mathcal{H}_A = \mathcal{H}_{\mathbb{R}^2}/A \), if \( A \geq 2 \). In case of \( A = 1 \), if we choose just one coherent state for any unit cell, then all the state excluding the vacuum state may be taken as a complete and minimal system of states. Bars [13] recently has discussed the solutions with size smaller than this extremal one.

By using \( |q\rangle = \exp\left(\frac{-i|q|^2}{2} + a^\dagger q + \frac{a^2}{2}\right)|0\rangle \), we find the sum in wave function (7) equals the \( \theta \) function
\[ \langle pq|0\rangle = \sqrt{\frac{l}{2\pi}} e^{\frac{-i\tau_2}{2\pi} p^2} \theta\left(\frac{p + iq}{l}, \frac{A}{\tau}\right), \]  
(10)

where \( \tau = \tau_1 + i \tau_2 \). Then by modular transformation of (10) or directly from (7) by using the Poisson resummation formula we obtain as in [10]:
\[ C_0(p, q) \equiv \langle pq|0\rangle = \frac{1}{\pi i l} \exp\left(-\frac{\tau}{2\pi\tau_2} p^2 + ipq\right)\theta_{00}\left(\frac{q + p\tau}{l}, \frac{A}{\tau}\right). \]  
(11)

The orthonormality condition (5) becomes
\[ \delta_{j_1,0} \delta_{j_2,0} = \int_0^{2\pi} dp \int_0^l dq e^{-ij_1lp + ij_2lq_2} |\Psi(p, q)|^2 = \int_0^{2\pi} dp \int_0^l dq e^{-ij_1lp + ij_2lq_2} |\tilde{c}(p, q)|^2 \sum_{n=0}^{A-1} |C_0(p, q + n\frac{l}{A})|^2. \]  
(12)
Thus, the paper [10] found that
\[
\langle pq \mid \psi \rangle \equiv \Psi(p, q) = \tilde{c}(p, q)C_0(p, q) = \frac{C_0(p, q)}{\sqrt{2\pi/A^2 \sum_{n=0}^{A-1} |C_0(p, q + nl/A)|^2}}.
\]

(13)

Now we turn to the further orbifolding. Let \( W_1 = U_1^{\frac{1}{n}} \), then
\[
C_1(p, q) \equiv \langle pq \mid W_1 \mid 0 \rangle = \langle p, q + \frac{l}{n} \mid 0 \rangle = \frac{1}{\pi^{\frac{1}{4}} \sqrt{l}} \exp(-\frac{\tau p^2}{2i\tau_2^2} + ip(q + \frac{l}{n}))\theta_0,1_{\frac{1}{n}}(\frac{q + \frac{\tau p}{l}}{A}, \frac{\tau}{A}),
\]

(14)

here we have chosen \( A \) and \( n \) relatively prime.

Similarly
\[
C_\beta(p, q) \equiv \langle pq \mid W_1^\beta \mid 0 \rangle = \frac{1}{\pi^{\frac{1}{4}} \sqrt{l}} \exp(-\frac{\tau p^2}{2i\tau_2^2} + ip(q + \frac{\beta l}{n}))\theta_0,0_{\frac{1}{n}}(\frac{q + \frac{\tau p}{l}}{A}, \frac{\tau}{A}),
\]

(15)

here \( z \equiv y_1 + iy_2 \), the map \( p, q \) to \( z \) is given as in [10] by the Weyl Moyal transformation. The effect of \( W_1 \) is to shift \( z \) to \( z + \frac{1}{n} \), or equivalently shift the characteristic \( \beta \) of \( \theta_{\alpha, \beta} \) to \( \beta + \frac{1}{n} \).

Constructing
\[
\langle pq \mid \Psi_\beta \rangle = \frac{C_\beta(p, q)}{\sqrt{2\pi^2 \sum_{n=0}^{A-1} |C_\beta(p, q + nl/A)|^2}},
\]

(16)

then \( |\Psi_{\beta+n} \rangle = e^{-ip} |\Psi_\beta \rangle \) and \( |\Psi_\alpha \rangle (\alpha = 0, 1, \cdots, n-1) \) are linearly independent.

Let
\[
P_\beta = \sum_{j_1, j_2} U_1^{j_1} U_2^{j_2} |\Psi_\beta \rangle \langle \Psi_\beta | U_2^{-j_2} U_1^{-j_1},
\]

(17)

then
\[
P_\beta^2 = P_\beta
\]

(18)

\[
W_1^{\beta_1} P_{\beta_2} W_1^{-\beta_1} = P_{\beta_1 + \beta_2}.
\]

(19)

As in paper [15], We may also orbifold the torus along the \( \tau \) direction \( n \) times. Let \( W_2 = U_2^{\frac{1}{n}} \), then
\[
C_\alpha'(p, q) \equiv \langle pq \mid W_2^\alpha \mid 0 \rangle = e^{i\alpha \frac{\tau_2 l}{n}} \langle p + \frac{\tau_2 l}{n}, q \mid 0 \rangle
\]

\[
= \frac{1}{\pi^{\frac{1}{4}} \sqrt{l}} \exp(-\frac{\tau p^2}{2i\tau_2^2} + i(p + \frac{\alpha \tau_2 l}{n})\theta_0,0_{\frac{1}{n}}(\frac{q + \frac{\tau p}{l}}{A}, \frac{\tau}{A}).
\]

(20)

Obviously, the \( W_2 \) shifts \( z \) to \( z + \frac{\tau}{n} \), i.e. shift the \( \alpha \) of \( \theta_{\alpha, \beta} \).
Subsequently, we may construct $|\psi'_\alpha\rangle$ and $P'_\alpha$ as in (16) and (17). But the set $P'_\alpha$, ($\alpha = 1, 2, \cdots, n$) are not independent from the set $P_\beta$, ($\beta = 1, 2, \cdots, n$), and in either basis the $W_1$ or (and) $W_2$ matrix are not constant. Actually, since the target space $\mathcal{H}_T$ of $P$ on total torus $T$ has been subdivided into $\mathcal{H}_n$ on torus $T_n$ as described by [15]. One may find a basis of $\mathcal{H}_n$ such that $W_1$ and $W_2$ become the $n \times n$ irreducible matrix representation of the Heisenberg group

$$W_1W_2 = W_2W_1\omega, \quad \omega^n = 1, \quad W_1^n = W_2^n = 1. \quad (21)$$

In next section we will construct explicitly this basis in terms of $\theta_{\alpha\beta}$ functions. But before that, let us compare with the case of N.C. plane, where the generic $n$ solitons solution is $\prod_{i=1}^{n}(a_i^\dagger - z_i)|0\rangle$, with $n$ soliton centers at $z_i$. Similarly, we will introduce the $z_i$ for the location of the centers of $n$-solitons solutions on the torus $T$, such that the moduli space of the $n$-soliton solution is $T^{\otimes n}/S_n$. 

### 3 Noncommutative algebra $\mathcal{A}_n$, Hilbert space $\mathcal{H}_n$, Heisenberg group $Z_n \times Z_n$ and $su(n)$ algebra, in case $[U_1, U_2] = 0$

#### 3.1 The Hilbert space

As in [15], usually the Hilbert space $\mathcal{H}_T$ can be written as the direct product of a $su(n)$ trivial bundle $V_n$ and an $U(1)$ line bundle $\mathcal{L}$: $\mathcal{H}_T = V_n \otimes \mathcal{L}$. On the $V_n$ acts the $W_1$ and $W_2$ matrices, on the $\mathcal{L}$ acts the covariant deference operators $\hat{V}_i$ (the notation as that in [14]), and $W_i \otimes \hat{V}_i = U_i$. Thus, the operators $U_i$ are matrix difference operators acting on vector functions $v_{a_i}$ ($a = 1, 2, \cdots, n$). But now it happens that in case of commutative $U_1$ and $U_2$, this space $V_n$ is identical to the whole $\mathcal{H}_n$ on subtorus $T_n$ in the following way.

