\[
\begin{align*}
\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta A + c_{nf}d - c_{nf}n (d + \lambda) &= c_{nf}L,
\end{align*}
\]

where is the spatial part of the Lorentz index.

\[
\begin{align*}
\phi_{\alpha_1}\phi_{\beta_1}\phi_{\gamma_1}\phi_{\delta_1} &= \phi_{\alpha_2}\phi_{\beta_2}\phi_{\gamma_2}\phi_{\delta_2} \\
\theta_{\alpha_3}\theta_{\beta_3}\theta_{\gamma_3}\theta_{\delta_3} &= \theta_{\alpha_4}\theta_{\beta_4}\theta_{\gamma_4}\theta_{\delta_4} \\
\end{align*}
\]

\section*{Superfluidity}

We will introduce explicitly new feature of superfluidity associated to the chiral flavor currents and collective modes.

\section*{Introduction}

In this paper we consider dissipative hydrodynamics with spontaneous symmetry breaking.
Since we are dealing with a perfect fluid the entropy conservation must be satisfied. Actually it is already contained in the second above equation when projected along the direction of \( u^\nu \): \[ u_\mu \partial_\mu T^{\mu \nu} = 0 \Rightarrow \partial_\mu (s u^\mu) = 0 \]

Now, we can go further and add dissipative terms in the hydrodynamic equations. The method, described for example in [7], consists in introducing fluxes such as :

\[
\partial_\mu (n_0 u^\mu - V^2 \partial_\mu \phi + \nu^\mu) = 0 \tag{5}
\]

\[
\partial_\mu \left( (\epsilon + p) u^\mu u^\nu - p g_\mu^\nu + V^2 \partial_\mu \partial_\nu \phi + \tau^{\mu \nu} \right) = 0 \tag{6}
\]

\[ u^\mu \partial_\mu \phi = -\mu \tilde{\nu} = \phi^0 \tag{7} \]

Moreover, our choice for the hydrodynamic velocity \( u^\mu \) imposes that \( u_\mu \tau^{\mu \nu} = 0 \) and \( u_\mu \nu^\mu = 0 \) (the Eckart choice leads to other constraints, see [11] for example).

With these equations, we are now able to derive explicitly the new equation for the entropy. Writing \( u_\mu \partial_\mu T^{\mu \nu} = 0 \), we obtain :

\[
\partial_\mu (s u^\mu - \frac{\mu u^\mu}{T_0} + \nu^\mu) = -\nu^\mu \partial_\mu \frac{\mu}{T_0} + \frac{\phi^0}{T_0} \partial_\mu (V^2 \partial_\mu \phi) + \frac{\tau^{\mu \nu}}{T_0} \partial_\mu u_\nu \tag{8}
\]

This equation has the generic form : \( \partial_\mu S^\mu = \sigma \) where \( S^\mu \) is the total four-flow entropy and \( \sigma \) the entropy production. \( \sigma \) is a bilinear form between fluxes and hydrodynamical forces. In the hydrodynamic regime, we are by definition near the global equilibrium so that we can express linearly the relation between these fluxes and these forces. The coefficients of proportionality are called the transport coefficients. Their physical meaning is thus to characterize the magnitude of the response of the system (flows) to a certain disturbance (thermodynamic forces).

Moreover, because of thermodynamics, \( \sigma \) must be positive. This constraint when combined with the Onsager principle leads to :

\[ \nu^\mu = \kappa (g^{\mu \nu} - u^\mu u^\nu) \partial_\nu \left( \frac{\mu}{T_0} \right) \tag{9} \]

\[ \tau^{\mu \nu} = (g^{\mu \nu} - u^\mu u^\nu) \left[ C_1 \partial_\lambda (V^2 \partial_\lambda \phi) + C_2 - \frac{2}{3} \eta \partial_\nu u^\nu \right] + \eta \left[ (g^{\mu \lambda} - u^\mu u^\lambda) \partial_\lambda u^\nu + (g^{\nu \lambda} - u^\nu u^\lambda) \partial_\lambda u^\mu \right] \tag{10} \]

