The Rotation and Shear of a String.

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Abstract

Whether a string has rotation and shear can be investigated by an analogy with the point particle. Rotation and shear involve first covariant spacetime derivatives of a vector field and, because the energy-momentum tensor for both the point particle and the string have no such derivatives, the best vector fields can be identified by requiring the conservation of energy-momentum. It is found that the best vector field is a non-unit accelerating field in $x$, rather than a unit non-accelerating vector involving the momenta; it is also found that there is an equation obeyed by the spacetime derivative of the Lagrangian

$$\mathcal{L}^\mu = k(\sqrt{-\gamma})^\mu = (p + 1) \Theta P_\alpha^\mu$$

using notation which will be defined in the paper.

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1 Introduction

In general relativity the covariant derivative of a vector field can be decomposed into parts called the rotation, shear and so forth, see for example [2] and the appendix §5: these reduce to the same objects in Newtonian fluid dynamics. The vector field can be thought of as being tangent to the world-line of a particle and so it is of particular interest what these objects are for a point particle whose motion has a Lagrangian description. It is not immediate what the best vector field to chose is: one could choose the unit vector field \( P^\mu/m \), which would considerably simplify subsequent calculations; or one could choose the non-unit vector field \( \dot{x}^\mu \), which has acceleration. Here it is suggested that the best choice is \( \dot{x}^\mu \), as the conservation of energy-momentum then takes the simplest form. The analysis is then extended from the point particle to the string [3] and membrane [1]. Two topics not looked at are: firstly boundary conditions, in fact only the rotation and shear of a part of the string are looked at, whether the whole object has these properties is left open; secondly its classification, the existence, or otherwise, of shear and rotation, can signify the classification, for example Petrov type, of the spacetime under consideration; this can be thought of as a classification of the covariant derivatives of the vector field: thus in the present case there could be a classification of the spacetime covariant derivative and the internal covariant derivative, and perhaps a mixture of the two. The notation used is that of [2], except that \( p \) is the dimensions of the brane, \( P \) denotes the momentum, and \( \mathcal{P} \) is the pressure, \( \gamma \) is the internal metric, \( \Gamma \) is used for the \( \gamma \)-equation of state.
2 The Point Particle.

2.1 The Point Particle’s Action.

The coordinate space action of a point particle is

\[ S = \int_{\tau_1}^{\tau_2} d\tau L, \]

(1)

for the simplest case of a non-interacting particle, the specific lagrangian is

\[ L = -m\ell, \quad \ell = \sqrt{-\dot{x}^2}, \quad \dot{x}^\alpha = \frac{D}{d\tau} x^\alpha. \]

(2)

This is sometimes also called the non-linear form, and sometimes the determinant form. Varying the velocity \( \dot{x}^\alpha \) one can interchange the variation and the covariant derivative

\[ \delta \dot{x}^\alpha = \delta \frac{D}{d\tau} x^\alpha = D\frac{\delta x^\alpha}{d\tau}, \]

(3)

So that the variation of the action is

\[ \delta S = \frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu|_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau \delta x^\mu \frac{D}{d\tau} \frac{\partial L}{\partial \delta \dot{x}^\mu}. \]

(4)

The first term of the varied action 4 can be made to vanish by using suitable boundary conditions, a simple choice is

\[ \delta x^\mu|_{\tau_1} = \delta x^\mu|_{\tau_2} = 0, \]

(5)

which are obeyed in the particular case

\[ \delta x^\mu|_{\tau_1} = \delta x^\mu|_{\tau_2} = 0. \]

(6)

The second term of the varied action 4 vanishes when the equations of motion

\[ \frac{D}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} = 0, \]

(7)

are obeyed. Substituting \( \partial L/\partial \dot{x}^\mu \) for the particular Lagrangian defined in 2 gives

\[ m \frac{D}{d\tau} \frac{\dot{x}^\mu}{\ell} \equiv 0 \]

(8)

where in this and subsequent cases \( \equiv \) signifies that the equality holds only for a specific Lagrangian in this case 2. The covariant nature of \( D/d\tau \) ensures that 8 is the geodesic equation.
2.2 Introducing Momenta and the Hessian.