The basis vectors of the Hilbert space $\mathcal{H}_n$ are

$$v_a = \sum_{b=1}^{n} F_{-a,b}(a = 1, 2, \cdots, n),$$

$$F_{\alpha} \equiv F_{\alpha_1, \alpha_2} = e^{i\pi \alpha_2} \prod_{j=1}^{n} \sigma_{\alpha_1, \alpha_2}(z_j - \frac{1}{n} \sum_{k=1}^{n} z_k), \quad (22)$$

here $\alpha \equiv (\alpha_1, \alpha_2) \in Z_n \times Z_n$, and

$$\sigma_\alpha(z) = \theta \left[ \frac{1}{2} + \frac{\alpha_1}{n} \right] (z, \tau).$$

(In the following, we use the modulus $\tau$ to represent the $\frac{\tau}{n}$ in section 2).

Remark: The $\Psi (13)$ has been normalized, but it is not entire, i.e. it has poles and is nonanalytic. Our basis $v_a$ is entire, they have $n$ zeros in a torus, i.e. they span the space of weight $n$ quasiperiodic functions (sections) [23], i.e. the space of $n$ soliton sections of some $su(n)$ bundle.

Now we turn to show that the $W_1^{\alpha_1}, W_2^{\alpha_2}$ with $(\alpha_1, \alpha_2) \in Z_n \times Z_n$ acting on this $\mathcal{H}_n$ generates the N.C. algebra $\mathcal{A}_n$ on $T_n$. 


3.2 Noncommutative algebra $A_n$ on fuzzy torus $T$ as the Heisenberg Weyl group $Z_n \times Z_n$

From section 2 we know that the effects of the noncommutative Wilson loop [24, 25] $W_1 = \frac{1}{n} U_1$ and $W_2 = (U_2^\frac{1}{n})$ on the $i$-th soliton is to translate its position, from $z_i$ to $(z_i + \frac{1}{n}\tau - \delta_{in}\tau)$ and $(z_i + \frac{1}{n} - \delta_{in})$ respectively, or equivalently shift all $z_i$ by $\frac{1}{n}$ or $\frac{2}{n}$ mod the torus $T$, furthermore equivalent to shift the coordinate origin $u$ by $\frac{1}{n}$ of $\frac{2}{n}$ in opposite direction. Substituting in (22), we find

$$\mathbf{V}_1 v_a(z_1, \cdots, z_n) = (\prod_{i=1}^{n-1} T_{\frac{i}{n}}^{(i)}) T_{\frac{n-1}{n}} v_a(z_1, \cdots, z_n)$$

$$= v_{a-1}(z_1, \cdots, z_n) = W_1 v_a(z_1, \cdots, z_n),$$

$$\mathbf{V}_2 v_a(z_1, \cdots, z_n) = v_a(z_1 + \frac{1}{n}, \cdots, z_n + \frac{1}{n} - 1)$$

$$= (-1)^{n+1} e^{2\pi i z_1} v_a(z_1, \cdots, z_n) = W_2 v_a(z_1, \cdots, z_n),$$

where

$$T_{\frac{i}{n}} f(z) \equiv e^{\frac{\pi i z_1^2}{\frac{n}{2}} + 2\pi i z_1} f(z_1, \cdots, z_1 + a\tau, \cdots, z_n).$$

Using basis $v_a$ (22), we have transform the action of the N.C. Wilson loop of $T_n$ from the shift in functional space form (23) and (25) into the matrix operator form (24) and (26), i.e.

$$(\mathbf{V}_1)_{ab} = \delta_{a+1,b}; \quad (\mathbf{V}_2)_{ab} = \delta_{ab} \omega_a, \quad \omega = e^{2\pi i n} = e^{i\theta_n}$$

As expected this $\mathbf{V}_1$ and $\mathbf{V}_2$ satisfy the relations

$$\mathbf{V}_1^\dagger \mathbf{V}_1 = \mathbf{V}_2^\dagger \mathbf{V}_2 = 1, \quad \mathbf{V}_1^n = \mathbf{V}_2^n = 1, \quad \mathbf{V}_1 \mathbf{V}_2 = \mathbf{V}_2 \mathbf{V}_1 e^{2\pi i \theta_n},$$

here $\theta_n = \frac{1}{n}$ is the noncommutative parameter for the $n$ 2-branes [15].

Remark: There are some subtle and crucial points in our paper which is liable to cause confusion. It seem worth to be stressed and clarified here. Originally for $H_n$ (22), we find $v_1 = T_{\frac{n}{2}} v_1$, where $v_1$ corresponds to shift in functional space form (23) and (26), i.e.

$$W_1 = T_{\frac{n}{2}} W_1 = T_{\frac{n}{2}} v_1 = T_{\frac{n}{2}} v_1 e^{2\pi i \theta_n}.$$
that the $\hat{V}_\alpha \left(z_c \rightarrow z_c + \frac{\alpha + \alpha^*}{n}z\right)$ yield a matrix transform on our basis function, so it is equivalent to the operator form of $W_\alpha$. Thus for the total covering torus $\mathcal{T}$ the $W_\alpha$ and $\hat{V}_\alpha^{-1}$ cancels, $U_\alpha = W_\alpha \otimes \hat{V}_\alpha^{-1}$ are commutative, as the (6.54) in [15].

The functional and the operator forms are related by Weyl Moyal transformations on torus (see Appendix B of [10]), functions $v_\alpha$ behaves as the coherent "symbols", Fock-Bargmann representation functions [22] on torus.

On the space $\mathcal{H}_n = \{v_\alpha | a = 1, 2, \cdots, n\}$, the noncommutative algebra $\mathcal{A}_n$ generated by $W_1$ and $W_2$ has been truncated and becomes a $n \times n$ dimensional unital $C^*$ algebra with $n^2$ basis $W_1^{\alpha_1}W_2^{\alpha_2} = W_1^\alpha$ and the define relations of $\mathcal{A}_n$

$$W_1^{\alpha_2}W_2^{\alpha_1} = 1, \quad W_1^{\alpha_2}W_2^{\alpha_1}W_3^{\alpha_3} = W_1^{\alpha_2}W_2^{\alpha_1}W_3^{\alpha_3}W_4^{\alpha_4} = W_1^{\alpha_2}W_2^{\alpha_1}W_3^{\alpha_3}W_4^{\alpha_4}W_5^{\alpha_5} = \cdots = W_1^{\alpha_2}W_2^{\alpha_1}W_3^{\alpha_3}W_4^{\alpha_4} \cdots W_\alpha W_\alpha = 1. \quad (31)$$

Here and hereafter to represent in matrix (operator) form the basis $W_1^{\alpha_1,\alpha_2}$ of $\mathcal{A}_n$ we introduce the usual $Z_n \times Z_n$ matrix $(I_\alpha)_{ab} = (I_{\alpha_1,\alpha_2})_{ab} = \delta_{\alpha_1,\alpha_2}\omega^{b\alpha_2}$ [23]. This algebra in difference operator form is the Heisenberg group $G_\mathcal{H}(n) \times Z_n \times Z_n$ in ordinary $\theta$ function theory [23], including both the shift $\frac{\alpha_1}{n} + \frac{\alpha_2}{n}$ of arguments and the change of phases.

