The Radiative Return at small angles: virtual corrections

1 Introduction

The complete NLO corrections have currently been presented in a formal [5], and the NLO corrections have currently been presented in a formal [5], and the NLO corrections have currently been presented in a formal [5], and the NLO corrections have currently been presented in a formal [5].
with tagged photons, i.e. at least one photon was required to be emitted under large angles.

An important ingredient in the extension of the NLO Monte Carlo program PHOKHARA to small photon angles is the evaluation of the virtual corrections to the reaction (1) in the limit $m^2_{2}/s \ll 1$, which are equally valid for large and small angles. Compact results for the one-loop two-, three- and four-point functions that enter this calculation can be found in the literature [16,17] for arbitrary values of $m^2_{2}/s$. However, the combination of these analytical expressions with the relevant coefficients is numerically unstable in the limit of small mass and angles. A compact, numerically stable result, valid for an arbitrarily small photon angle, is therefore required. As a consequence of the highly singular kinematic coefficients, terms proportional to $m^2_{2}$ and even $m^2_{e}$ must be kept in the expansion, which after angular integration will contribute to the total cross section even in the limit $m^2_{2}/s \rightarrow 0$.

The present paper extends the analysis of Ref. [18] where the corrections from virtual and soft photons were presented for the case of large angles. In section 2 we recall the basic definitions and describe the systematic procedure used in the expansion of the results for small $m^2_{2}/s$ and small angles simultaneously. The analytic results for real and imaginary parts of the leptonic tensor, expressed in an angular momentum basis, are presented in section 3 and compared with results for related quantities that can be found in the literature. After summation over the polarizations of the virtual photons, our result agrees with the one of Berends, Burgers and van Neerven [20,21]. The result of Kuraev, Merenkov and Fadin [22] for the real part of the tensor, which was obtained for virtual Compton scattering $\gamma^{*} + e^{-} \rightarrow \gamma^{*} + e^{-}$ is related to our case by proper analytic continuation; and indeed after analytic continuation we find agreement for the real part1. Section 4 contains our summary and the conclusions. The mass-dependent terms proportional to $m^2_{2}$ and $m^2_{e}$, expressed in the Cartesian basis, are given in Appendix A. The scalar loop integrals needed for the calculation are listed in Appendix B.

2 The leptonic tensor for the radiative return

Consider the $e^{+} e^{-}$ annihilation process

$$e^{+}(p_1) + e^{-}(p_2) \rightarrow \gamma^{*}(Q) + \gamma(k_1) ,$$

(2)

where the virtual photon decays into a hadronic final state, $\gamma^{*}(Q) \rightarrow$ hadrons, and the real one is emitted from the initial state. The differential rate can be cast into the product of a leptonic and a hadronic tensor and the corresponding factorized phase space

$$d\sigma = \frac{1}{2s} L_{\mu\nu} H_{\mu\nu} d\Phi(p_1, p_2; Q, k_1) d\Phi_{\gamma}(Q; q_1 \ldots q_n) \frac{dQ^2}{2\pi} .$$

(3)

1 We disagree, however, with eq.(30) of the translated version [23] which contains a misprint.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Initial-state radiation in the annihilation process $e^{+} e^{-} \rightarrow \gamma^{*} \rightarrow$ hadrons at the Born level.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{One-loop corrections to initial-state radiation in $e^{+} e^{-}$ annihilation with the emission of a virtual photon.}
\end{figure}

where $d\Phi_{\gamma}(Q; q_1 \ldots q_n)$ denotes the hadronic $n$-body phase space, including all the statistical factors coming from the hadronic final state.

For an arbitrary hadronic final state, the matrix element for the diagrams in Fig. 1 can be written

$$M_{\gamma} = M_{\gamma}^{\mu} J_{\mu} ,$$

where $J_{\mu}$ is the hadronic current and $M_{\gamma}^{\mu}$ is the leptonic current in lowest order. Summing over the polarizations of the real photon, averaging over the polarizations of the initial $e^{+} e^{-}$ state, and using current conservation, $Q_{\mu} J^{\mu} = 0$, the leptonic tensor

$$L_{\mu\nu}^{\gamma} = \frac{M_{\gamma}^{\mu} M_{\gamma}^{\nu}}{M_{\gamma}^{\gamma}}$$

can be written in the following form:

$$L_{\mu\nu}^{\gamma} = \frac{(4\pi \alpha/s)^2}{q^\mu q^\nu} \left[ \left( \frac{2m^2 q^2 (1 - q^2)}{y_1^2 y_2^2} - \frac{2q^2 + y_1^2 + y_2^2}{y_1 y_2} \right) g^{\mu\nu} - \frac{8m^2}{y_1 y_2} \frac{p_1^\mu p_1^\nu}{s} + \left( \frac{8m^2}{y_1 y_2} - \frac{4q^2}{y_1 y_2} \right) \frac{p_1^\mu p_1^\nu}{s} \right] ,$$