The momentum and its absolute derivative are defined by

\[ P_\mu = \frac{\delta S}{\delta \dot{x}_\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = \frac{m}{\ell} \dot{x}_\mu, \quad \ddot{P}_\mu = \frac{D}{d\tau} P_\mu, \]  

(9)

The boundary condition 5 can be expressed in terms of the momenta

\[ \delta x^\mu P_\mu|_{\tau_1} = \delta x^\mu P_\mu|_{\tau_2} = 0, \]  

(10)

as can the equation of motion

\[ \dot{P}_\mu = 0. \]  

(11)

The Hessian is defined as

\[ W_{\mu\nu} \equiv \frac{\delta^2 S}{\delta \dot{x}_\mu \delta \dot{x}_\nu} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_\mu \partial \dot{x}_\nu} = \frac{2}{\ell} h_{\mu\nu}. \]  

(12)

\( h_{\mu\nu} \) is the projection tensor defined below 14. This equality suggests that the Hessian can be thought of as a generalized projection. For the particular Lagrangian 2 the geodesic equation 8 or 11 can be expressed in terms of the specific projection tensor

\[ \dot{P}_\mu = W^{\mu\nu} \ddot{x}_\nu = \frac{2}{\ell} h^{\mu\nu} \ddot{x}_\nu = 0, \]  

(13)

where the specific projection tensor familiar from general relativity is

\[ h^{\mu\nu} \equiv g^{\mu\nu} - \frac{\dot{x}^\mu \dot{x}^\nu}{\dot{x}^2} = g^{\mu\nu} + \frac{P_\mu \dot{x}^\nu}{m\ell} = g^{\mu\nu} + \frac{P_\mu P_\nu}{m^2}, \]

\[ h^{\mu\alpha} h_{\nu\alpha} = h^{\nu} = g^{\mu\nu} h_{\nu\alpha}, \]

\[ h^{\mu\nu} \ddot{x}_\nu = 0, \quad h = d - 1 \]  

(14)

2.3 Some Standard Equations.

The metrical stress, also called energy-momentum, is

\[ T^{\mu\nu} = -2 \frac{\delta S}{\delta g_{\mu\nu}}. \]  

(15)

In the present case this energy-momentum is restricted to the portion of the spacetime interior to the minimal surface, outside there is an exterior
energy-momentum which is often a vacuum. It is assumed that the energy-
momentum tensor is conserved

\[ T_{\mu\nu} = 0, \]  

(16)

For the point particle 2 the energy-momentum tensor is

\[ T_{\mu\nu} = -\ell^2 W_{\mu\nu} \equiv -m\ell h_{\mu\nu} \equiv -m\ell g_{\mu\nu} - P_{\mu}\dot{x}_\nu \]  

(17)

The perfect fluid energy-momentum tensor Eq.3.8[2] is

\[ T_{\mu\nu} = (\mu + P)V_\mu V_\nu + P g_{\mu\nu}, \quad = (\mu + P)h^{\mu\nu} - \mu g_{\mu\nu}, \quad V_\mu V^\mu = -1, \]  

(18)

in this case \( h^{\mu\nu} \) is given by the first equality of 14 with \( V^\mu = \dot{x}^\mu/\ell \). 18 can encompass the energy-momentum of many systems, such as scalar fields, essentially this is because of the freedom to choose an equation of state. When \( \mu = 0 \) the perfect fluid 18 has energy-momentum tensor

\[ T_{\mu\nu} = Pf_{\mu\nu}, \]  

(19)

comparing with the second equality of 17 the energy-momentum of a point particle can be thought of as a perfect fluid with density \( \mu = 0 \) and pressure

\[ P \equiv -m\ell. \]  

(20)

From the properties of the projection tensor 14, 17 gives

\[ \dot{x}^\nu T_{\mu\nu} \overset{2}{=} 0, \quad h^{\mu\alpha} T_{\alpha\nu} \overset{2}{=} -m\ell h^{\mu\nu} \]  

(21)

The first of these equations has as a consequence that one cannot usefully take a definition such as

\[ P^\mu = V_\nu T^{\mu\nu}, \]  

(22)

as taking \( V^\nu \) proportional to \( \dot{x}^\nu \) gives \( P^\nu = 0 \).

### 2.4 Derivatives of the Stress

Stress conservation 16 is explicitly

\[ T_{\mu\nu} = -\frac{m}{\ell} \left( \dot{x}^\alpha (\dot{x}_{\alpha} - \dot{x}_{\alpha}) + \dot{x}^\mu \dot{x}_{\alpha} - \frac{\dot{x}^\alpha \dot{x}^\beta}{x^2} \dot{x}_{\alpha\beta} \dot{x}^\mu \right), \]  

(23)

and is satisfied under the stringent condition

\[ \ddot{x}_{\alpha\beta} = 0. \]  

