The negative index of refraction demystified

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Abstract. We study electromagnetic wave propagation in mediums in which the effective relative permittivity and the effective relative permeability are allowed to take any value in the upper half of the complex plane. A general condition is derived for the phase velocity to be oppositely directed to the power flow. That extends the recently studied case of propagation in mediums for which the relative permittivity and relative permeability are both simultaneously negative, to include dissipation as well. An illustrative case study demonstrates that in general the spectrum divides into five distinct regions.

PACS numbers: 41.20.Jb, 42.25.Bs, 42.70

Accepted for: Eur. J. Phys.
1. Introduction

Materials with either negative real relative permittivity or negative real relative permeability at a certain frequency are not uncommon. Consideration of materials for which both these quantities are simultaneously negative and real–valued, commenced with Veselago’s paper of 1968 [1]. Although he pointed out many unusual properties of such materials, including inverse refraction, negative radiation pressure, inverse Doppler effect, over three decades ago, the considerations were completely speculative in view of the lack of a material, or even a nonhomogeneous composite medium, with a relative permittivity having a negative real part and a very small imaginary part. A breakthrough was achieved by Smith et al. [2], who, developing some earlier ideas by Pendry et al. [3, 4, 5], presented evidence for a weakly dissipative composite medium displaying negative values for the real parts of its effective permittivity and effective permeability. Their so–called meta–material consists of various inclusions of conducting rings and wires embedded within printed circuit boards. Their conclusions were based on observations from three separate composite mediums.

- **Medium 1** consisted of a lattice of ring–like inclusions, which for a certain field configuration was presumed to have a resonant relative permeability $\mu_{\text{eff}}(\omega)$ of the form [2, 5]

$$\mu_{\text{eff}}(\omega) = 1 - \frac{F \omega^2}{\omega^2 - \omega_0^2 + i\omega \Gamma},$$

where the resonance frequency $\omega_0$ depends principally on the geometry of the rings. In this model, dissipation is facilitated by $\Gamma$, and $F$ ($0 < F < 1$) is the ratio of the area occupied by a ring and that of a unit cell. For weak dissipation, the real part of $\mu_{\text{eff}}$ is negative for $\omega_0 < \omega < \omega_0/\sqrt{1-F}$.

- **Medium 2** consisted of an included matrix of wires. The effective relative permittivity $\epsilon_{\text{eff}}$ of this composite medium supposedly displays plasma–like behaviour according to

$$\epsilon_{\text{eff}}(\omega) = 1 - \frac{\omega_p^2}{\omega^2},$$

the effective plasma frequency $\omega_p$ being dependent on the geometry. In such a medium, $\epsilon_{\text{eff}}$ is negative for $\omega < \omega_p$.

- **Medium 3** combined the first two, it being postulated that the combination would exhibit negative real–valued permeability and negative real–valued permittivity within a certain frequency range. Although some numerical and experimental evidence was presented in 2000 [2], the most striking report appeared the following year [6] and gave preliminary indications of the observation of the so–called negative index of refraction.

Other types of nanostructural combinations with similar response properties can also be devised [7].
The emergence of a clear terminology is often a difficult process with regards to scientific findings relating to novel effects, something that is also apparent in the present instance. The original classification of the materials exhibiting the effects discussed labelled them left–handed electromagnetic materials [1]. But chiral materials are important subjects of electromagnetics research and the terms left–handedness and right–handedness have been applied to the molecular structure of such materials for well over a century [8]. The continued use of the term left–handed materials (LHMs) for achiral materials in, for example, [2, 6, 9] will thus confuse the crucial issues. Other authors [10] are promoting the term backward medium (BW) which presumes the a priori definitions of forward and backward directions. In the most recent contribution [11] (which also provides the most extensive theoretical and numerical analysis of the negative index of refraction to date), the authors introduce the technical term double negative (DNG) medium to indicate that the real parts of both permittivity and permeability are negative. While sensible enough, such nomenclature conceals the importance of dissipative effects.

In time, a consensus about terminology will undoubtedly emerge; and it is not our aim to contribute to this particular discussion. Instead the purpose of this note is pedagogical. In the first instance, it is important that dissipation be included in the analysis. This was largely neglected in the reports cited earlier, with the exemption of the most recent study [11].

Secondly, it is desirable to derive the general condition for the type of anomalous propagation that is characteristic of the considered materials: namely, where the phase velocity is directed oppositely to the power flow.