(4)
with \( y_n = \frac{2k_1 \cdot p_n}{s} \), \( m^2 = \frac{m^2}{s} \), \( q^2 = \frac{Q^2}{s} \) \(^{(5)}\). It is symmetric under the exchange of the electron and the positron momenta. Expressing the bilinear products \( y_n \) by the photon emission angle in the cms frame

\[
y_n,2 = \frac{1 - q^2}{2}(1 \mp \beta \cos \theta), \quad \beta = \sqrt{1 - 4m^2},
\]

and rewriting the two-body phase space

\[
\Phi_2(p_1, p_2; Q, \frac{q}{s}) = \frac{1 - q^2}{32\pi^2} d\Omega,
\]

it is evident that expression (4) contains several singularities: soft singularities for \( q^2 \to 1 \) and collinear singularities for \( \cos \theta \to \pm 1 \). The former are avoided by requiring a minimal photon energy. The latter are regulated by the electron mass.

The physics of the hadronic system, whose description is model-dependent, enters only through the hadronic tensor

\[
H_{\mu\nu} = J_\mu J^{\nu*},
\]

where the hadronic current has to be parametrized through form factors. For two-charged pions in the final state, the current

\[
J_{\pi}^\mu = ieF_2(Q^2) (q_{\pi^+} - q_{\pi^-})^\mu,
\]

where \( q_{\pi^+} \) and \( q_{\pi^-} \) are the momenta of the \( \pi^+ \) and \( \pi^- \) respectively, is determined by only one function, the pion form factor \( F_2 \) [24]. The hadronic current for four pions exhibits a more complicated structure and has been discussed in [5, 25, 26].

At NLO, the leptonic tensor receives contributions both from one-loop corrections arising from the insertion of virtual photon lines in the tree diagrams of Fig. 1 and from the emission of an extra real photon from the initial state. In this paper, we consider only the emission of soft photons. The implementation of these results in the program PHOKHARA and the discussion of their physical consequences will be discussed in a separate work [27].

At NLO, the leptonic tensor has the general form:

\[
L_{\mu\nu} = \frac{(4\pi\alpha/s)^2}{q^4} \left[ a_{00} g_{\mu\nu} + a_{11} p^{\mu}_1 p^{\nu}_1 + a_{22} p^{\mu}_2 p^{\nu}_2 + a_{12} p^{\mu}_1 p^{\nu}_2 + a_{21} p^{\mu}_2 p^{\nu}_1 + i\pi a_{-1} \left( p^{\mu}_1 p^{\nu}_2 - p^{\mu}_2 p^{\nu}_1 \right) \right].
\]

Terms proportional to \( Q^2 \) are absent as a consequence of current conservation. The scalar coefficients \( a_{ij} \) and \( a_{-1} \) allow the following expansion

\[
a_{ij} = a_{ij}^{(0)} + \frac{\alpha}{\pi} a_{ij}^{(1)}, \quad a_{-1} = \frac{\alpha}{\pi} a_{-1}^{(1)} \quad \text{[7]}
\]

The LO coefficients \( a_{ij}^{(0)} \) can be directly read from eq. (4)

\[
a_{00}^{(0)} = \frac{2m^2 q^2 (1 - q^2)^2}{y_1^2 y_2^2} - \frac{2q^2 + y_1^2 + y_2^2}{y_1 y_2},
\]

\[
a_{11}^{(0)} = \frac{8m^2}{y_2^2} - \frac{4q^2}{y_1 y_2}, \quad a_{22}^{(0)} = a_{22}^{(1)}(y_1 \leftrightarrow y_2),
\]

\[
a_{12}^{(0)} = a_{12}^{(0)} + \frac{8m^2}{y_1 y_2}.
\]

The imaginary asymmetric piece proportional to \( a_{-1} \) appears for the first time at NLO. The leptonic tensor therefore remains fully symmetric only at LO.