### 3.3 $su(n)$ algebra $g$ on N.C. torus

It is shown [26] that the level $l$ representation of the Lie algebra $sl_n(\mathcal{T})$ on the elliptic curve $\mathcal{T}$ can be written as following:

$$E_\alpha = (-1)^{\alpha_1}\sigma_\alpha(0) \sum_j \prod_{k \neq j} \sigma'_\alpha(z_{jk}) \left[ l \sum_{i \neq j} \frac{\sigma'_\alpha(z_{ji})}{\sigma_\alpha(z_{ji})} - \partial_j \right], \quad (32)$$

and

$$E_0 = -\sum_j \partial_j, \quad (33)$$

here $\alpha \equiv (\alpha_1, \alpha_2) \in Z_n \times Z_n$ and $\alpha \neq (0,0) \equiv (n,n)$, $z_{jk} = z_j - z_k$, $\partial_j = \frac{\partial}{\partial z_j}$. $E_0$ commutes with $E_\alpha$, $sl_n(\mathcal{T})$ includes only the $E_\alpha$ with $\alpha \neq 0$. After a complicate calculation, we obtain the commutation relation:

$$[E_\alpha, E_\gamma] = (\omega^{-\alpha_1} - \omega^{-\alpha_2})E_{\alpha+\gamma}, \quad (34)$$

or in usual $sl(n)$ with $i,j$ label the Chan Paton indices basis, let

$$E_{ij} \equiv \sum_{\alpha \neq 0} (I^\alpha)_{ij}E_\alpha, \quad (35)$$

then

$$[E_{jk}, E_{lm}] = E_{jm}\delta_{kl} - E_{lk}\delta_{jm}. \quad (36)$$

Remark: The representation (22) and the commuation rules (34) can also be obtained [26] from a quasiclassical limit from the representation of the $Z_n \times Z_n$ Sklyanin albebra [27].
Sklyanin and Takebe [28] give the elliptic \( sl(2) \) by using double periodic Weierstrass functions. The high spin \( l \) representations is given also. In this paper, it is restricted to the \( l = 1 \) representation of \( sl_n(T) \) by holomorphic sections on \( T^{\otimes n}/S_n \).

### 3.3.1 Automorphism of \( E_\beta \in su_2(T) \) by noncommutative gauge transformation \( W^\alpha \in A \)

Since the Wilson loops \( W_1 \) and \( W_2 \) acting on the noncommutative covering torus \( T \) is to shift \( z_i \) to \( (z_i + \frac{\tau}{n} - \delta_{in}) \) and \( (z_i + \frac{1}{n} - \delta_{in}) \) respectively, so \( E_\alpha \) in (22) will be changed into

\[
W_1 \cdot (E_\alpha(z_i)) = W_1 E_\alpha(z_i) W_1^{-1} = E_\alpha(z_i + \frac{\tau}{n} - \delta_{in}) = \omega^{-\alpha_2} E_\alpha(z_i),
\]

\[
W_2 \cdot (E_\alpha(z_i)) = W_2 E_\alpha(z_i) W_2^{-1} = E_\alpha(z_i + \frac{1}{n} - \delta_{in}) = \omega^{\alpha_1} E_\alpha(z_i).
\]

(37)

or more generally

\[
W^{\beta_1, \beta_2} \cdot E_\alpha = W^{\beta_1, \beta_2} E_\alpha (W^{\beta_1, \beta_2})^{-1} = \omega^{\alpha_1 \beta_2 - \alpha_2 \beta_1} E_\alpha.
\]

(38)

This could be compared with the matrix model on noncommutative torus [14], where

\[
U_i X_j U_i^{-1} = X_j + \delta_{ij} 2\pi R_j
\]

or more exactly with the covariant derivatives

\[
U_i \nabla_j U_i^{-1} = \delta_{ij} \nabla_j
\]

i. e.

\[
E_\alpha \implies \exp\left(\frac{i\pi}{n} \nabla_1\right)^{\alpha_1} \exp\left(\frac{i\pi}{n} \nabla_2\right)^{\alpha_2}.
\]

### 3.3.2 Isomorphism of \( su_n(T) \) and \( A_n \) on \( H_n \)

let \( E_\alpha \in g \) to act on \( v_\alpha \), we find that

\[
E_\alpha v_\alpha = \sum_b (I_\alpha)_{ba} v_b.
\]

(39)

As in section 3.2 we already learn that \( W^\alpha v_\alpha = \sum (I_\alpha)_{ba} v_b \), so on \( H_n \) we establish the isomorphism

\[
su_n(T) \implies A

E_\alpha \implies W^\alpha
\]

(40)

Obviously this is correspondent with the noncommutative plane case, where one have the homomorphism of the operators \( \partial \) and \( [ , * ] \)

\[
ie_{ji} \partial_x f(x) \implies [x, * f(x)]
\]

(41)
and the isomorphism upon acting on Fock space $\mathcal{H}$

$$\partial_z \rightarrow a, \quad \partial_{\bar{z}} \rightarrow a^\dagger \quad (42)$$

Compare the eq. (41) with the eqs. (37) (38) and compare the eq.(42) with the eq. (32) and (39), it is easy to see that the infinitesimal translations $\partial_i$ on plane $\mathbb{R}^2$ corresponds to $E_\alpha$ of the $su_n(T)$ and the algebra $\mathcal{A}_{\mathbb{R}^2}$ of $a, a^*$ corresponds to the algebra $\mathcal{A}_n$ of $W_i$. Meanwhile the local adj operation for $\mathcal{A}_{\mathbb{R}^2}$ (41) changes to the global Adj operation (gauge transformation [24, 25]) for $\mathcal{A}_n$ (37).

### 3.3.3 $\hat{V}_i$ and $E_\alpha$ as generators of the Weyl reflection group

Kac [29] has shown that the $\theta$ function with characteristic $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_n \otimes \mathbb{Z}_n$ transforms under affine Weyl reflections as the Heisenberg group generated by $\hat{V}_i$ (23), (25). As we have shown in (24), (26) and (39), both $\hat{V}_0$ and $E_\alpha$ act on $v_\alpha$ as $I_\alpha$, operators $\hat{V}_1$ and $E_1$ as the coxeter element $I_1$, operators $\hat{V}_2$ and $E_2$ as the cyclic element $I_2$; operator $E_{ij}(i \neq j)$ permutes $i, j$ i.e. gives the reflection $\hat{r}_{ij}$ reversing the root $e_i - e_j$.

Here, let us stress the relations of the geometrical N.C. algebras $\mathcal{H}_n$, $\mathcal{A}_n$ and $su(n)$ with the physical properties of N.C. solitons. Obviously, since in (22) each term $F_\alpha$ are symmetrical with respect to permutation $S_n$ of the centers $z_i$ of solitons, so the vectors $v_\alpha$ is defined on the module $\mathcal{T}_n \otimes_{\mathbb{Z}_n} \mathcal{S}_{\mathbb{Z}_n}$. The operator $\hat{V}_1$ shifts each soliton $z_i$ from the $i$-th covering of $T$ over $\mathcal{T}_n$ i.e. from the $i$-th brane to the next one, or equivalently changes each brane to the previous one. The operator $\hat{V}_2$ shifts the $U(1) \otimes_{\mathbb{Z}_n}$ phase of $i$-th brane by $\omega^i$. The Chan Paton $su(n)$ of $n$ branes is generated by $E_{ij}$. Acted on $\mathcal{H}_n$, both $\hat{V}_i$ and $E_{ij}$ generate the Weyl reflection. The offdiagonal elements give permutations of the branes, the diagonal elements generates the phase shifts of soliton wave functions.