2. Plane Wave Propagation

Consider a plane wave propagating along the $z$ axis in a linear, homogeneous, isotropic, dielectric–magnetic medium whose relative permittivity and relative permeability are denoted by $\epsilon_r$ and $\mu_r$, respectively. An $\exp(-i\omega t)$ time–dependence is assumed here. Then

\[ E(z) = A \exp(ik_0nz)u_x, \]
\[ B(z) = \frac{1}{i\omega} \nabla \times E(z) = \frac{nk_0}{\omega} A \exp(ik_0nz)u_y, \]
\[ H(z) = \frac{n}{\mu_r \eta_0} A \exp(ik_0nz)u_y, \]

where $k_0$ is the free–space wavenumber, $\eta_0$ is the intrinsic impedance of free space, and $n^2 = \epsilon_r \mu_r$. Consequently, the Poynting vector is parallel to the $z$ axis and its time–average is given as

\[ P_z(n) = \frac{1}{2} u_z \cdot \text{Re}[E(z) \times H^*(z)] = \text{Re} \left[ \frac{n}{\mu_r} \frac{|A|^2}{2\eta_0} \exp(-2k_0\text{Im}[n]z) \right], \]

where $\text{Re}[\cdot]$ and $\text{Im}[\cdot]$, respectively, denote the operations of taking the real and the imaginary part, whilst $^*$ indicates complex conjugation.
Let us now assume a Lorentzian model for $\varepsilon_r$ and $\mu_r$. This will include the specified forms (1) and (2) as special cases. Dissipation results from the imaginary parts of $\varepsilon_r$ and $\mu_r$ whilst causality dictates that $\text{Im}[\mu_r] > 0$ and $\text{Im}[\varepsilon_r] > 0$, so that $\varepsilon_r$ and $\mu_r$ lie in the upper half of the complex plane.

However, there are two resultant complex refractive indexes, $n_\pm = \pm \sqrt{\varepsilon_r \mu_r}$, of which $n_+$ lies in the upper half of the complex plane and $n_-$ in the lower half. The situation is summarized in Figure 1 for which the resonant form of (1) was used as representative of both $\varepsilon_r(\omega)$ and $\mu_r(\omega)$. Of course, the resonances of $\varepsilon_r(\omega)$ and $\mu_r(\omega)$ are unlikely to coincide, so that for a particular value of $\omega$, the arrows corresponding to $\varepsilon_r$ and $\mu_r$ will not be necessarily parallel. Only the upper half of the complex plane is shown in the figure.

Now $n_\pm$ may be written as

$$n_\pm = \pm n_0 \exp i \phi_n,$$

where

$$n_0 = \pm \sqrt{\varepsilon_r |\mu_r|}, \quad \phi_n = \frac{\phi_\varepsilon + \phi_\mu}{2}.$$

Here $\phi_\varepsilon$ and $\phi_\mu$, representing the arguments of $\varepsilon_r$ and $\mu_r$ respectively, must obey the conditions $0 \leq \phi_\varepsilon, \phi_\mu \leq \pi$. Consequently, $0 \leq \phi_n \leq \pi$. We then always have

$$\text{Re} \left[ \frac{n_+}{\mu_r} \right] > 0 \quad \text{i.e.} \quad P_z(n_+) > 0$$

and also

$$\text{Re} \left[ \frac{n_-}{\mu_r} \right] < 0 \quad \text{i.e.} \quad P_z(n_-) < 0.$$

Thus the choice $n_+$ always relates to power flow in the $+z$ direction, whilst $n_-$ always relates to power flow in the $-z$ direction. Since necessarily $\text{Im}[n_+] > 0$ and $\text{Im}[n_-] < 0$, power flow is always in the direction of exponential decrease of the fields’ amplitudes.