As an alternative one can replace the Cartesian basis (eq. (9)) by a basis derived from the three circular polarization vectors of the virtual photon \( \varepsilon_L \) and \( \varepsilon_{\pm} \), defined through

\[
\varepsilon_L = \frac{2}{\sqrt{\mathbf{q}^2(1 - q^2)}} k_{1L},
\]

\[
\varepsilon_{\pm} = \frac{2}{\sqrt{\mathbf{q}^2(1 - q^2)}} \varepsilon_{\eta_{\eta \rho}} k_{1P} \varepsilon_{L}, \quad i, j = L, \pm .
\]

Only four of the scalar coefficients are independent

\[
a_{L-} = a_{L+}, \quad a_{L-} = a_{L+} = a_{L+}^L, \quad a_{\pm} = a_{\pm}^L.
\]

The trace of the leptonic tensor

\[
L_{\mu\nu}(q_\mu q_\nu - g_{\mu\nu} q^2) = \frac{(4\pi\alpha/s)^2}{q^4} \left[ a_{L+}^L + a_{L+} \right]
\]

is related to the cross section after angular averaging of the hadronic tensor.

The relations between the components in the Cartesian and the circular basis read as follows:

\[
a_{LL} = -a_{00} + \frac{1}{4q^2(1 - q^2)} \left[ (y_2 - q^2 y_1)^2 a_{11} + (y_1 - q^2 y_2)^2 a_{22} + 2(y_1 - q^2 y_2)(y_2 - q^2 y_1) a_{12} \right],
\]

\[
a_{LL} = \frac{\beta \sin \theta}{4\sqrt{q^2(1 - q^2)}} \left[ (y_2 - q^2 y_1) a_{11} + (y_1 - q^2 y_2) a_{22} + (1 + q^2)(y_1 - q^2 y_2) a_{21} - i\pi a_{-1} \right],
\]

\[
a_{L+} = \frac{\beta \sin^2 \theta}{8} \left[ a_{11} + a_{22} + 2a_{12} \right],
\]

\[
a_{L+} = a_{L+} - a_{00}.
\]
Conversely
\[ a_{00} = a_{+} - a_{+} , \]
\[ a_{11} = \frac{4}{(1 - q^2)^2} \left[ q^2 (a_{LL} + a_{+} - a_{+}) - 2 \frac{\sqrt{2} q^2 (y_1 - q^2 y_2)}{1 - q^2} \text{Re}(a_{L+}) \right. \]
\[ + \left. \frac{2 (y_1 - q^2 y_2)^2}{1 - q^2} a_{+} \right] , \]
\[ a_{22} = a_{11}(y_1 + y_2) , \]
\[ a_{12} = \frac{4 a_{+} - 4 a_{+}}{2 - \beta^2 q^2 \sin^2 \theta} , \]
\[ a_{+1} = \frac{4 \sqrt{2} q^2}{\pi (1 - q^2) \beta \sin \theta} \text{Im}(a_{L+}) . \]

The scalar coefficients in the circular basis are given at LO by
\[ a_{00}^{(0)} = \frac{2 q^2 \beta^2 \sin^2 \theta}{y_1 y_2} , \]
\[ a_{L+}^{(0)} = \frac{\beta \sin \theta}{y_1 y_2} (1 + q^2 - \frac{2 m^2 (1 - q^2)^2}{y_1 y_2}) , \]
\[ a_{+}^{(0)} = \frac{\beta^2 \sin^2 \theta}{y_1 y_2} (1 - q^2 + \frac{m^2 (1 - q^2)^2}{y_1 y_2}) , \]
\[ a_{+}^{(0)} = a_{+}^{(0)} + \frac{2 q^2 y_1 + y_2}{y_1 y_2} - \frac{2 m^2 q^2 (1 - q^2)^2}{y_1 y_2} . \]

The one-loop matrix elements (Fig. 2) contribute to the leptonic tensor through their interference with the lowest order diagrams (Fig. 1). They contain ultraviolet (UV) and infrared (IR) divergences, which are regularized using dimensional regularization in $D = 4 - 2 \epsilon$ dimensions. The UV divergences are renormalized in the on-shell scheme. The IR divergences are cancelled by adding the contribution of an extra soft photon emitted from the initial state and integrated in the phase space up to an energy cutoff $E_\gamma < w \sqrt{s}$ far below $\sqrt{s}$. The result, which is finite, depends, however, on this soft photon cutoff. Only the contribution from hard photons with energy $E_\gamma > w \sqrt{s}$ would cancel this dependence.