## 4 Affine algebra on N.C. torus, integrable elliptic Gaudin and Calogero Moser models [26]

Now we will show that the cotangent bundle for N.C. elliptic $su_n(T)$ (32) with twisted loop $W_i$ realizes the elliptic Gaudin model on N.C. torus, which is related to critical level twisted $su(n)$ WZW i.e. $A_{n-1}$ affine algebra on N.C. torus. Then use the elliptic Vandermonde determinant to gauge transform it into elliptic C.M. model which is the Hamiltonian reduction of the cotangent bundle for the algebra of the semidirect product of the Cartan torus and the Weyl group [30].

### 4.1 N.C. Elliptic Gaudin Model

The elliptic Gaudin model on commutative space [31] is defined by the transfer matrix (quantum lax operator):

$$L_{ij}(u) = \sum_{\alpha \neq (0,0)} w_\alpha(u) E_\alpha(I_\alpha)_{ij}, \quad (43)$$
where
\[ w_\alpha(u) = \frac{\theta'(0)\sigma_\alpha(u)}{\sigma_\alpha(0)\sigma_0(u)}, \] (44)

\( E_\alpha \) and \( I_\alpha \) are the generators of \( su(n) \) (or \( A_{n-1} \) Weyl) and \( G_H(n) \) in the quantum and auxiliary space respectively. Using the general defining relations of \( su(n) \) (34) and \( G_H(n) \) (31), we find that the commutators of \( L \) (quantum version of fundamental Poisson bracket):

\[ [L^{(1)}(u_1), L^{(2)}(u_2)] = [r^{(1,2)}(u_1, u_2), L^{(1)}(u_1) \oplus L^{(2)}(u_2)], \] (45)

where the classical Yang-Baxter matrix

\[ r(u)_{i,j}^{k,l} = \sum_{\alpha \neq (0,0)} w_\alpha(u)(I_\alpha)_{i}^{k} \otimes (I_\alpha^{-1})_{j}^{l} \] (46)

is antisymmetrical

\[ r_{i,j}^{k,l}(u) = r_{j,i}^{l,k}(-u) \] (47)

and satisfies the classical Yang-Baxter equation:

\[ [r_{12}(u_{12}), r_{13}(u_{13})] + [r_{12}(u_{12}), r_{23}(u_{23})] + [r_{13}(u_{13}), r_{23}(u_{23})] = 0, \] (48)

where \( u_{ij} = u_i - u_j \). Thus this system is integrable. Indeed, we can find from the quantum determinant of \( L \) the complete set of Hamiltonian. Some examples may be found in [31, 32, 33]. It was obtained also as the nonrelativistic limit of the Ruijsenaars-Macdonald operators which has been described in the part two of our unsubmitted paper [34]. As a lattice model, the common eigenfunctions and eigenvalues of Gaudin model is solved in terms of Bethe ansatz [35], which has been expressed by the conformal blocks of the twisted WZW models on the torus [36, 37].

To relate these well known results about this usual Gaudin model with this newly introduced Gaudin model on the fuzzy torus \( T_n \) (section 3), let us substitute the differential representation (32) of \( su_n(T) \) \( E_\alpha \) into the Gaudin \( L \) (43), then the \( L \) turns to be in a factorized form [26]:

\[ E_0 + \sum_{\alpha \neq (0,0)} w_\alpha(u)E_\alpha(I_\alpha)_{j}^{i} = L(u)_{j}^{i} = \sum_{k} \phi(u, z)_{j}^{i} \phi^{-1}(u, z)_{j}^{k} \partial_u - l \sum_{k} \partial_u \phi(u, z)_{j}^{i} \phi^{-1}(u, z)_{j}^{k}, \] (49)

where the factors \( \phi_{k}^{i} \) are the vertex face intertwiner

\[ \phi(u, z)_{j}^{i} = \theta_{i}^{j} - \frac{n}{4} \frac{1}{\theta_{i}^{j}}(u + nz_j - \sum_{k=1}^{n} z_k + \frac{n-1}{2}, n\tau). \] (50)

We can also show that (49) is a representation of \( L \) by substituting it directly into (45). Now one may find that the expression of each elements of \( r \)-matrix becomes

\[ r_{i,j}^{l,k}(u) = \delta_{i,j}^{l,k} \left\{ (1 - \delta_{i}^{j}) \frac{\theta_{j}^{i}(0)\theta_{i}^{j}(u)}{\theta_{i}^{j}(u)\theta^{i}(u)} + \delta_{i}^{j} \left( \frac{\theta_{i}^{j}(u)^{'} - \theta(u)^{'}}{\theta_{i}^{j}(u)} \right) \right\}, \] (51)

12
where \( \theta^i(u) \equiv \theta_{\frac{i}{2} - \frac{1}{2}}(u, n\tau) \). It is easy to find that it is \( Z_n \otimes Z_n \) symmetrical, \( r(u) = (I_\alpha \times I_\alpha) r(u)(I_\alpha^{-1} \otimes I_\alpha^{-1}) \) and turns to be equal to the sum in (46), i.e. actually the factorized differential operators \( L^j_i \) (49) realizes a representation of the defining relation (45).

The intertwiner "factor"s (50), now intertwines the Chan Paton \( su(n) \) index \( i \) for the brane (vertex model index) with the dynamical indices \( j \) (face model indices) of dynamical soliton position \( z_j \) on the world sheet.

Feigin et. al [35] has established the relation between the critical level \( su(n) \) WZW models and the rational Gaudin model, that is the expressions of \( L(u) \) and \( R(u) \) being similar as (43) and (46), but instead of the elliptic \( w_\alpha(u) \) in (44) now with a rational \( w_\alpha^R(u) = \frac{1}{u}, u \in C \).

Kuroki and Takebe [37] find the same relation of \( RLL \) for the elliptic case, and point out that the WZW is twisted, i.e. is defined on a twisted bundle \( g^{I\!\!W} \):

\[
g^{I\!\!W} := (C \times g)/ \sim
\]

where the equivalence relation \( \sim \) are \( (u, g) \sim (u + 1, I_1 g I_1^{-1}) \sim (u + \tau, I_2 g I_2^{-1}) \). Correspondingly the \( w_\alpha(u) \) (44) is a meromorphic section of the twisted bundle with single pole at \( u = 0 \) and residue 1. While wrapping around different one cycle (+1 or +\( \tau \)) on the base torus, the global gauge transformation \( U_1 \) or \( U_2 \) for sections in the fibre are noncommutative, but the sections are holomorphic functions of \( u \) on base torus, which has only a complex conformal structures, it has neither metric nor symplectic structure, thus remains to be commutative.

For our N.C. torus we have both \( z = y_1 + iy_2 \) and \( \bar{z} = y_1 - iy_2 \) which are N.C., i.e. \( y_i \) are orthonormal with respect to the metric and N.C. symplectic at the same time, it causes both the local N.C. \( z, \bar{z} \) and the global N.C. \( W_1, W_2 \).