We can now identify when the phase velocity is opposite to the direction of power flow. This occurs whenever $\text{Re}[n_+] < 0$ (and consequently $\text{Re}[n_-] > 0$, also). After setting

$$\varepsilon_r = \varepsilon'_r + i\varepsilon''_r, \quad \mu_r = \mu'_r + i\mu''_r,$$

(where $\varepsilon'_r, \varepsilon''_r$ and $\mu'_r, \mu''_r$ are the real and imaginary parts of the relative permittivity and the relative permeability, respectively), the following condition is straightforwardly derived for such propagation:

$$\left[ + \left( \varepsilon'_r^2 + \varepsilon''_r^2 \right)^{1/2} - \varepsilon'_r \right] \left[ + \left( \mu'_r^2 + \mu''_r^2 \right)^{1/2} - \mu'_r \right] > \varepsilon''_r \mu''_r.$$

Before turning to a fully illustrative example in the proceeding section, let us investigate some immediate repercussions of the inequality (12) which is central to this paper.
Consider, in the first instance, the behaviour at a resonance of the relative permittivity, i.e. $\varepsilon_r' = 0$, $\varepsilon_r'' > 0$. Then, (12) reduces to
\[
\left[ +\left( \mu_r'^2 + \mu_r''^2 \right)^{1/2} - \mu_r' \right] > \mu_r'',
\]
an inequality that is always fulfilled when $\mu_r' < 0$. Likewise, at a resonance of the relative permeability, i.e. $\mu_r' = 0$, $\mu_r'' > 0$, (12) is fulfilled whenever $\varepsilon_r' < 0$.

Further insight into inequality (12) can be gained by requiring that
\[
+\left( \varepsilon_r'^2 + \varepsilon_r''^2 \right)^{1/2} > \varepsilon_r' + \varepsilon_r''
\]
and
\[
+\left( \mu_r'^2 + \mu_r''^2 \right)^{1/2} > \mu_r' + \mu_r''
\]
simultaneously hold. Consequently, (12) is definitely satisfied. It should be remarked though, that the parameter space of the permittivity and permeability that fulfils (14) and (15) is only a subset of the one fulfilling (12). In any case, (14) holds if and only if $\varepsilon_r' < 0$, and (15) holds if and only if $\mu_r' < 0$ (we remind the reader that $\varepsilon_r'' > 0$, $\mu_r'' > 0$ because of causality requirements). We note that $\varepsilon_r' < 0$ and $\mu_r' < 0$ can only occur close to absorption resonances (as discussed in the previous item).

Finally, consider an electromagnetic wave propagating in a plasma below the plasma frequency ($\varepsilon_r' < 0$, $\mu_r' = 1$) and in which dissipation is very small ($\varepsilon_r'' \ll 1$, $\mu_r'' \ll 1$). Straightforward Taylor expansions reduce inequality (12) to
\[
|\varepsilon_r'| > \frac{\varepsilon_r''}{\mu_r''}.
\]
Therefore, the existence of the type of anomalous propagation being studied here depends in this case crucially on the ratio of the imaginary parts of the relative permittivity and relative permeability. Whether the criterion (16) is satisfied or not, the power flow in this case is in any case small.

### 3. A detailed illustrative case study

Let us exemplify the foregoing in detail by an explicit invocation of the Lorentz model for both $\varepsilon_r$ and $\mu_r$; thus,
\[
\varepsilon_r(\lambda_0) = 1 + \frac{p_e}{1 + \left( N_e^{-1} - i\lambda_e\lambda_0^{-1} \right)^2}, \tag{17}
\]
\[
\mu_r(\lambda_0) = 1 + \frac{p_m}{1 + \left( N_m^{-1} - i\lambda_m\lambda_0^{-1} \right)^2}. \tag{18}
\]

Here $\lambda_0 = 2\pi/k_0$ is the free-space wavelength, $p_{e,m}$ are the oscillator strengths, $\lambda_{e,m}(1 + N_{e,m}^{-2})^{-1/2}$ are the resonance wavelengths, while $\lambda_{e,m}/N_{e,m}$ are the resonance linewidths.
Figures 2a and 2b comprise plots of the real and imaginary parts of $\varepsilon_r$ and $\mu_r$ as functions of $\lambda_0$, when $p_e = 1$, $p_m = 0.8$, $N_e = N_m = 100$, $\lambda_e = 0.3$ mm and $\lambda_m = 0.32$ mm. Clearly, five separate spectral regions can be identified in Figure 2. At the either extremity of the horizontal axis are the two regions wherein $\varepsilon'_r > 0$ and $\mu'_r > 0$. In the neighbourhood of $\lambda_0 = 0.22$ mm, $\varepsilon'_r < 0$ but $\mu'_r > 0$. Both $\varepsilon'_r < 0$ and $\mu'_r < 0$ in the neighbourhood of $\lambda_0 = 0.25$ mm. Finally, $\varepsilon'_r > 0$ but $\mu'_r < 0$ around $\lambda_0 = 0.31$ mm. Of course, both $\varepsilon''_r > 0$ and $\mu''_r > 0$ for all $\lambda_0$.