The algebraic manipulations have been carried out with the help of the FeynCalc Mathematica package [28]. Using standard techniques [29], it automatically reduces the evaluation of the one-loop contribution to the calculation of a few scalar one-loop integrals and performs the Dirac algebra.

Since we consider the small angular region, mass terms proportional to $y_1$ and even $y_1$ arise. Terms proportional to $m^2$ and even $m^2$ must be kept, if they are multiplied by $y_1^2$ and $y_1^3$ respectively. In the expansion of the one-loop integrals, functions that depend on the ratio $m^2 / y_1$ cannot be expanded, in contrast to functions of $y_1$ or $y_1$ separately. To arrive at a systematic approach we therefore make the replacements $m^2 \rightarrow \lambda m^2$, $y_1 \rightarrow \lambda y_1$, perform the expansion for small $\lambda$ up to the appropriate order, and set $\lambda = 1$ at the end.

### 3 The NLO leptonic tensor

Combining the one-loop and the soft contribution we now arrive at the leptonic tensor in NLO. It will be convenient to split the coefficients $a_{ij}^{(1)}$ into a part that contributes at large angles and a part proportional to $m^2$ and $m^3$, denoted by $a_{ij}^{(1,0)}$ and $a_{ij}^{(1,1)}$ respectively:

\[ a_{ij}^{(1)} = a_{ij}^{(0)} \left[ - \log(4 \theta^2) \left[ 1 + \log(m^2) \right] \right. \]
\[ - \frac{3}{2} \frac{m^3}{4 \theta^2} \frac{m^2}{4 \theta^2} - \frac{3}{2} \left] + a_{ij}^{(1,0)} + a_{ij}^{(1,1)} . \]

The factor proportional to the LO coefficients $a_{ij}^{(0)}$ contains the usual soft and collinear logarithms. The expressions are particularly compact in the circular basis. For completeness we also repeat the results for $a_{ij}^{(1,0)}$, which can be found in [18]3, albeit in the Cartesian basis:

\[ a_{L+}^{(1,0)} = \frac{2 q^2 y_1}{y_1 y_2} \left( - \frac{y_1}{1 - y_2} - 2 \log(q^2) + 2 L(y_1) \right. \]
\[ + \left. \left[ 1 - \frac{y_1}{1 - y_2} - \frac{q^2}{(1 - y_2)^2} \right] \log\left( \frac{y_1}{q^2} \right) \right] , \]

\[ a_{L+}^{(1,0)} = \frac{\beta \sin \theta}{4 \sqrt{2} q^2 (1 - q^2)} \left( - \frac{y_1}{1 - y_2} - 2 \log(q^2) + 2 \frac{1}{y_2} \right. \]
\[ - \frac{2 q^2 (1 + q^2)}{y_2} \log(q^2) + \left. \left[ 1 - \frac{y_1}{1 - y_2} - \frac{q^2}{(1 - y_2)^2} \right] \log\left( \frac{y_1}{q^2} \right) \right] \]
\[ - \frac{2 q^2}{y_2} \log\left( \frac{y_1}{y_2} \right) - \frac{1}{1 - y_2} - \frac{2 q^2}{y_2} \log\left( \frac{y_1}{y_2} \right) \]
\[ + \frac{1}{1 - y_2} + \frac{2 q^2}{(1 - y_2)^2} \left[ - \log\left( \frac{y_1}{y_2} \right) \right] , \]

\[ a_{L+}^{(1,0)} = \frac{\beta \sin \theta}{4 \sqrt{2} q^2 (1 - q^2)} \left( - \frac{y_1}{1 - y_2} - 2 \log(q^2) + 2 \frac{1}{y_2} \right. \]
\[ + \frac{1}{1 - y_2} + \frac{2 q^2}{(1 - y_2)^2} \log\left( \frac{y_1}{y_2} \right) + \left. \frac{2}{1 - y_2} \left[ 1 - \frac{y_1}{y_2} \right] \log\left( \frac{y_1}{y_2} \right) \right] , \]

