Evolution of Gaussian Wave Packet and Nonadiabatic Geometrical Phase for the time-dependent Singular Oscillator

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The geometrical phase of a time-dependent singular oscillator is obtained in the framework of Gaussian wave packet. It is shown by a simple geometrical approach that the geometrical phase is connected to the classical nonadiabatic Hannay angle of the generalized Harmonic oscillator.

PACS numbers: 03.65.Bz, 03.65.Ge, 03.65.Sq

Explicitly time-dependent problems present special difficulties in classical and quantum mechanics. However, they deserve detailed study because very interesting properties emerge when, even for simple linear systems, some parameters are allowed to vary with time. For instance, particular recent interest has been devoted to systems in which evolution originates geometric contributions [1-6]. One of these, the generalized harmonic oscillator has invoked much attention to study the nonadiabatic geometric phase for various quantum states, such as Gaussian, number, squeezed or coherent states, which can be found exactly [7-10]. Recently, the geometric phase for a cyclic wave packet solution of the generalized harmonic oscillator and its relation to Hannay’s angle were studied by Ge and Ghild [7]. They introduce the

$$\Phi(x,t) = \exp \left( \frac{-i}{\hbar} \left[ -\alpha(x - q)^2 + ip(x - q) + k \right] \right)$$  \hspace{1cm} (1)

centred around the classical guiding trajectory \((q, p)\), and proceed to derive equations of motion for the complex or real parameters \((\alpha(t), q(t), p(t),\text{ and } k(t))\) which serve to specify a complete quantum wave packet.

On the other hand, the number of exactly solvable quantum time-dependent problems is very restricted, one of the rare examples admitting exact solutions of the Schrödinger equation and have been studied intensively lately [12-19] is the quantum time-dependent generalized singular oscillator

$$H = \frac{1}{2} \left[ Z(t)p_x^2 + Y(t)(p_x x + xp_x) + X(t)x^2 + \frac{Z(t)l^2}{x^2} \right]$$  \hspace{1cm} (2)

where \(x\) and \(p_x = -i\hbar\partial/\partial x\) are the quantum operators, \(X(t), Y(t),\) and \(Z(t)\) are an arbitrary function of time, and \(l\) is an arbitrary constant which could be zero. A distinguished role of the Hamiltonian (2) is explained by the fact that, in a sense, it belongs to a boundary between linear and nonlinear problems of classical and quantum mechanics. For this reason, it was used in many applications in different areas of physics. For example, it served as an initial point in constructing interesting exactly solvable models of interacting N-body systems [12-13]. It was also used for modeling diatomic and polyatomic molecules [14]. It can have some relation to the problem of the relative motion of ions in electromagnetic traps [19].

The aim of this letter is to explore Gaussian wave function dynamics for the Hamiltonian (2) with nonadiabatic time dependence, and formulate a geometrical approach to derive a nonadiabatic geometric phase effect in
quantum and classical mechanics. For that purpose we introduce a class of
wave function of the form

\[ \Psi_l(x, t) = x^{\left(1/2 - \sqrt{(l^2)/\hbar^2} + 1/4\right)} \exp\left\{ \frac{1}{\hbar} \left( \frac{1}{2}(l+ipq) \left[ \frac{x-q}{q} \right]^2 + 2\frac{x-q}{q} \right) + k \right\} \]

(3)
given as products of squeezed Gaussian wave packet of the type (1) and a
function \( x \) of order \( (1/2 - \sqrt{(l^2)/\hbar^2} + 1/4) \). Inserting Eq. (3) into Schrödinger
equation

\[ i\hbar \frac{\partial \Psi_l}{\partial t} = H \Psi_l, \]

(4)
and then compare between the coefficients of various powers of \((x-q)\). This
lead to

\[(x-q)^2: \quad i\dot{\beta} = 2Z\beta^2 - 2iY\beta + \frac{X}{2}, \]