Detailed calculations confirm that the spectral region wherein the inequality (12) is satisfied is larger than the middle region (wherein both $\varepsilon'_r < 0$ and $\mu'_r < 0$). The former cover parts of the adjoining regions, in which either (14) or (15) holds.

In the five spectral regions identified, the isotropic dielectric–magnetic medium would respond differently to monochromatic electromagnetic excitation. Suppose that a plane wave is normally incident on a half–space occupied by this medium. The reflectance $R(\lambda_0)$ is then given by the standard expression

$$R(\lambda_0) = \left| \frac{\sqrt{\mu_r(\lambda_0)/\varepsilon_r(\lambda_0)} - 1}{\sqrt{\mu_r(\lambda_0)/\varepsilon_r(\lambda_0)} + 1} \right|^2,$$

where $0 \leq R \leq 1$ for all $\lambda_0$ by virtue of the principle of conservation of energy. The reflectance spectrum calculated with the constitutive parameters used for Figures 2a and 2b is shown in Figure 3. The reflectance is markedly high in the two regions wherein $\varepsilon'_r$ and $\mu'_r$ have opposite signs, but not in the other three regions. The reflectance is particularly low in the leftmost and the rightmost regions ($\varepsilon'_r > 0$ and $\mu'_r > 0$) because the ratio $\mu_r(\lambda_0)/\varepsilon_r(\lambda_0)$ is close to unity therein. However, the reflectance is somewhat higher in the central region ($\varepsilon'_r < 0$ and $\mu'_r < 0$) because $|\mu_r(\lambda_0)/\varepsilon_r(\lambda_0)| < 0.25$.

4. Conclusions

In this pedagogical note, we have derived a general condition for the phase velocity to be oppositely directed to the power flow in isotropic dielectric–magnetic mediums in which the only constraints on the values of the relative permittivity and relative permeability are those imposed by causality. In this regard, the topical case of mediums in which $\varepsilon_r$ and $\mu_r$ have negative real parts is seen to be a sufficient, but not necessary, condition for such propagation, as noted in the comments succeeding (12). An illustrative case study has shown that there are, in general, five distinct spectral regions, characterized by the various sign combinations of the real parts of $\varepsilon_r$ and $\mu_r$.

5. Acknowledgement

MWM acknowledges the support of the Engineering and Physical Sciences Research Council of Great Britain (EPSRC grant no. GR/R55078/01).
References

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Figure Captions

Figure 1. Argand diagram parametrically displaying $\mu_{\text{eff}}(\omega)$ from equation (1), with $\Gamma = 0.1\omega_0$ and $F = 0.5$. On taking (1) as a model resonance form for meta–materials, the plot can also be regarded as displaying the effective permittivity $\epsilon_{\text{eff}}$, for which the resonance at $\omega = \omega_0$ is unlikely to coincide, and hence the arrows indicating $\epsilon_{\text{eff}}$ and $\mu_{\text{eff}}$ do not coincide in general. The complex number $n_+ = +\sqrt{\epsilon_{\text{eff}}\mu_{\text{eff}}}$, while the corresponding index $n_- = -\sqrt{\epsilon_{\text{eff}}\mu_{\text{eff}}}$ is not shown. The dots indicate equi–spaced frequencies from $\omega = 0$ to $\omega = 2\omega_0$.

Figure 2. (a) Real parts of the relative permittivity and relative permeability according to equations (17) and (18), respectively, when $p_e = 1$, $p_m = 0.8$, $N_e = N_m = 100$, $\lambda_e = 0.3$ mm and $\lambda_m = 0.32$ mm. The significance of the identified five regions of the spectrum is explained in the text. (b) Imaginary parts of the relative permittivity and relative permeability according to equations (17) and (18), respectively, when $p_e = 1$, $p_m = 0.8$, $N_e = N_m = 100$, $\lambda_e = 0.3$ mm and $\lambda_m = 0.32$ mm.

Figure 3. Plane wave reflectance $R(\lambda_0)$ calculated with the constitutive parameters depicted in Figures 2.
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