3 The result for the imaginary part of $a_{L+}$ differs from $a_{L+}$ in the original version of [18].
\[ a_{+}^{(1,0)} = \frac{q^2}{(1 - q^2)^2} \left( \frac{1 + q^4}{2q^4(1 - y_1)} - \frac{1 - y_2 \log(q^2)}{y_1} \right) - \left[ 1 + \frac{(1 - q^2)^2}{y_2} - \frac{q^2(1 - q^2)}{1 - y_2} - \frac{1 + q^4}{2(1 - y_2)^2} \right] \times \log\left(\frac{y_1}{q^2}\right) - \left[ 2 + \frac{(1 - q^2)^2}{y_2} \right] L(y_1) + [y_1 \leftrightarrow y_2] \right), \]

with

\[ L(y) = L_i(1 - \frac{y \log(y)}{q^2}) \]

where \( L_i \) is the Spence or dilogarithm function. The result in the Cartesian basis has been given in Ref. [18]. The terms proportional to powers of \( m^2 \) are given by

\[ a_{+}^{(1,0), L_L} = 0, \]

\[ a_{+}^{(1,0), L_L} = \frac{\beta^2 \sin^2 \theta}{8} \left( \frac{4m^2}{y_1} \left[ \log(q^2) \log\left(\frac{y_1^2}{m^2 q^2}\right) + 4L_i(1 - q^2) \right] \right) \]

\[ + \frac{m^2}{y_1} \left[ L_i(1 - \frac{y_1 - m^2}{m^2}) - \frac{\pi^2}{6} - \frac{m^2}{y_1} \left[ 1 - \log\left(\frac{y_1}{m^2}\right) \right] \right] + \frac{m^2}{y_1} \left[ L_i(1 - \frac{y_1 - m^2}{m^2}) - \frac{\pi^2}{6} \right] + \frac{q^2}{2} n(y_1, \frac{1 - 3q^2}{y_1}) \]

\[ + [y_1 \leftrightarrow y_2], \]

and

\[ a_{+}^{(1,0), L_L} = \left\{ \begin{array}{l}
\beta^2 \sin^2 \theta \left( \frac{4m^2}{y_1} \left[ \log(q^2) \log\left(\frac{y_1^2}{m^2 q^2}\right) + 4L_i(1 - q^2) \right] \right)
+ \frac{m^2}{y_1} \left[ L_i(1 - \frac{y_1 - m^2}{m^2}) - \frac{\pi^2}{6} - \frac{m^2}{y_1} \left[ 1 - \log\left(\frac{y_1}{m^2}\right) \right] \right]
+ \frac{m^2}{y_1} \left[ L_i(1 - \frac{y_1 - m^2}{m^2}) - \frac{\pi^2}{6} \right] + \frac{q^2}{2} n(y_1, \frac{1 - 3q^2}{y_1})
+ \frac{1}{y_1} \left( \frac{3 - q^2}{1 - q^2} + \frac{m^2}{m^2(1 - q^2) - y_1} \right) N(y_1)
+ [y_1 \leftrightarrow y_2] \right. \end{array} \}

The coefficient \( a_{+} \) is antisymmetric with respect to the exchange \([y_1 \leftrightarrow y_2]\), while all the others are symmetric. Only \( a_{+}^{(1,0)} \) contributes to the imaginary part. Notice that the mass-suppressed terms are all real. The functions \( n(y_1, z) \) and \( N(y_1) \) are defined through

\[ n(y_1, z) = \frac{m^2}{y_1 (m^2 - y_1)} \left[ 1 + z \log\left(\frac{y_1}{m^2}\right) \right] + \frac{m^2}{(m^2 - y_1)^2} \log\left(\frac{y_1}{m^2}\right), \]

\[ N(y_1) = \log(q^2) \log\left(\frac{y_1}{m^2}\right) + L_i(1 - q^2) \]

\[ + L_i(1 - \frac{y_1}{m^2}) - \frac{\pi^2}{6}. \]

The apparent singularity of \( n \) inside the limits of phase space is compensated by the zero in the numerator. For the numerical evaluation in the region \( y_i \) close to \( m^2 \) we use

\[ n(y_1, z) \bigg|_{y_i \rightarrow m^2} = \frac{1}{y_1} \left[ 1 + z \log\left(\frac{y_1}{m^2}\right) \right] - 1 \sum_{n=0} \left( \frac{1}{n + 2} + \frac{z}{n + 1} \right) \left(1 - \frac{y_1}{m^2}\right)^n. \]