For the usual Gaudin [31, 32] on a marked commutative torus, the \( z_i \) is the marked points, i.e. the poles of \( L(u) = \sum_1^\alpha \sum_\alpha \ w_\alpha(u - z_i)E_\alpha I_\alpha \). The \( p_i \) is the conjugate momentum: classically \( [p_i, q_j] = \delta_{ij}, \) quantume \( [p_i, q_j] = \delta_{ij} \). So, the \( L \)-matrix is endowed with dynamics, by given the Poisson (equivalently Konstant-Kirrilov) brackets at first, then by quantized the \( p_i \sim -i \frac{\partial}{\partial q_i} \) to find the matrix differential form of \( L \) [26]. While now for the Gaudin on noncommutative torus, the \( z_i \) are the center (position) of the \( i \)-th soliton, \( \partial_i \) as its infinitesimal translation is equivalently to the \( [\bar{z}_i,*] \). Here we should stress that the key point to translate the usual integrable models on commutative space to that on N.C. space is to notice that since in N.C. space \( [\bar{z}_i, z_j] = \delta_{ij} \), so N.C. plane automatically become the symplectic manifold corresponding to the quantum phase space. \( p_i \to \bar{z}_i, q_i \to z_i \) i.e in the holomorphic Fock Bargmann formalism \( [\bar{z}, f] \) automatically becomes \( \partial_{\bar{z}} f \) just like \( p \to \frac{\partial}{\partial q} \). Apparently, the complex structure with given metric determines naturally a symplectic structure, moreover it is automatically "quantized", as in the geometrical quantization.

Let us show, that the dynamical equation, induced by N.C. this way, really gives the interaction between N.C. solitons assumed in [10], by studying the potential and the spectral curves of C.M. model.
4.2 Elliptic Calogero Moser model and its equivalence with elliptic Gaudin model

The elliptic C.M. model is defined by the Himiltonian:

$$H = \sum_{i=1}^{n} \partial_i^2 + \sum_{i \neq j} g\varphi(z_{ij}), \quad (52)$$

where $\varphi(z) = \partial^2 \sigma(z)$. This quadratic and other higher Hamiltonian are generated by the Krichver Lax matrix

$$L_{Kr}(u)^i_j = \partial_i \delta^i_j + (1 - \delta^i_j) \sqrt{g} \frac{\sigma(u + z_{ji})}{\sigma(u)\sigma(z_{ji})}. \quad (53)$$

where $z_{ji} = z_j - z_i$. By the Poisson transformation (classically $\partial_i \sim p_i, z_i \sim q_i$)

$$p_i \rightarrow p_i - \frac{\partial}{\partial q_i} \ln \Pi(q), \quad (54)$$

here

$$\Pi(q) = \prod_{i < j}^{\sigma(q_{ij})}, \quad \sqrt{g} = -\frac{l}{n}\sigma'(0), \quad (55)$$

and $L_{Kr}(u)^i_j$ becomes

$$L_{CM}(u)^i_j = (p_i - \frac{l}{n}\frac{\partial}{\partial q_i} \ln \Pi(q)) \delta^i_j - \frac{l}{n}\sigma'(0)(1 - \delta^i_j) \frac{\sigma(u + q_{ji})}{\sigma(u)\sigma(q_{ji})}. \quad (56)$$

This may be further gauge transformed into the factorized $L$ (49) of Gaudin model by gauge transformation matrix

$$G(u; q)^i_j \equiv \frac{\phi(u; q)^i_j}{\prod_{l \neq j}^{\sigma(q_{jl})}}, \quad \text{no summation of } i \text{ in } \phi^i_j. \quad (57)$$

It is well known that C.M. model gives the dynamics of a long distance interaction between $n$-bodies located at $z_i, \ (i = 1, 2, \cdots, n)$. Now we will show that it is the interaction between $n$-solitons on the fuzzy torus while $z_i$ becomes the positions of the centers of each soliton. According to paper [10], for N.C. multisitons the potential term is argumented to be the Laplacian of a Kähler potential $\mathcal{K}$, which is the logarithm of a Vandermonde determinant. Actually we have

$$\sum_{i \neq j} \varphi(z; j) = \sum_{i} \partial_i^2 \log \prod_{j \neq k}^{\sigma(z_j - z_k)} \equiv \sum_{i} \partial_i^2 \mathcal{K}(u, z), \quad (58)$$

$$e^{\mathcal{K}(u, z)} = \prod_{j \neq k}^{\sigma(z_j - z_k)\sigma(nu + \frac{n(n - 1)}{2})} = \det(\phi^i_k) \equiv \sigma(nu + \frac{n(n - 1)}{2}) \prod_{i \neq j}^{\sigma(z_{ij})}. \quad (59)$$
The variable $u$ of the marked torus is the so called spectral parameter or evaluation parameter of Lax matrix $L_{Kr}(u)^i_j$.

As [38], replacing the "spin" index $i,j$ elements $E_{ij}$ (35) in the Lax matrix $L_{Kr}(u)^i_j$ of the C.M. model by soliton exchange $z_i \leftrightarrow z_j$ permutation $s_{ij}$ in $A_{n-1}$ Weyl reflection, $L_{Kr}(u)^i_j$ becomes like the Dunkle operators [39]. This "spin" index exchange equivalence with particle (soliton) exchange is obtained if restricted to be acted on the space of wave functions which is totally (anti)symmetric in both spins and positions. This is the $A_{n-1}$ Weyl symmetry for the $su_n(T)$ Chan Paton index of the branes and for the positions $z_i, z_j$ of solitons.

Here the determinant of the vertex-face interaction is the determinant of our gauge transformation matrix from Gaudin to C.M. We should mention that its $u$ independent factor $\prod (z)$ is the Weyl antiinvariant ground state wave function of C. M. It is the "phase functions" part of the conform block of the twisted WZW model on elliptic curve related to the Gaudin model [35, 36, 33]. Dividing by this phase the KZB equation of the elliptic Gaudin Model reduces [36, 33] to the heat equation associated to the elliptic C. M. equation.

5 C. M. and Hitchin systems. Moment map and BPS equation, Spectral curves and brane configuration

5.1 Hitchin system for the cotangent bundle of $W \otimes T_n$ [30]

Instead as in section 4.2, where we obtained $L_{CM}$ from $L_{Gaudin}$, now the C.M. $L_{CM}$ operators (56) as the Weyl reflection of solitons [38] may be derived directly as a Hitchin systems for a group $G$ on root space. $G$ is an extension of the semidirect product of the Weyl group and the Cartan maximum torus of the roots. The cotangent bundle of $G$ consists of $su(n)$ connection $A$ and Hermitean $n \times n$ matrix forms scalar field $\varphi$.

The moment map condition: for $A$, the transition functions on root part (offdiagonal) is zero; for root part of $\varphi$ ($\varphi_{\text{offdiagonal}}$) is: $\text{res}_{u=0}(\varphi_{\text{offdiagonal}}) = a$ Weyl invariant constant $\sqrt{g}$. The solution turns to be

\[
A = \text{diag}(q_1, q_2, \ldots, q_n) \quad (60)
\]

\[
\varphi(u)^i_j = p_i \delta^i_j + (1 - \delta^i_j) \sqrt{g} \frac{\sigma(u + q_{ji})}{\sigma(u) \sigma(q_{ji})}, \quad (61)
\]

here \{\{p_i, q_j\} = \delta_{ij}.