(5)
where \( \beta = -ip/(2q) - l/(2q^2) \),

\[(x-q)^1: \quad -2i(\beta - \frac{l}{2q^2})(\dot{q} - Zp - Yq) + (\dot{p} + Xq + Yp - \frac{Zl^2}{q^3}) = 0, \]

(6)
\[(x-q)^0: \]

\[ p\dot{q} + ik = \frac{1}{2} \left[ Zp^2 + 2Ypq + Xq^2 + \frac{Zl^2}{q^2} \right] - \frac{Z(t)}{q^2} + 2\hbar(1/2 - \sqrt{(l^2)/\hbar^2} + 1/4)(Z\beta - i\frac{Y}{2}). \]

(7)
The \((x-q)^2\) condition determines \( \beta \) by a nonlinear equation of the
Riccati form, which can be transformed to a linear system by introducing a
two dimensional vector \( \vec{v} \equiv (Q, P) \) and

\[ \beta \equiv -\frac{i}{2} \frac{P}{Q}, \]

(8)
where $Q$ and $P$ may be complex. In order that $\beta$ satisfies (5), it is sufficient that $\vec{v}$ obey the Hamilton’s equation

$$\vec{v} = \begin{pmatrix} \dot{Q} \\ \dot{P} \end{pmatrix} = \begin{pmatrix} Y & Z \\ -X & -Y \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} = -\mathcal{H} \vec{v}. \tag{9}$$

The $(x-q)^1$ condition makes sense if

$$\dot{q} = Zp + Yq$$
$$\dot{p} = -Xq - Yp + \frac{Zl^2}{q^3}, \tag{10}$$

and determines the complex guiding trajectory associated to the classical Hamiltonian

$$H(q, p, t) = \frac{1}{2} [Z(t)p^2 + Y(t)(pq + qp) + X(t)q^2 + \frac{Z(t)l^2}{q^2}] \tag{11}.$$  

The $(x-q)^0$ condition determines the time dependent global phase and normalization included in $k$ which can be rearranged in the form

$$k(t) - k(0) = i \int_0^t \left( L(t') + \frac{Z(t')l^2}{q^2} - 2\hbar(1 - \sqrt{(l/\hbar)^2 + 1/4}) (Z\beta - iY/2) \right) dt',$$  

where $L(t) = p(q, \dot{q})\dot{q} - H[q, p(q, \dot{q}), t]$. Examining the three terms in the expression (12) for $k$, we see that the two first give $i\{(pq) - p(0)q(0)\}/2$. The remaining terms in $k/\hbar$, namely $-2i(1 - \sqrt{(l/\hbar)^2 + 1/4}) \int_0^t (Z\beta - iY/2) dt'$ is recognized as the angle (or phase) accumulated in the nonadiabatic evolution, and contain a dynamical part $\theta_d(t)$ and a geometrical one $\theta_H(t)$ (Hannay’s angle).

In the literature, $\theta_H(t)$ is usually defined in relation with the introduction of time dependent canonical transformations. However, a simple
geometrical approach (hence no calculatory) may be formulated to deduce the decomposition of the total phase factor. Let us consider the classical equation (9). The main property of this evolution is that is linear and area preserving. This implies that any initial conditions \((Q(0), P(0))\) being at \(t = 0\) on a centered ellipse \(E(0)\) in phase space evolve at time \(t\) on a similar ellipse \(E(t)\) of the same area. A little thought show that, more precisely, two points \(M_0, N_0\) on \(E(0)\) whose parameters differ by \(\Delta \varphi\) evolve in points \(M_t, N_t\) on \(E(t)\) with the same difference of parameters so that the area \(\overrightarrow{OM}_t \wedge \overrightarrow{ON}_t\) remains constant. The reason is that the standard parameter \(\varphi(\varphi \in [0, 2\pi])\) which parametrizes a point on an ellipse is such that it is proportional to the area swept by the vector \(\vec{v} = \overrightarrow{OM}\). The natural origins of these family (homothetical centered) ellipses are the points associated with \(\varphi = 0\), and the action angle coordinates \((I, \theta)\) of a point \(M\) in phase space, with respect to the family associated with \(E(t)\), are respectively the area (divided by \(2\pi\)) and the usual angular variable defined on this ellipse. There exist a natural transport (with respect to the symplectic structure in phase space) for a family of (homothetical centered) ellipses. More precisely, let \(M_t\) on \(E(t)\) a point of coordinates \((I, \theta(t))\) and \(\widetilde{M}_{t+dt}\) its transported on \(E(t + dt)\). Clearly, this transport must preserve areas, but this is not sufficient since it remains to precise how one point (for example the origin) is transported: for this, one simply requires that the area which is swept by the vectors \(\vec{v}(\varphi)\) (on an ellipse) during the transport has, when averaged over \(\varphi\), a mean value equal to zero. Therefore the transport is defined by