For the conversion from the circular to the Cartesian basis, and to ensure finite results in the limit \( \sin \theta \rightarrow 0 \), it is important that \( a_{+} \) and \( a_{+} \) vanish at \( \sim \sin \theta \) and \( \sin^2 \theta \) respectively. This corresponds to the requirement that the factors in curly brackets do not diverge for small \( \sin \theta \), i.e. in the limit \( (m^2(1 - q^2) - y_i) \rightarrow 0 \). This is guaranteed by the behaviour of \( N(y_1) \) for \( y_i \rightarrow m^2(1 - q^2) \):

\[ \frac{m^2 N(y_1)}{m^2(1 - q^2) - y_i} \bigg|_{y_i \rightarrow m^2(1 - q^2)} = \frac{-\log(1 - q^2) - \log(q^2)}{1 - q^2}, \]

The results for \( a_{+}^{(1,0)} \) in the Cartesian basis are listed in Appendix A.

We note that the imaginary part of \( L_{\mu \nu} \), which is present in the coefficients \( a_{\mu+} \) or \( a_{-} \) only, is of interest for those cases where the hadronic current receives contributions from different amplitudes with non-trivial relative phases. This is possible, e.g. for final states with three or more mesons or for pp̅ production.

4 Tests of the result

After summation over the polarizations of the virtual photon the differential rate is given by the properly contracted
leptonic tensor:
\[
\frac{q^2}{(4\pi\alpha/s)^2} L^\mu\nu(\eta_\mu Q_\nu - g_{\mu\nu} q^2) = a_{LL} + 2a_{++}
\]
\[
-3\alpha g_{\eta\eta} + \frac{1}{4\eta^2}(1 - y_1^2 - m^2) a_{11}
\]
\[
+ \frac{1}{4\eta^2}(1 - 2y_1) a_{12}
\]
\[
+ \frac{1}{2}(1 - y_1)(1 - y_2) - (1 - 2m^2) a_{12}
\]
\[
= L_0 \left\{ 1 + \alpha \pi \left[ -\log(4u^2)[1 + \log(m^2)] \right]
\right.
\]
\[
- \frac{3}{2} \log_2\frac{m^2}{\eta^2} - 2 + \frac{\alpha^2}{2} \right\} + \frac{\alpha}{\pi}\left\{ \frac{1}{2} \log_2\frac{1 + (1 - y_2)^2}{2y_2} + L(y_1)
\right.
\]
\[
+ \frac{1}{2}(1 - y_1 - 2y_2) \log(q^2)
\]
\[
+ \log_2\frac{1 - 1}{y_1} + \frac{1 - y_2}{1 - y_1}
\]
\[
+ \left[ y_1 \leftrightarrow y_2 \right] - 2a_{++}^{(1,m)}
\right\} , \quad (31)
\]

where

\[
L_0 = \frac{q^2}{(4\pi\alpha/s)^2} L_0(\eta_\mu Q_\nu - q^2 g_{\mu\nu}) =
\]
\[
- \frac{2}{y_1 y_2}\left( 2q^2 + y_1^2 + y_2^2 - 2m^2 q^2(1 - y^2)^2 \right), \quad (32)
\]

\( L_0^{\mu\nu} \) being the leptonic tensor at LO. In \( L_0 \) only the relevant terms in the limit \( m^2 \to 0 \) have been kept. Large angle terms and mass corrections are in agreement with Berends et al. [20, 21]. After proper analytic continuation and using the substitutions

\[
t \to -y_1 + i\eta, \quad s \to y_1 y_2 + i\eta, \quad u \to y_1 + i\eta, \quad \eta
\]

the leptonic tensor in eq. (9) is closely related to the tensor \( T_0^{\mu\nu} \), which describes “virtual Compton scattering” and has been calculated by Kuraev et al. [22]. After performing this analytical continuation, the results are in agreement (However, the result for \( T_{12} \) printed in [23] contains a typo and would give rise to a discrepancy.)

5 Conclusions

Compact analytical formulae have been obtained for the virtual corrections to the process \( e^+e^- \to \gamma^* \), which are valid for photon emission under both large and small angles. After proper analytic continuation the results are in agreement with those obtained in [22] for the reaction \( \gamma^* + e^- \to \gamma + e^- \). In polarization averaged form they are in agreement with those for \( \gamma^* + e^- \to \gamma + Z \) from [20, 21].