\[ \phi^0 = C_1 \partial_\mu u^\mu + C_2 \partial_\mu (V^2 \partial_\mu \phi) \tag{11} \]

where \( \kappa \) is proportional to the thermal conductivity and \( \eta, C_1, C_2, C_3 \) are the shear and bulk viscosities (notations are those defined in [7]). The positivity of \( \sigma \) implies that \( \eta, C_1, C_2, C_3 \) are positive and \( C_1 \leq C_2 C_3 \). The sign of \( C_1 \) has to be determined by physical considerations only (here, it can be checked by comparison with [7] that \( C_1 \) is positive as well).

### CHIRAL DYNAMICS

**Perfect fluid**

In this section, we are going to recall the main results of hydrodynamics with chiral \( SU(2) \) symmetry spontaneously broken [2]. The hydrodynamic degrees of freedom in a chiral fluid are the densities of the conserved quantities, namely entropy density \( s \), momentum density \( T^{0i} \), baryonic number density \( n \), and left and right-handed charge densities written as \( SU(2) \)-matrices \(\rho_L \equiv \rho_L \gamma_1/2 \) and \(\rho_R \equiv \rho_L \gamma_1/2 \) and finally the variables associated to the Goldstone modes. For chiral symmetry, these modes are the pions. Then, following [2], we can write the new hydrodynamic variables as a \( SU(2) \)-matrix \( \Sigma \equiv e^{i \theta_r x^r / \pi} \) and, by analogy with the superfluid, denote \( \Sigma \) as "phases". The energy density \( T^{00} \) is a function of all these variables and of the first partial derivatives of \( \Sigma \) (assuming \( \Sigma \) varies slowly). Following again [2] we thus write the energy density \( T^{00} \) as :

\[ T^{00} = \epsilon_0 (s, n, T^{0i}) + \epsilon_1 \]

where \( \epsilon_0 \) is the "normal fluid" part and \( \epsilon_1 \) contains all the non trivial terms, at lowest order, compatible with chiral symmetry :

\[
\epsilon_1 = \frac{3}{4} \left( \delta_{ij} \frac{1}{1 - v^2} \frac{1}{1 - v^2} \right) \mathrm{tr} \partial_4 \Sigma \partial_j \Sigma^i + \frac{1}{\gamma_0} \left( \frac{1}{1 - v^4} \right) \mathrm{tr} (\rho_L + \Sigma \partial_4 \Sigma^i)^2 \]

\[
+ \left( \frac{1}{\gamma_0} \right) \left( \frac{1}{1 - v^4} \right) \mathrm{tr} (\rho_L + \Sigma \partial_4 \Sigma^i)^2 - \frac{1}{\gamma_0} \left( \frac{1}{1 - v^4} \right) \gamma_0 \mathrm{tr} (\rho_L \Sigma \partial_4 \Sigma^i + \Sigma^i \partial_4 \Sigma \rho_L) \]

\[ \]
where \( v_x \equiv f_x / f_t \) is the pion velocity and \( f_s, f_L, f_R \) and \( f_t \) are functions to be determined by thermodynamics of the underlying fundamental theory, namely QCD.

We have fourteen hydrodynamics variables. Therefore we must have fourteen hydrodynamics equations. With the hamiltonian density above, it is possible to show \([2]\) that the equations can be written in a covariant way only if we make some combinations of the initial variables. Finally the result is:

\[ \partial_j (\pi \sigma u^j) = 0 \]  
(12)

\[ \partial_j T^{\mu 
u} = 0 \]  
(13)

\[ \partial_j (\alpha u^j) + \frac{1}{2} [\mu \sigma \partial_j \mu \sigma^\dagger, \alpha] = 0 \]  
(14)

\[ i \partial_j ((f_1^2 - f_2^2) \mu u^\nu \Sigma \partial_j \mu \sigma^\dagger + f_1^2 \Sigma \sigma \partial_j \Sigma \sigma^\dagger) + [\mu \sigma \partial_j \mu \sigma^\dagger, \alpha] = 0 \]  
(15)

with \( T^{\mu 
u} = (\epsilon + p) u^\mu u^\nu - pg^{\mu 
u} + f_1^2 \text{tr}(\partial_j \Sigma \sigma \partial_j \Sigma \sigma^\dagger) + [\mu \sigma \partial_j \mu \sigma^\dagger, \alpha] \).