\[
\langle \overrightarrow{OM}_t \wedge M_t \overrightarrow{M}_{t+dt} \rangle = 0. \tag{13}
\]
This transport associates to $M_t$ of coordinates $\theta(t)$, the point \( \tilde{M}_{t+dt} \) of coordinates $\theta(t) + d\theta_H(t)$ ($\theta_H(t)$: Hannay’s angle).

The angular coordinate on $E(t + dt)$ of evolved point $M_{t+dt}$ of $M_t$ is $\theta(t) + d\theta(t)$. The difference $d\theta(t) - d\theta_H(t) = d\theta_D(t)$ does not depend on the chosen point $M_t$ on $E(t)$. The quantity $I d\theta_D = I(d\theta - d\theta_H)$ is the area $\overrightarrow{OM}_{t+dt} \wedge \overrightarrow{M}_{t+dt} M_{t+dt}$ and can be written as the difference $\overrightarrow{OM}_t \wedge M_t M_{t+dt} - \overrightarrow{OM}_t \wedge M_t \tilde{M}_{t+dt}$ of the area swept by the vector $\overrightarrow{OM}_t$ during its evolution and of the one swept by $\overrightarrow{OM}_t$ in the geometrical transport. Averaging these two areas on the ellipse $E(t)$, one can see that $I d\theta_D$ can be interpreted as the mean value of the swept area during the motion. This justifies the appellation of dynamical angle for $\theta_D(t)$.

To translate analytically the previous geometrical remarks, let $\vec{E}$ be a complex two dimensional vector, it is known (for instance from optics) that one can describe (homothetical centered) ellipses as the set of vectors $\overrightarrow{OM}_t = [\vec{E}(t) e^{-i\varphi}]$ and those obtained by the transport are represented by the set of vectors $\overrightarrow{OM}_{t+dt} = [\vec{E}(t+dt) e^{-id\theta_H} e^{-i\varphi}]$. Using the area preserving property (proportional to $i \vec{E}^* \wedge \vec{E}$), one can easily verify that the transport defined by Eq. (13) is also written as $\vec{E}^* \wedge (d\vec{E} - id\theta_H \vec{E}) = 0$ or

$$\dot{\theta}_H = \frac{\vec{E}^* \wedge \vec{E}}{i(\vec{E}^* \wedge \vec{E})}.$$  \hspace{1cm} (14)

Obviously Eq. (14) gives new expression for the nonadiabatic Hannay angle of the generalized harmonic oscillator. Within such formalism the above remarks justify that the general solution of the Hamilton equation (9) may
be looked for in the form

$$\vec{v}(t) = A e^{-i(\theta(t)+\varphi)} \vec{E}(t), \quad (\theta(0) = 0),$$  

(15)

with $i\vec{E}^* \wedge \vec{E}$ conserved and $A$ and $\varphi$ fixed ($A$ and $\varphi$ are the conditions measured with respect to the family $\vec{E}(0)$). The angular drift of the origins points of $\vec{E}(0)$ (measured with respect to $\vec{E}(t)$) is naturally decomposed in a geometrical part (Hannay's angle) and a dynamical one

$$\dot{\theta} = \frac{\vec{E}^* \wedge \vec{E}}{i(\vec{E}^* \wedge \vec{E})} + \frac{\vec{E}^* \wedge \mathcal{H}\vec{E}}{i(\vec{E}^* \wedge \vec{E})}. $$  

(16)

(This relation is obtained by inserting $\vec{v}(t)$ in Hamilton equation (9) and making the wedge product with $\vec{v}^*(t)$).