(24)
The equation 24 can be thought of as
\[ \dot{x}^\alpha_{;\beta} = \nabla_\beta \frac{D}{d\tau} x^\alpha = \frac{D}{d\tau} \nabla_\beta x^\alpha = \frac{D}{d\tau} \left( \delta^\alpha_{\beta} + \Gamma^\alpha_{\beta\gamma} x^\gamma \right) = \frac{D}{d\tau} \Gamma^\alpha_{\beta\gamma} x^\gamma \] (25)
implying that either the metric is flat or the coordinates are special. At first
sight the first term in 23 looks as if it is related to the vorticity and the second
term to the expansion of \( \dot{x}^\alpha \); however \( \dot{x}^\alpha \) is not of unit size
i.e. \( \dot{x}^\alpha \dot{x}^\alpha \neq 1 \), or even constant vector, so that the
identification is not straightforward, and the expression appears not to simplify.
Simplification is more easily achieved
with the mixed \( x, P \) form of the stress, see the last equality of 17 to get the
conservation equation
\[ T^{\mu\nu} = 2 m \ell^\mu - P^{\mu}_{\nu} \dot{x}^\nu + P^{\nu}_{\mu} \dot{x}^\nu. \] (26)
Taking the absolute derivative to be given by
\[ \frac{D}{d\tau} = \dot{x}^\nu \nabla_\nu, \] (27)
although it could be a function times this, the second term in 26 is the \( \tau \)
covariant derivative of \( P \), i.e.
\[ P^{\mu}_{\nu} \dot{x}^\nu = \dot{P}^\mu, \] (28)
and this vanishes by the equations of motion 11. This term would still vanish
if there was an additional function on the right of the right hand side of 27.
The last term in 26 can be taken to be the expansion of \( x \)
\[ \dot{x}^\nu_{;\nu} = \check{x}. \] (29)
Substituting 11, 16, 28 & 29 into 26 gives
\[ m \ell^\mu = \check{x} P^\mu, \] (30)
which equates the expansion of \( x \) to the change in the length \( \ell \). Now
\[ m \ell^\mu = -\mathcal{L}^\mu, \] (31)
so that
\[ -\mathcal{L}^\mu = \check{x} P^\mu. \] (32)
Expanding 30 for \( x \) gives
\[ \dot{x}^\alpha_{;\mu} \dot{x}^\mu = -\ddot{x}^\alpha_{;\mu} \dot{x}^\mu = -\ddot{x}^\mu + 2 \dot{x}^\alpha \dot{x}_{[\mu]}. \] (33)
which shows that the result 17 could not be directly obtained by differentiating the Lagrangian, neither does it follow immediately from 23. Collecting together 30, 32 and 33 gives

\[
L^{\mu} = -m \ell^{\mu} = \frac{m}{\ell} \dot{x}^\alpha \dot{x}_\mu^\alpha = \frac{m}{\ell}(\ddot{x} - 2\dot{x}^\alpha \dot{x}_{[\mu\alpha]}) = -\frac{m}{\ell} \dot{x}_\alpha^\alpha \dot{x}^\mu = -\Theta P^\mu. \tag{34}
\]

The preceding way of requiring stress conservation occurs in two parts separately is not the same as the standard approach when a vector field is present, for example for a perfect fluid. There one requires \( V^\alpha T^{\alpha\beta} \) and \( h^\alpha \beta T^{\beta\gamma} \) to vanish separately. To see what happens if this is tried in the present case notice the form of stress conservation

\[
T^{\mu\nu} = -m \ell^{\mu} h^{\mu\nu} - m \ell h^{\mu\nu}. \tag{35}
\]

Transvecting with \( V^\mu \) gives the second term equal to zero, substituting back into 35 gives the first term equal to zero, so that \( \ell \) is a constant, which implies a very restricted system. The middle term of 26 is going to vanish by the equations of motion and 30 is just what remains, so that this method imposes no restriction on the system.

### 2.5 The Phase Space Action.

The phase space action is

\[
S = \int_{\tau_1}^{\tau_2} d\tau L_{\text{ph}}, \quad L_{\text{ph}} = 2 \dot{x} \cdot P + \lambda_1 \sigma^1. \tag{36}
\]

this is sometimes called the first order form, and sometimes the Hamiltonian form. Varying with respect to \( \lambda, P^\alpha \) and \( x^\alpha \) gives

\[
\frac{\delta S}{\delta \lambda_1} = P^2 + m^2, \quad \frac{\delta S}{\delta P^\alpha} = \dot{x}^\alpha + 2\lambda_1 P^\alpha, \tag{37}
\]

\[
\int_{\tau_1}^{\tau_2} d\tau P_\alpha \delta \dot{x}^\alpha = \int_{\tau_1}^{\tau_2} d\tau \left[ \frac{D}{d\tau} (P_\alpha \delta x^\alpha) - \delta x^\alpha \frac{D}{d\tau} P_\alpha \right] = P_\alpha \delta x^\alpha |_{\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} d\tau \delta x^\alpha \dot{P}_\alpha,
\]

respectively. The first equality of the \( \delta x^\alpha \) variation follows from 3. The first term of the \( \delta x^\alpha \) variation is just the boundary condition 4 and so can be satisfied by 10. The second term of the \( \delta x^\alpha \) variation gives the equation of motion 11.