Correspondingly for our case of N.C. soliton, this classically Poisson conjugate $p_i$ and $q_i$ in phase space should be automatically quantized and be replaced by quantum $\tilde{\partial}_i, \tilde{z}_i$ and $z_i$ with $[\tilde{\partial}_i, z_j] = \delta_{ij}$ ($\sim \tilde{\partial}_i$) and $z_i$ on N.C. $T$ and (61) becomes the matrix differential operator

\[
\varphi(u)^i_j = L_{Kr}(u)^i_j. \quad (62)
\]
5.2 Brane picture

Kapustin and Sethi [40] consider the case of a single D4 wrapped on $T^2$ probing by $n$ D-0 branes with FI (Fayet-Iliopoulos) deformation. The Higgs branch is given by the moduli space of solutions of the $D$-flat condition, but with impurity source (moment map equation Douglas Moore [41])

$$\partial \varphi + [A_\mu, \varphi] = \zeta (1 - n|v\rangle\langle v|)\delta^2(u), \quad \langle v| = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1), \quad (63)$$

here the $|v\rangle\langle v|$ term comes from the fundamental hypermultiplet of the single 0-4 string, $\zeta \sim$ the FI deformation parameter. Remark: The paper [40] consider the case of more 0-4 strings (impurities), such that the r.h.s. of (63) includes the sum of $\sigma^2(u_i)$, which will corresponds to the marked points $u_i$ on torus [32], to the lattice points $u_i$ for Gaudin models [35], or to the vertices for conformal block and KZB equations [36, 33].

Since the $n$ D0 branes dissolved into $T$ as the B-flux, the torus becomes N.C. [42]. In fact, recently, Susskind et al. [43, 44, 45] proposing the noncommutative Chern-Simons theory of the QHE on 2-brane, use this equation as a constrain equation (moment map equation). Here $|v\rangle\langle v|$ comes from the boundary field [44] or equivalently as the Wilson loop [46] related to the quasiparticle excited states.

Compare with [43, 44] we see that the commutator equation (the first one in (31)) of $U_i$ generating the $A_n$ algebra in section 3 generalizes their vortex free (ground state) constrain equation, while the elliptic C. M. moment map equation (63) generalizes their constrain equation with vortex (quasiparticle) from their case of $N \subset R^2$ to our case of N. C. torus.

Before study the QHE in next section, let us sketch the algebraic geometrical picture of the branes.

Krichver [47] defined the spectral curve $\Gamma$ of elliptic C.M. model by

$$\Gamma(k, u) \equiv \Gamma_{p, q}(k, u) \equiv \text{det}[k - L_{K_i}(u)] = \prod(k - K_i(u, p, q) = 0, \quad (64)$$

where the spectral parameter $u \in \Sigma_0 = \mathbb{C}/\mathbb{Z}^*$. The characteristic function $\Gamma(k, u)$ is an order $n$ polynomial of $k$, the coefficients or the eigenvalue $K_i(u, p, q)$ for any given $u$ depends on the modules $p_i, q_i$. The explicit solution of $K_i$ has been given by D’Hoker and Phong [48, 49]. The curve $\Gamma$ covers the bare $\Sigma_0$ $n$ times. Upon the marked point $u = 0$, $K_i(u) \sim \frac{1}{u}$, for generic $u$, $\Gamma(k, u)$ has $n$ zeros $K_i$ seperately on the $i$-th sheet. The number of quadratic branch points as the zeros of $\frac{\partial \Gamma}{\partial k}$ is $2(n - 1)$ i.e. the genus of $\Gamma$ is $n$.

In Witten’s construction of DHWW brane [50, 51, 52] via M theory, the components on the worldvolume of the brane wrapping on a compactified target spans a two dimensional surface $\Sigma$ embedding in a 4 dimensional manifold $Q(x_4, x_5, x_6, x_{10})$.

The $N = 2$ supersymmetry gives $Q(k, u)$ the complex structure $k = x_4 + ix_5$, $u = x_6 + ix_{10}$, then $\Sigma(u)$ is a Riemann surface. While the 4-manifold $Q$ is the moduli space of the ALF metric generated by $n$ parallel 6-branes transverse to $Q$. The ALF metric (e.g. Taub-Nut metric), being a hyperquotient, endows the manifold $Q$ a complex symplectic structure [53].
Donagi and Witten [54, 52] find that for the integrable complex Hamiltonian systems the Σ will move in Q such that there exist a line bundle LΣ(u) on Σ(u), which can be deformed into LΣ(u′) on Σ(u′) ⊂ Q, (u ≡ u(t₀), u′ ≡ u(t₀ + ε)), by the Hamiltonian flow from time t₀ to t₀ + ε. If we find a brane such that its Σ in Q has the bundle section LΣ equals the Lax matrix LKr of C.M., then the SW curve Σ(k, u) of this brane configuration can be identified with the spectral curve Γ(k, u) of integrable C.M. systems in the following way. Since the D flat equation (63) [40] (or BPS like equation [55]) for brane or SUSY [40, 41, 55] is the same as the C.M. moment map equation, which reduces the N.C. flat equation (63) [40] (or BPS like equation [55]) for brane or SUSY [40, 41, 55] is the same as the C.M. moment map equation, which reduces the N.C. flat equation, the embedding target field φ(u) j i matrix of brane, corresponds to the offdiagonal element of the C.M. Lax operator LKr(u)i j.

The LKr(u) is the section of the bundle on the spectral curve Γ(k, u) after pulled back u on Σ₀ by (64) to (k, u) on Γ(k, u). Thus the spectral curve Γ(k, u) is identified with the two dimensional surface Σ(k, u) ("target direction" x₄ + ix₅ = k, "world sheet" x₆ + ix₁₀ = u) wrapping the target brane. Thus, the map from the SW Σ to the spectral Γ is a symplectic map [56, 57] from the SW modules to the C.M. modules pᵢ, qᵢ.

Here it is crucial that the world sheet coordinates x₆, x₁₀ of base torus become noncommutative by the B flux, consequently pᵢ ≜ ∂i(∼ ẑi)), qᵢ ≜ ẑi; Poisson bracket in phase space \{pᵢ, qᵢ\} ≜ N.C. \{ẑₖ, *zₖ\} on brane world sheet; C.M. module pᵢ, qᵢ becomes the central position of N.C. solitons on the brane. The finite shift along two N.C. direction \(\frac{1}{n}, \frac{z}{n}\) causes the two U(n) matrices of Wᵢ wrapping two cycles around the compactified sub-brane becomes noncommutative, so the target become a twisted suₙ(T) bundle section. Then the sources by the piercing transverse strings, "impurities" further causes a U(n) symmetry breaking moment map, which reduces the N.C. suₙ(T) cotangent section field into the N.C. LCM. The modules of brane is identified to the N.C. soliton solution qᵢ(t). These explain the mysteries, why the SW curve corresponds to the spectral curve of C.M. equation.

The Hamiltonian flows will determine the evolution of the brane Σ in the ALF manifold Q as \(θ(Uz + Vt + W) = 0\) [56, 58], where the period matrix of the Riemann θ function is the periods of B-cycles of Σ; U, V are some constant vectors [56, 58, 59] in the Jacobian of Σ, determined by the Abelian map of meromorphic forms on Σ and W is the Riemann vector. Then the explicit soliton solution \(z_t(t)\) [59] will give the evolution of the brane \(φ_j^i(z(t))\) \(61\) in space time. We have consider only the second order interaction as GMS [10]. That is the V is a second jet, corresponds to the second order C.M. Hamiltonian tr|L²|, so the \(z(t)\) is determined by the C.M. equation. The exact Hamiltonian form should be determined by the action of brane.