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A The leptonic tensor in the Cartesian basis

For the convenience of the reader we list the mass-suppressed terms \( a_{ij}^{(1,m)} \) also in the Cartesian basis. The large-angle contributions have been given in [18]. The component proportional to \( g_{\mu\nu} \) reads

\[
a_{ij}^{(1,m)} = -a_{++}^{(1,m)} , \quad (34)
\]

see eq. (25). The coefficient \( a_{++} \) does not contribute at this order to \( a_{00} \). The remaining components are given by

\[
a_{11}^{(1,m)} = \frac{q^2}{1 - q^2} \left\{ \frac{4m^2}{y_1} \left[ 1 - \log\frac{y_1}{m^2} \right]
\right.
\]
\[
+ \frac{m^2}{y_1} \left( L_1(1 - y_1 - m^2) - \frac{\pi^2}{6} \right) - n(y_1, 1)
\]
\[
+ \frac{2m^2 q^2}{y_1}\left[ \frac{1}{y_1} \log_2\frac{y_1}{m^2} + \log_2\frac{y_1}{y_2} \right]
\]
\[
+ \left( 1 + \frac{m^2}{y_1} \right) N(y_1) \right\} +
\]
\[
+ \frac{1}{1 - q^2}\left[ \frac{4m^2(1 - q^2)}{y_2} \log_2\frac{y_2^2}{m^2 q^2} \right]
\]
\[
+ 4L_2(1 - q^2) + 2 \left( L_1(1 - y_1 - m^2) - \frac{\pi^2}{6} \right)
\]
\[
+ \frac{4m^2 q^2}{y_2} \left[ 1 - \log_2\frac{y_2^2}{m^2} + \left( 1 + \frac{m^2}{y_2} \right) \left( L_2(1 - y_2^2) - \frac{\pi^2}{6} \right) \right]
\]
\[
- \frac{1}{1 - q^2} n(y_1, 1, y_2, 3 - 8q^2 + 6q^2)
\]
\[
+ \frac{2m^2}{y_2}(1 - q^2) - y_2\left[ \frac{1}{y_2} \log_2\frac{y_2^2}{m^2} + \log_2\frac{y_2}{y_1} \right]
\]
\[
+ \left( 3 + \frac{m^2}{y_2} \right) N(y_2) \right\} , \quad (35)
\]

\[
a_{22}^{(1,m)} = a_{11}^{(1,m)}(y_1 \leftrightarrow y_2) , \quad (36)
\]

\footnote{We thank N.P. Menenkov for drawing our attention to this reference.}
\[ a_{12}^{(1,m)} = - \frac{q^2}{1 - q^2} \left( \frac{4m^2}{y_1^2} \left[ 1 - \log \left( \frac{y_1}{m} \right) \right] \right) + \left( \frac{1}{2} + \frac{2y_1}{y_1^2} \right) \left( L_2(1 - \frac{y_1}{m^2}) - \frac{\pi^2}{6} \right) \]

\[ - \frac{1 - q^2}{q^2} n(y_1, \frac{1}{1 - q^2} + \frac{2m^2}{y_1^2}) \times \left[ \frac{1}{q^2} \log \left( \frac{y_1}{m} \right) + \log \left( \frac{q^2}{y_1} \right) \right] + \left( 2 + \frac{m^2}{y_1^2(1 - q^2) - y_1} \right) N(y_1) \right\} + [y_1 \leftrightarrow y_2]. \] (37)

and

\[ a_{-1}^{(1,m)} = 0. \] (38)

### B Scalar one-loop integrals

A few two-, three-, and four-point scalar one-loop integrals enter our calculation. Expression for the two-point scalar integrals are simple and well known. The notation from [18], where the corresponding results valid for large photon angles \((m^2 \ll 1, q^2, y_1)\) can be found, is used in the following. The general three-point scalar one-loop integral is defined by

\[ C_3(p_3^2, (p_a - p_b)^2, y_1^2, m_1^2, m_2^2, m_3^2) = \text{in} \, 16 \pi^2 \mu^{4-D} \times \frac{1}{(2\pi)^D} \int \frac{d^Dk}{[k^2 - m_1^2][(k - p_1)^2 - m_2^2][(k - p_2)^2 - m_3^2]} \] (39)