At first sight the treatment of the energy-momentum is different from the coordinate space approach. The absence of the square root changes the
first term by a factor of 2 and the constraint gives an extra term so that by
substituting
\[ T^{\mu\nu} = -2u[^{\mu}P^\nu] - 2\lambda_1 P^\mu P^\nu + g^{\mu\nu} \mathcal{L}, \] (38)
and this is the same as before 17.

2.6 The Second Order Action
Removing \( P^\alpha \) from the Lagrangian 36 using 38 the Lagrangian reduces to its second order form
\[ \mathcal{L}_2 = \frac{1}{2} (\eta^{-1} \dot{x}^2 - \eta m^2), \quad 2\lambda_1 = -\eta. \] (39)
Requiring that the variation of 39 with respect \( \eta \) to vanish gives
\[ \eta = \pm \sqrt{-\dot{x}^2}, \] (40)
requiring the variation of 39 with respect to \( \dot{x}^\mu \) to vanish gives
\[ \frac{D}{d\tau} \left( \frac{\dot{x}^\alpha}{\eta} \right) = 0, \] (41)
substituting the value of \( \eta \) 40 gives the equation of motion 8. The energy-
momentum tensor 15 is
\[ T^{\mu\nu} = \frac{1}{\eta} \left( -\dot{x}^\mu \dot{x}^\nu + \frac{1}{2} g^{\mu\nu} (\dot{x}^2 - m^2 \eta^2) \right), \] (42)
substituting the value of \( \eta \) 40 back into 42 gives the stress 17.

2.7 The Rotation and Shear of a Point Particle.
The identification of the preferred vector field as \( \dot{x}^\alpha \) in §2.4 remains the same as the stresses \( T^{\mu\nu} \) 17, 38, 42 and hence \( T^{..\nu}_{\cdot\cdot\nu} \) are unaltered. Thus under the transformation \( \tau \rightarrow \tau' \) the preferred vector field just transforms as \( \frac{dx^\alpha}{d\tau} \rightarrow \frac{dx'^\alpha}{d\tau'} \).

Having identified the vector \( x'^\alpha \), rather than say \( P^\alpha \), as the preferred one, one can use the standard decomposition [2] and the appendix §5 to calculate the rotation and shear.
3 The String.

A point particle can be thought of as a minimal surface of zero dimension $p = 0$, and a string a minimal surface of dimension one $p = 1$, and a membrane a minimal surface of arbitrary $p < d$ the dimension of the spacetime. In this section the simplest string action and in the next section the simplest membrane action are investigated.

3.1 The Nambu-Goto Action.

The action of the string [Eq.I.16 of [3]] can be taken in a form that corresponds to

$$S = \int_{\tau_1}^{\tau_2} \int_0^\pi d\tau d\sigma L, \quad (43)$$

where the standard specific Lagrangian is the Nambu-Goto Lagrangian

$$L = - \frac{A}{2\pi \alpha'}, \quad (44)$$

the area $A$ is a generalization of the length $\ell$ 2

$$A \equiv \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}, \quad (45)$$

also in 43, $\tau$ is the evolution parameter, $\sigma$ is the kinematic parameter, $\alpha'$ is the string tension, $\dot{x}^\alpha = D x^\alpha / d\tau$, $x' = D x^\alpha / d\sigma$. The velocities $\dot{x}$ and $x'$ can be varied in a similar manner to 3 to give the varied action

$$\delta S = - \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \delta x^\mu \left[ \frac{D}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} + \frac{D}{d\sigma} \frac{\partial L}{\partial x'^\mu} \right]$$

$$+ \int_0^\pi d\sigma \frac{\partial L}{\partial \dot{x}^\mu} \delta x^\mu|_{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \frac{\partial L}{\partial x'^\mu} \delta x^\mu|_{\sigma=0}. \quad (46)$$