The hyperquotient manifold Q is obtained by the moment map, with the moments (level) \(μ_i ∼ \) the distance of 6-branes \(\hat{r}_i - \hat{r}_{i+1}\), it is \(A_{n-1}\) Weyl invariant. We conject [41] there existing an ALF orbifold such that \(μ_i\) can be identified with the moments (FI source of D-flat equation) of C.M. That is it is reasonable to conject the Kronheimer like relation for the moduli space of a ALF orbifold with the moduli space of N.C. multisoliton \(T^{⊗n}/S_n\).

The N.C. elliptic algebra geometry will facilitate to investigate explicitly this correspondence and other properties of the ALF orbifold and the K3 \(Q(k, z)\) manifold.
6 Quantum Hall effect on torus, Laughlin wave function of quasiparticle, Bethe ansatz of elliptic Gaudin

The QHE on $\mathbb{R}^2$ is described [43, 44] by matrix (N.C.) Chern-Simons theory. The area preserving gauge transformation for the incompressible electron fluid without source is generated by the Gaussian constrain $[x_1, x_2] = \frac{iB}{2\pi \rho_0} \equiv i\theta B = \frac{i}{\nu}$, here $\rho_0$ is the density of the carrier and $\nu$ the filling fraction.

After adding a quasiparticle source, the Gaussian constrain for finite matrix Chern-Simons [44] turns to be

$$B[x_1, x_2] = \frac{i}{\nu} (1 - n|v\rangle \langle v|), \quad (65)$$

here the source is the boundary state $|v\rangle = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)$ or equivalently a Wilson line [46].

Hellerman and Raamsdonk [61] show the one-to-one correspondence in symmetry pattern of Laughlin wave functions and the minimal basis of matrix CS theory. Later Karabali and Sakita [17] find that, after separating the total antisymmetrical Vandermonde determinant representing vacuum state, the matrix CS eigenfunction with given symmetry pattern is identified to corresponding Jack polynomial representing the CM eigenfunction.

In case of QHE on torus, the global shift $U$ should be considered as in case of cylinder by Polychronakos [62]. Now, both the global shift $U_1, U_2$ and the local shift $X_1, X_2$ becomes N.C. The Noncommutativity of $U_1, U_2$ implies the global $su(n)$ structure as in section 3. While the holomorphic and antiholomorphic combination of local shifts becomes respectively the connection form $\theta + A$ and hermitian $n \times n$ matrix $\varphi$. Thus the constrain with a quasiparticle source turns to be (63) which is the moment map equation of the elliptic C.M. Furthermore, it is gauge (57) equivalent to the elliptic Gaudin with single marked point $u = 0$. So, we could find the quantum wave function explicitly by using the plentiful known results of elliptic Gaudin-Calogero theory.

Etingof and Kirrilov [63, 64] has shown that if the KZB equation of the conformal block of twisted bundle on torus is restricted to the one dimensional zero weight space of $S_{\ln C^n}$, then it will be degenerate into the elliptic C.M. equation. Subsequently, Felder and Varchenko find the eigenfunctions as the integrand of the integral representations of the solutions of KZB equation [65]; and later [21] as the Bethe ansatz for the elliptic Gaudin, as the following theorem given by them.

**Theorem.** For an irreducible highest weight module with highest weight $\Lambda = \sum_j m_j \alpha_j$ and highest weight vector $v_\Lambda$, Set $m = \sum_j m_j$ and let the ”color function” $c$: $\{1, \cdots, m\} \rightarrow \{1, \cdots, n\}$ be the unique non-decreasing function such that $c^{-1}\{j\}$ has $m_j$ elements, for all $j = i, \cdots, n$. Then the function parameterized by $t \in \mathbb{C}^m$

$$\psi(t, z) = e^{2\pi i c(z)} \sum_{\alpha \in S_n} w_{\alpha, c}(t, \alpha_1(z), \cdots, \alpha_r(z)) f_{c(\sigma(1))} \cdots f_{c(\sigma(n))} v_\Lambda \quad (66)$$

18
is an eigenfunction of $H_{\text{Sug}}$ (69), if the parameters $t_1, \cdots, t_n$ are a solution of the set of $n$ equations ("Bethe ansatz equations")

$$
\left( \sum_{l \neq j} \frac{\theta'(t_j - t_l)}{\theta(t_j - t_l)} \alpha_c(l) - \frac{\theta'(t_j)}{\theta(t_j)} \Lambda + 2\pi i \zeta, \alpha_c(j) \right) = 0, \quad j = 1, \cdots, n. \quad (67)
$$

The corresponding eigenvalue $\epsilon$ is

$$
\epsilon = 4\pi^2 (\zeta, \zeta) - 4\pi i \frac{\partial}{\partial \tau} S(t_1, \cdots, t_n, \tau),
$$

$$
S(t_1, \cdots, t_m, \tau) = \sum_{i < j} (\alpha_c(i), \alpha_c(j)) \ln \theta(t_i - t_j) - \sum_i (\Lambda, \alpha_c(i)) \ln \theta(t_i).
$$

Here, the "weight function" in (66) $w_{\sigma, c}(t, z) \equiv w(t_{\sigma(1)}, \cdots, t_{\sigma(n)}, z_{c(\sigma(1))}, \cdots, z_{c(\sigma(n))})$ and

$$
w(t, z) = \prod_{j=1}^{m} \frac{\theta(z_1 + \cdots + z_j - t_j + t_{j+1})}{\theta(z_1 + \cdots + z_j) \theta(t_j - t_{j+1})},
$$

and $\zeta_i, (i = 1, \cdots, n)$ are the twisting parameters

$$
(\zeta, \zeta) = \sum_i \zeta_i \zeta_i, \quad (\zeta, z) = \sum_i \zeta_i z_i, \quad (\zeta, \alpha) = \zeta_j - \zeta_{j+1}.
$$

The Sugawara Hamiltonian is

$$
H_{\text{Sug}} = -\sum_{i=1}^{n} \partial_{z_i}^2 + \sum_{\alpha \in \Delta} \varphi(\alpha(z)) e_\alpha e_{-\alpha} + \text{constant}.
$$

(69)

where $\Delta$ is the set of roots of $su(n)$, $e_\alpha \in sl(n)$ is the corresponding root vector. For the primary root $\alpha_i$ ($i = 1, \cdots, n - 1$), $e_i \equiv e_{\alpha_i}$, $f_i \equiv e_{-\alpha_i}$, $\alpha \in \Delta_+$, $f_i \equiv e_{-\alpha_i}$.

For given color function $c$, the Hermite-Bethe variety $HB(c)$ is defined in [66] by eliminating $\zeta$ from the BA equation (67). They show also that if $t_j$ is replaced by $t_j + l + m\tau$ ($l, m \in Z$), the $\zeta$ is shifted by $-m\alpha_{c(j)}$, and these replacements do not change the eigenfunction $\psi$. Therefore there is a map $\zeta: HB(c) \to h^*/Q$ ($Q$: root lattice of $A_{n-1}$) mapping $t$ to $\zeta$, and $HB(c)$ parameterizes eigenfunctions $\psi$ such that

$$
\psi(z + \omega) = e^{2\pi i \zeta(\omega)} \psi(z),
$$

(70)

here $\omega \in p^\vee$, $p^\vee = \{ \omega \in h|\alpha(\omega) \in Z \}$. Thus $\zeta$ determines the quasiperiodic twisting angle $\theta$ of phase shift of $\psi(z)$ under a real periods of $z$ on torus. It is wellknown that for the 8-vertex and $Z_n \times Z_n$ Belavin model, the twisting angle $\theta$ between different sectors on different "vacuum" of Bethe vector equals $2\pi i \zeta(\omega) \eta \equiv 2\pi i \lambda(\omega) \eta$: crossing parameter, $\lambda = (\lambda_1, \cdots, \lambda_n) \in h^*$, dynamical "heights" of different "vacuum". In the trigonometric limit ($\tau \to i\infty$), this twisting gives the exponents of the boundary matrix factor [67] of monodromy matrix for the twisted
period boundary condition. This is caused by the boundary term in the Hamiltonian. In Gaudin limit it contributes a linear term $\zeta \delta_{ij}$ in $L_{ij}$ [68], and a shift $\zeta(\alpha_j)$ in Bethe ansatz [28] as in (67). This also explains the contributions $(\zeta, \zeta)$ for the eigenvalue (68) of $H_{Sug}$ (69).