Four different three-point scalar one-loop integrals are needed

\[ C01 = C_3((p_1 - k_1)^2, 0, m_1^2, m_2^2, m_3^2) , \]
\[ C02 = C_3((p_2 - k_2)^2, m_1^2, 0, m_2^2, m_3^2) , \]
\[ C03 = C_3((p_3 - k_3)^2, Q^2, m_1^2, m_2^2, m_3^2) , \]
\[ C04 = C_3(Q^2, 0, m_1^2, m_2^2, m_3^2) , \]

\[ i = 1, 2, \text{ and one scalar box} \]

\[ D0 = - i16\pi^2 \mu^{4-D} \int \frac{d^Dk}{(2\pi)^D} \times \frac{1}{k^2[(k + p_1)^2 - m_1^2][(k + p_2 - k_1)^2 - m_2^2][(k - p_3)^2 - m_3^2]} \] (41)

with \( j \neq i \).

The following simple expressions are used, from where the limits \((m^2 \ll 1, q^2, y_1)\) or \((m^2 \ll 1, q^2 \text{ but } m^2 \sim y_1)\) are obtained:

\[ C01 = \frac{s^{-1}}{y_1} \left[ L_2\left(1 - \frac{y_1}{m^2}\right) - \frac{\pi^2}{6}\right] , \]
\[ C02 = \frac{s^{-1}}{\beta} \left[ \frac{\sqrt{\Delta - 2 \log(\beta)}}{2} \log(c) \right] , \]
\[ - 2L_2 \left( c \right) - \frac{2 \pi^2}{3} + i \pi \left( \Delta - 2 \log(\beta) \right) , \]
\[ C04 = \frac{s^{-1}}{c_q} \left[ \frac{\log(c)}{2} - \frac{\log(c_q)}{2} + i \pi \log\left( \frac{c}{c_q} \right) \right] , \]
\[ D0 = \frac{s^{-1}}{\beta y_1} \left[ \frac{\sqrt{\Delta + 2 \log\left( \frac{m}{y_1} \right)}}{2} \log(c) + \log^2(c) \right] \]
\[ + 2L_2 \left( 1 - \frac{c_q}{c} \right) + 2L_2 \left( 1 - c \right) - L_2\left( 1 - c_q \right) - \pi^2 \]
\[ - i \pi \left( \Delta + 2 \log\left( \frac{m}{c y_1} \right) \right) . \] (42)

with

\[ \Delta = \frac{(4\pi)^2}{\varepsilon} \left( \frac{\mu^2}{s} \right)^2 , \] (43)

and

\[ \beta = \sqrt{1 - 4m^2} , \]
\[ c = \frac{1 - \beta}{1 + \beta} , \]
\[ c_q = \frac{1 - \beta}{1 + \beta} . \] (44)

Our expression for the C03 function is rather cumbersome:

\[ C03 = \frac{s^{-1}}{q^2} \log\left( \frac{z_3 - z_1}{z_3 - z_2} \right) \log\left( \frac{1 - z_1}{1 - z_2} \right) \]
\[ + \left( L_2\left( \frac{1}{z_1} \right) \log(1 - \frac{z_2}{z_1}) \right) \log(1 - \frac{1}{z_1}) \]
\[ + L_2\left( \frac{1 - z_1}{z_3 - z_1} \right) - L_2\left( \frac{1 - z_1}{z_2 - z_1} \right) - L_2\left( \frac{1 - z_2}{z_4 - z_2} \right) - L_2\left( \frac{1 - z_4}{z_2 - z_4} \right) - [z_1 \leftrightarrow z_2] \] (45)

where

\[ z_{1,2} = \frac{q^2 + y_1}{2q^2} \left( 1 \pm \sqrt{1 - 4(m^2 - i\eta)q^2} \right) , \]
\[ z_{3,4} = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4(m^2 - i\eta)q^2} \right) , \] (46)

being

\[ Im(C03) = \frac{\pi}{q^2(z_1 - z_2)} \log\left( \frac{(z_3 - z_1)(z_4 - z_3)}{(z_3 - z_2)(z_4 - z_2)} \right) . \] (47)
References

1. [LEP Collaborations], hep-ex/0103048.
10. A. Akoisio et al. [KLOE Collaboration], hep-ex/0402023.
12. M. Adinolli et al. [KLOE Collaboration], hep-ex/0000330.