Choosing initial and final positions on the string to be fixed so that 6 is obeyed the second term vanishes. For open strings the third term vanishes when the edge condition c.f. eq.I.18 of [3]

$$\frac{\partial L}{\partial x'^\mu}|_{\sigma=0} = \frac{\partial L}{\partial x'^\mu}|_{\sigma=\pi} \quad (47)$$

is obeyed. For closed strings $x^\mu(\tau, \sigma + 2\pi) = x^\mu(\tau, \sigma)$ c.f. §II.7 of [3]. The vanishing of the first term gives the string equation of motion

$$\frac{D}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} + \frac{D}{d\sigma} \frac{\partial L}{\partial x'^\mu} = 0 \quad (48)$$
3.2 Reduction to the Point Particle.

The theory of the string, a $p = 1$ minimal surface, must somehow incorporate that of a $p = 0$ minimal surface, which describes a point particle. To see how use the reduction equation

$$\frac{\partial L}{\partial x^{\alpha \prime}} = 0,$$

the string equation of motion 48 takes the same form as 7. For the specific Lagrangian 44, the reduction equations

$$\dot{x} \cdot x^{44} \equiv 0 \quad \& \quad x^{2} \equiv 1,$$

reduce the area $\mathcal{A}$ 45 to the length $\ell$. The reduction equation

$$m = \frac{\lambda}{2\pi\alpha'}$$

equates the coupling constants so that the point particles equation of motion 8 is recovered.

3.3 Introducing Momenta.

One can define momenta

$$P_{\tau \mu} \equiv \frac{\delta S}{\delta \dot{x}^{\mu}} = \frac{\partial L}{\partial \dot{x}^{\mu}}, \quad P_{\sigma \mu} \equiv \frac{\delta S}{\delta x^{\prime \mu}} = \frac{\partial L}{\partial x^{\prime \mu}}.$$  (52)

These allow the edge condition 47 to be put in the form

$$P_{\sigma}^{\mu}|_{\sigma=0} = P_{\sigma}^{\mu}|_{\sigma=\pi},$$

and the equations of motion 48 to be put in the simple form

$$\dot{P}_{\tau}^{\mu} + P_{\tau}^{\mu} = 0,$$  (54)

where $\dot{P}_{\tau}^{\mu} = DP_{\tau}^{\mu}/d\tau$ and $P_{\sigma}^{\mu} = DP_{\sigma}^{\mu}/d\sigma$. For the specific Lagrangian 43 the momenta 52 are

$$-2\pi\alpha' A P_{\tau}^{\mu} \equiv \dot{x} \cdot x^{\prime} P_{\tau}^{\mu} - x^{2} \dot{x}^{\mu}, \quad -2\pi\alpha' P_{\sigma}^{\mu} \equiv \dot{x} \cdot x^{\prime} P_{\sigma}^{\mu} - \dot{x}^{2} x^{\prime \mu},$$

and using 50 the reduction equation 49 becomes

$$\dot{x}^{2} x^{\prime \mu} = 0,$$  (56)

and $P_{\tau}^{\mu}$ reduces to $P_{\tau}^{\mu}$ and $P_{\sigma}^{\mu}$ vanishes. Inverting 55

$$\mathcal{A} \dot{x}^{\mu} = -2\pi\alpha' (\dot{x}^{2} P_{\tau}^{\mu} + \dot{x} \cdot x^{\prime} P_{\tau}^{\mu}) = 2\pi\alpha' (P_{\tau}^{2} P_{\tau}^{\mu} - P_{\tau} \cdot P_{\sigma} P_{\tau}^{\mu}),$$

$$\mathcal{A} x^{\prime \mu} = -2\pi\alpha' (x^{2} P_{\sigma}^{\mu} + \dot{x} \cdot x^{\prime} P_{\sigma}^{\mu}) = 2\pi\alpha' (P_{\tau}^{2} P_{\sigma}^{\mu} - P_{\tau} \cdot P_{\sigma} P_{\tau}^{\mu}).$$  (57)
3.4 Introducing the Hessian.