The Bethe ansatz solution (66, 67, 69) is valid for generic $\zeta$. Now we turn to its specialization in QHE. The papers [63, 36, 65] find the solution of C.M. equation from asymptotic limit of the solution of KZB equation. Here the crucial points is the "zero weight" and the vanishing condition [36, 65]. The Weyl invariance and vanishing condition is the condition for solution of KZB equation, devided by the Weyl-Kac denominator to be a conformal block of the $su(n)$ WZW (i.e. twisted $\hat{su}(n)$ bundle), it requires [65] $\zeta$ to be dominant weight of $su(n)$. These conformal block gives the Bethe ansatz solution [35] of Gaudin model since the Gaudin Hamiltonian (include e.g. the quadratic $H_{Sug}$) spans the center of $\hat{su}(n)$ at critical level, this center is Poisson isomorphic with the classical W algebra (include e.g. the Virasoro $H_{Sug}$). Their common eigenfunction is given by the Bethe ansatz.

Now we show that to be consistant with the Weyl invariant of the $L$-matrix and moment map, the $\zeta$ vector should be $\zeta(1, \cdots, 1)$. In the algebra-geometric formulation [56] the eigenvector $\psi$ (66) corresponds to the Baker Akhiezer function $\psi(t, z)$ of the Schrödinger equation

$$\left(\partial_t - \partial^2_{z_i} + 2 \sum_{i=1}^{n} \varphi(z - z_i(t))\right)\psi = 0. \quad (71)$$

The solution is expressed in terms of $n$ linear independent double-Bloch solution with simple poles $z = z_i(t)$, and

$$\psi = \sum_{i=1}^{n} c_i(t, k, u) w(z, z_i(t)) e^{kz + k^2t}, \quad (72)$$

here $u$ is the spectral parameter (of $\Sigma_0$) and $k$ the conjugate eigenvalue parameter in the spectral curve $\Sigma(k, u) \sim \Gamma(k, u)$, $z = z_i(t), (i = 1, \cdots, n)$ is the $n$ independent solutions of classical C.M. equation. In fact the vanishing of the double poles in (71) gives

$$(L_{CM}(t, u) - kI)c = 0, \quad c = (c_1, \cdots, c_n). \quad (73)$$

Here, the $L_{CM}$ is identical to (56), as $\dot{z}_i = 2p_i$. Then as described in the last subsection, the $L_{CM}(u)$ at $u = 0$ has the form

$$\zeta(1 - n|v\rangle\langle v|)^{-1}u + O(1), \quad (74)$$

which is the consequence of the moment map condition (63), i.e. the quasipaticle source of the Gauss constrain for QHE.

Finnally let us match with the Laughlin wave functions. On the zero weight space the KZB connection for heat equation becomes [36] the quadratic Gaudin Hamiltonian $H_{Sug}$, and on the module $S^{ln}C^n$, i.e. $\Lambda = l \sum_{j=1}^{n-1} (n - j) \alpha_j$, the zero weight space is one dimensional [64], and $H_{Sug}$ reduces [63, 36] to the C.M. Hamiltonian (52) with coupling constant $g = \zeta^2 \cong l^2$. Then it is easy to see that the $(l + 1)$th power of the total antisymmetrical Kac-Weyl denominator
is a horizontal section for the KZB equation, i.e. the vacuum vector (ground state of QHE) of the quantum C.M. equation. The quantum coupling constant of C.M. is \( l(l+1) \), and \( \frac{1}{l(l+1)} \) is the filling fraction in QHE.

At the critical level the B.A. eigenfunction (66, 67, 69) turns to be that of C.M. \( H \) (52) or \( H_{\text{Sug}} \) (69) respectively for the \( S^{k}C^{n} \) or the adjoint representation. The zero weight space of the latter, the adjoint representation is \( n \) dimensional.

The corresponding color function \( c \), gives the partition number of oscillators (creation operators \( \text{tr}((A^{\dagger})^{i}) \) of [61]), so determines the symmetrical pattern of the "elliptical" generalized Jack polynomial for the Laughlin wave function on torus, as has been conjectured by Hellerman and Raamsdonk [61] and explicitly given by Karabali and Sakita [69] for the N.C. \( R^{2} \) case.

7 Discussion

The aim of this paper is to show that once the N.C. integrable model has been found for N.C. torus (and sphere), then the various algebraic analytic and algebraic geometrical methods can be used to explicitly investigate the relevant physics for string-brane etc. Previously in a talk given by B. Y. Hou [70], we have given the multi-soliton solutions on N.C. torus in connection with the elliptic algebra for integrable models. The detail and expanded version of this talk has been put in the web as [34], which has never been submitted elsewhere. In the section 3 there in, the algebra \( A_{n} \) and the Hilbert space \( H_{n} \), in reality is relevant only for the vortex free (zero moment map) trivial bundle of the \( su(n) \) symmetry algebra. While the elliptic Gaudin and C.M. are \( su(n) \) bundle constrained by moment map with source. We included this as the section 3 and 4 of this paper.

Now this paper further display explicitly how these algebras work for brane and QHE. We have given the \( H_{n} \) and \( A_{n} \) of the LLL (lowest Landau level) on the \( S^{4} \) [71] in the vortex free case. We’re working of the LLL Laughlin wave functions in case with quasiparticles source both for the fuzzy Kähler \( S^{2} \) and for the hyperKähler \( S^{4} \).

Meanwhile we have a preliminary result of the projection operators of multisolitons on the \( Z_{k} \) orbifolding N.C. torus [72]. The result of this paper for brane, Laughlin wave function etc. will be easily generalized into the discrete orbifold and orientifold. Then, by pairing these moment map of impurity sources to the corresponding moment map of gravity centers of ALF, one may investigate various behavior including the dual properties of the brane configuration.

But to investigate the full Narain duality, we must consider the case of the more generic \( \theta \neq \frac{1}{n} \) case. We have argued in the second part of our unsubmitted paper [34], that the generic \( \theta \) will corresponding to the ratio of modulus of the two \( SL(2Z) \) in Narain’s \( SL(2Z) \otimes SL(2Z) \) and then to the crossing parameter \( \eta \) of \( Z_{n} \otimes Z_{n} \) Belavin and RSOS and to the coupling constant of the elliptic Ruijseenaar-Schneider model. These should be checked more carefully in connection with the brane theory.

At last we would like tentatively remind that in the Wakimoto construction for conformal
field, the zero mode is related to the N.C. part $A_1$ of Witten [4], i.e. the boundary conformal field in Seiberg-Witten limit. It is interesting to find the relation of the boson oscillators which realizes the $A_0$ part or the string brane interactions.

References


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