If one wants to explicitly calculate $\dot{\theta}_H$ and $\dot{\theta}$, we must find vector $\vec{E}(t)$. Before hand, one can impose the condition $-i(\vec{E}^* \wedge \vec{E}) = 4I$ (which fix the ellipse area) and take the first componant of $\vec{E}$ to be real. The general form of $\vec{E}(t)$ is thus

$$\vec{E}(t) = \left(\begin{array}{c}
\frac{\sqrt{QQ^*}}{2i\beta(t)\sqrt{QQ^*}}
\end{array}\right).$$  

(17)

It leads to the relations

$$\dot{\theta}_H = -i \frac{\dot{\beta}}{\beta + \beta^*} - i \frac{d}{2dt} \ln(QQ^*)$$  

(18)

$$\dot{\theta} = -2(Z\beta - iY/2) - \frac{i}{2} \frac{d}{dt} \ln(QQ^*).$$  

(19)

From the form of the time-dependent phase factors included in $k$ Eq. (12), it is clear that

$$\gamma_l(t) = -2(1 - \sqrt{(l/h)^2 + 1/4}) \int_0^t dt (Z\beta - iY/2)$$

(20)
we see that the logarithm term goes "downstairs" as time-dependent normalization factor in the $\Psi_l(x, t)$. The remaining term in $\gamma_l(t)$ is recognized as the phase factors acquired by the wave packet in its evolution.

Then, we can reach a simple relation between the geometrical phase for the quantum singular oscillator and the nonadiabatic Hannay angle associated to the generalized harmonic oscillator

$$\gamma^G_l(t) = (1 - \sqrt{(l/\hbar)^2 + 1/4}) \Delta \theta_H(t),$$

where the first part is independent of $\hbar$ and is equal to the Hannay angle, the second part depends on $\hbar$ and $l$.

Now we turn our attention to the Eq. (3) which can be rewritten as a simple Gaussian wave packet

$$\Psi_l(x, t) = \left(x \frac{Q(0)Q^*(0)}{Q(t)Q^*(t)} \right)^{1/2-\sqrt{(l/\hbar)^2+1/4}} e^{-\frac{1}{4} \ln \frac{Q(0)Q^*(0)}{Q(t)Q^*(t)}}$$

$$\exp \left\{ \frac{1}{\hbar} \left[ -\beta x^2 - 1/2(l - ip(0)q(0)) + k(0) + \right. \right.$$

$$\left. \hbar (1 - \sqrt{(l/\hbar)^2 + 1/4}) \int_0^t \left[ \frac{\vec{E}^* \wedge \vec{E}}{(\vec{E}^* \wedge \vec{E})} + \frac{\vec{E}^* \wedge \vec{H} \vec{E}}{(\vec{E}^* \wedge \vec{E})} \right] dt' \right\}$$

which is also a solution to the Schrödinger equation.

In conclusion, the squeezed Gaussian wave packet dynamics for the time-dependent singular oscillator has been obtained as a simple squeezed wave packet. The quantum phases are obtained explicitly and connected to the classical angle of the generalized harmonic oscillator. The classical version of the generalized harmonic oscillator has been discussed, and a new expression
for the nonadiabatic Hannay angle has been obtained by employing a geometrical approach of the evolution in phase space. In the literature, $\theta_H(t)$ is usually defined in relation with the introduction of time dependent canonical transformations. However, the geometrical part has been found by asking whether there exists a natural transport (with respect to the symplectic structure) for a family of (homothetical centered) ellipses. When the parameter $l$ vanishes, we see that the Gaussian wave packet (22) corresponds to the evolution of the ”ground” state of the time-dependent generalized harmonic oscillator and the geometrical phase is equal to one-half of the classical angle. This is just what was obtained in Ref. [7].
References