The specific projection tensor is defined by

\[ h_{\mu\nu}^{44} \equiv g_{\mu\nu} + \frac{1}{A^2} \times \left( \dot{x}^2 x^\mu x^\nu + x^\mu x^\nu \dot{x}^\nu - (\dot{x} \cdot x^\prime)(\dot{x}^\mu x^\nu + x^\mu x^\nu) \right), \]

\[ h = d - 2. \] (58)

this projection tensor is a generalization of the familiar one in general relativity 14. It can be written in several forms using momenta only or, momenta and coordinates, the most convenient is

\[ h_{\mu\nu}^{44} = g_{\mu\nu} + \frac{2\pi\alpha'}{A} \times \left( P^\mu_{\tau\nu} + P^\mu_{\sigma\nu} \right), \] (59)

which corresponds to the middle equality of the first line of 14. The projection tensor 58 obeys the equations

\[ h_{\mu\nu} x^\prime_{\nu} = x^\prime_{\mu} + \frac{1}{A^2} \times \left( (\dot{x}^2 x^\tau \cdot x^\prime - \dot{x} \cdot x^\tau \dot{x}^\prime) x^\mu + (x^2 \dot{x} \cdot x^\prime - \dot{x} \cdot x^\prime \cdot x^\prime) \dot{x}^\mu \right), \]

\[ h^{\mu\nu} x^\prime_{\nu} = 0, \quad h = d - 2, \quad h^{\mu\alpha} h_{\nu\alpha} = h_{\mu\nu}, \] (60)

with \( * = 0 \) or \( ' \) and \( h^{\mu\nu} x^\prime \) following by symmetry. The specific projection tensor allows the derivatives of the momenta to be expressed as

\[ 2\pi\alpha' A P^\tau_{\tau} = h^{\mu\nu}(x^2 \dot{x}^\prime_{\nu} - \dot{x} \cdot x^\prime \dot{x}^\prime_{\nu}) \pm \dot{x} \cdot x^\prime_{\nu} x^\prime_{\mu} \mp x^\prime_{\nu} x^\prime_{\mu} \] (61)

with the top sign in the last two terms for \( * = 0 \) and the bottom sign for \( *= ' \), and similarly for \( P^\mu_{\sigma} \). The Hessian is

\[ W_{\tau\tau}^{\mu\nu} = \frac{x^2 h^{\mu\nu}}{2\pi\alpha' A}, \quad W_{\sigma\tau}^{\nu\mu} = -\frac{1}{2\pi\alpha' A}(\dot{x} \cdot x^\prime h^{\mu\nu} + \dot{x}^\prime h^{\mu\nu} - x^\prime h^{\nu\dot{x}}), \]

\[ W_{\mu\nu}^{\sigma\sigma} = \frac{x^2 h^{\mu\nu}}{2\pi\alpha' A}, \quad W_{\tau\sigma}^{\nu\mu} = -\frac{1}{2\pi\alpha' A}(\dot{x} \cdot x^\prime h^{\mu\nu} + \dot{x}^\prime h^{\mu\nu} - x^\prime h^{\nu\dot{x}}) \] (62)

the equations of motion 54 become

\[ 2\pi\alpha' A(\dot{P}^\mu_{\tau} + P^\mu_{\sigma}) = 44 \quad 2\pi\alpha' A(x^\nu W^\mu_{\tau\nu} + x^\mu W^\nu_{\sigma\nu} + \dot{x}^\nu W^\mu_{\tau\nu} + W^\mu_{\tau\nu} + W^\mu_{\nu\tau}) \]

\[ = 44 \quad h^{\mu\nu}(x^2 \dot{x}^\prime_{\nu} + \dot{x}^2 x^\prime_{\nu} - 2\dot{x} \cdot x^\prime \dot{x}^\prime_{\nu}) = 0, \] (63)
3.5 The Stress and Its Derivatives.

The energy-momentum 15 for 44 is

\[ T_{\mu\nu} = -\frac{A}{2\pi\alpha'} (2h_{\mu\nu} - g_{\mu\nu}), \quad T = \frac{(4 - d)A}{2\pi\alpha'}, \quad (64) \]

and 64 is traceless in four dimensions. At first sight, using 18 for the perfect fluid equivalence of this energy-momentum one has

\[ p = \mu = \frac{A}{2\pi\alpha'}, \quad (65) \]

however this assumes that the \( h^{\mu\nu} \)'s 14 and 58 are the same which they are not. Choosing two interacting fluids with stress

\[ T^{\mu\nu} = (\mu_1 + p_1) V^\mu V^\nu + (\mu_2 + p_2) W_\mu W_\nu + (p_1 + p_2) g^{\mu\nu} + 2\xi V_\mu W_\nu \quad (66) \]

and choosing non-unit vectors \( V^\mu = \dot{x}^\mu / A, \ W^\mu = x'^\mu / A \) the interaction parameter is

\[ \xi = (\dot{x} \cdot x') \frac{A}{\pi\alpha'} \quad (67) \]

and there is one free function among the four \( \mu_1, \mu_2, p_1, p_2 \) letting it be \( p_2 \) gives

\[ p_1 = -\frac{A}{2\pi\alpha'} - p_2, \quad \mu_1 = p_2 + (1 - 2x'^2) \frac{A}{2\pi\alpha'}, \quad \mu_2 = -p_2 - \frac{\dot{x}^2 A}{\pi\alpha'}. \quad (68) \]

For investigation of the stress’s derivatives it is easiest to work with the mixed form of the projection tensor 59 giving

\[ T^{\mu\nu} = -2(P^\mu_\tau \dot{x}^\nu + P^\mu_\sigma x'^{\nu}) - \frac{Ag^{\mu\nu}}{2\pi\alpha'}. \quad (69) \]

Stress conservation 16 gives

\[ T^{\mu\nu}_{\mu \nu} = -\frac{A^\mu}{2\pi\alpha'} - 2(\dot{P}^\mu_\tau + P^\mu_\tau + \dot{\Theta} P^\mu_\tau + \Theta P^\mu_\tau) \quad (70) \]

where the absolute derivatives are taken to be given by

\[ \frac{D}{d\tau} = \dot{x}^\nu \nabla_\nu, \quad \frac{D}{d\sigma} = x'^\mu \nabla_\mu, \quad (71) \]

and the expansions are

\[ \dot{\Theta} = \dot{x}^\nu \Theta_\nu, \quad \Theta = x'^\nu. \quad (72) \]
After using the equations of motion 54 stress conservation gives
\[ A^\mu = -4\pi\alpha'(\Theta P^\mu_\tau + \Theta P^\mu_\sigma) \] (73)
which equates the expansion of \( x \) to the change in the area, compare 21.
Equations 72 suggest that there are two preferred vector fields \( \dot{x}^\mu \) and \( x'^\mu \), however \( \tau \) and \( \sigma \) are not unique and can transform, so that mixtures of \( \dot{x}^\mu \) and \( x'^\mu \) might be better preferred vector fields. The best way to approach this is through vector notation with \( a, b \ldots = \tau, \sigma \), this can be achieved with not much more effort by using the Dirac membrane.

4 The Dirac Membrane.

4.1 The Action.
One of several forms of the membrane action is that of Dirac [1]
\[ S_D = k \int_M d^{p+1}\xi (-\det\gamma_{ab})^{1/2}, \quad \gamma_{ab} = g_{\mu\nu}\partial_a x^\mu \partial_b x^\nu. \] (74)
Choosing
\[ p = 1, \quad k = -\frac{1}{2\pi\alpha'}, \quad a, b \ldots = \tau, \sigma, \quad d^2\xi = d\tau d\sigma, \] (75)
\[ \gamma = -\det(\gamma_{ab}) = -A^2, \quad \gamma_{ab} = \left( \begin{array}{cc} \dot{x}^2 & \dot{x} \cdot x' \\ x' \cdot \dot{x} & x'^2 \end{array} \right), \quad \gamma \gamma^{ab} = \left( \begin{array}{cc} x'^2 & -x' \cdot \dot{x} \\ -\dot{x} \cdot x' & \dot{x}^2 \end{array} \right), \]
this reduces to the Nambu-Goto action 44. The generalization of the projection tensors 14 and 58 is
\[ h^{\mu\nu} = g^{\mu\nu} - \gamma^{ab} \partial_a x^\mu \partial_b x^\nu, \quad h = d - 1 - p. \] (76)
The stress 15 generalizing 17 and 63 is
\[ T^{\mu\nu} = k(-\gamma)^{\frac{1}{2}} \left( (p + 1)h^{\mu\nu} - pg^{\mu\nu} \right), \quad T = k(-\gamma)^{\frac{1}{2}} \left( d - (p + 1)^2 \right) \] (77)
This energy-momentum 77 can be projected
\[ h_{\mu\rho}T^{\mu\nu} = k(-\gamma)^{\frac{1}{2}}h_{\mu\rho}, \quad \partial_a x_{\nu} T^{\mu\nu} = -k(-\gamma)p\partial_a x^\mu. \] (78)
The energy-momentum 77 is of the form of functions times \( g_{\mu\nu} \) and \( h_{\mu\nu} \) and so has a fluid analogy, assuming that there is a fluid of a form generalizing 18
\[ T^{\mu\nu} = (\mu + \mathcal{P})h^{\mu\nu} - \mu g_{\mu\nu}, \] (79)
which now has a much more complex projection tensor $h_{\mu\nu}$ given by 76. 77 gives the pressure and density

$$P = k\sqrt{-\gamma}, \quad \mu = kp\sqrt{-\gamma},$$

and this has a $\Gamma$-equation of state

$$P = (\Gamma - 1)\mu,$$

with

$$\Gamma = \frac{p + 1}{p}, \quad p \neq 0,$$

the $p = 0$ case being given by 20. For $p \neq 0$ the projection tensor involves more than one vector field and so does not have a familiar general relativistic interpretation of being orthogonal to a moving observer. The equation of motion resulting from varying 74 with respect to $x$ is

$$\partial_a[(-\gamma)^{1/2} \gamma^{ab} \partial_b x^\mu] = (-\gamma)^{1/2} \nabla^2 x^\mu = 0.\tag{83}$$

The momenta are

$$P^{\mu a} \equiv \frac{\delta S}{\delta \partial_a x_\mu} = \frac{\partial \mathcal{L}}{\partial \partial_a x_\mu} = k(-\gamma)^{1/2} \gamma^{ab} \partial_b x^\mu,\tag{84}$$

so that the equation of motion 76 can be expressed as

$$\nabla_a P^{\mu a} = 0.\tag{85}$$

In terms of the momenta 84 the projection tensor 76 generalizing 59 is

$$h^{\mu\nu} = g^{\mu\nu} - \frac{1}{k(-\gamma)^{1/2}} P^{\mu a} \partial_a x^\nu,\tag{86}$$

and the stress 77 generalizing 69 is

$$T^{\mu\nu} = k(-\gamma)^{1/2} g^{\mu\nu} - (p + 1) P^{\mu a} \partial_a x^\nu.\tag{87}$$

### 4.2 Derivatives of the Stress.

Covariantly differentiating with respect to $x^\nu$ gives the generalization of 70

$$T_{\cdot \cdot \cdot \cdot \nu} = k \left( (-\gamma)^{1/2} \right)^\mu -(p + 1) \left( \nabla_a P^{\mu a} + \partial_a \Theta P^\mu_a \right),\tag{88}$$
where the absolute derivatives are
\[ \nabla_a = \partial_a x^\nu \nabla_\nu, \] (89)
and the expansions are
\[ \Theta^a = \partial_a x^\nu. \] (90)
After using the equations of motion 85 stress conservation gives
\[ \mathcal{L}^\mu = k \left( (\gamma)^{\frac{3}{2}} \right)^\mu = \frac{k}{2} \sqrt{-\gamma} (\ln \gamma)^\mu = (p + 1) \Theta P_{a}^\mu, \] (91)
which equates the spacetime derivative of the Lagrangian to the expansions of \( x^\mu \), this generalizes 73 and 34. From 88 and 90 it is apparent that the best vector field for calculating shear and vorticity is
\[ V^\mu = \sum_{a=1}^{p} \partial_a x^\mu. \] (92)
using this vector one can use the standard decomposition [2], and also the appendix §5 below, to calculate the rotation and shear of a membrane.

5 Appendix

A vector field can be decomposed
\[ V_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3} \Theta h_{\mu\nu} \dot{V}_\mu V_\nu, \] (93)
where the vorticity tensor and vector are
\[ \omega_{\mu\nu} = h^{\alpha}_{\mu} h^{\beta}_{\nu} [V_{\alpha;\beta}], \quad \omega^\mu = \frac{1}{2} \eta^{\mu\alpha\beta} V_{\nu} \omega_{\alpha\beta}, \] (94)
the expansion tensor and scalar are
\[ \Theta_{\mu\nu} = h^{\alpha}_{\mu} h^{\beta}_{\nu} V_{\alpha;\beta}, \quad \Theta = V_{\cdot;\mu}, \] (95)
and the shear is
\[ \sigma_{\mu\nu} = \Theta_{\mu\nu} - \frac{1}{3} h_{\mu\nu} \Theta, \] (96)
6 Conclusion.

The vector $\mathbf{92}$, rather than say $\sum_{\alpha=1}^{\rho} P^{\mu\alpha}$, produces conservation equations $88$, terms of which can be removed by the equations of motion, leaving a simple relationship between the spacetime derivative of the Lagrangian and the expansion of $\mathbf{92}$: thus it is best for forming geometric objects, such as rotation and shear.

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References

    An Extendible Model of the Electron.

[2] Hawking, S. W. and Ellis, G. F. R. (1973) § 1, 2.3
    The large scale structure of space-time.
    Math. Rev. 54#12154
    Cambridge University Press.

[3] Scherk, J. (1975) § 1
    An Introduction to the theory of dual models and strings.
    Rev. Mod. Phys. 47, 123-164.