The nine equations which are specific to chiral dynamics \( \partial_j \mu_\pm L_R = 0 \) and the first order equation for \( \Sigma \) are actually contained in the first order equation (14) for \( \alpha \equiv (\rho_L + \Sigma \sigma \sigma^\dagger) / \gamma \) and in the second order equation (15) for \( \Sigma \) constructed from the combination \( (\rho_L - \Sigma \sigma \sigma^\dagger) / \gamma \).

Once again, starting from equation (13) and using all the other (hydrodynamic and thermodynamic) equations, it is possible to deduce the entropy conservation : \( \partial_j (s u^j) = 0 \) as it should be for a perfect fluid.

**Dissipation**

We are now in position to treat the dissipation case. As for the superfluid, we add some dissipative flux densities to conservation equations : \( \nu^\mu \) for baryonic number, \( j_{\mu \nu} \equiv j_{\mu \nu}^R \gamma \sigma_i / 2 \) for chiral charges and \( \tau^{\mu 
u} \) for energy-impulsion tensor; and we add also a dissipative term for the other non-conserved variables : \( \frac{1}{2} \Sigma \sigma \sigma^\dagger \equiv \frac{1}{2} \Sigma \sigma \sigma^\dagger \).

Then, performing the same combinations as in the previous section, we can finally write after some calculations:

\[ \partial_j (\pi \sigma u^j + \nu^\mu) = 0 \]  
(16)

\[ \partial_j ((\epsilon + p) u^\mu u^\nu - pg^{\mu 
u} + f_1^2 \text{tr}(\partial_j \Sigma \sigma \partial_j \Sigma \sigma^\dagger) + [\mu \sigma \partial_j \mu \sigma^\dagger, \alpha] = 0 \]  
(17)

\[ \partial_j (\alpha u^j) + \frac{1}{2} [A, \alpha] = -\partial_j j_\mu^R - \Sigma \partial_j j_\mu^R \sigma \sigma^\dagger \]  
(18)

\[ i \partial_j \left( (f_1^2 - f_2^2) \mu u^\nu (A - \frac{1}{2} \Sigma \sigma \sigma^\dagger) + f_1^2 \Sigma \sigma \partial_j \Sigma \sigma^\dagger \right) + [A, \alpha] = 2 \partial_j j_\mu^R - 2 \Sigma \partial_j j_\mu^R \sigma \sigma^\dagger \]  
(19)

with \( A = u^\mu \Sigma \partial_j \mu \sigma^\dagger \). With these new equations, it is a simple task to express the equation for the entropy:

\[ \partial_j \left( s_\mu u^\mu - \frac{\nu_\mu}{T_0} + \text{tr} \left( \frac{\mu L_0}{T_0} j_\mu^L + \frac{\mu R_0}{T_0} j_\mu^R + \frac{i A}{4 T_0} (f_1^2 - f_2^2) \sigma_\mu \sigma^\dagger \right) \right) = -\nu_\mu \partial_j (s_\mu) \]  
(20)

where \( s_\mu = 2 \alpha / f_t^2 - i A \) and \( \mu L_0 = \Sigma \sigma (2 \alpha / f_t^2 + i A) \Sigma \).

We have introduced the above shorthand notations \( \mu R_0 \) and \( \mu L_0 \) simply because they can be identified as chemical potentials but for the left and right chiral densities. On the right-hand side of the equation, the entropy production is, as usual, a bilinear form between “thermodynamic forces” and dissipative fluxes. Again, we can make linear
combinations with these thermodynamic forces and make the entropy production positive. These prescriptions allow us to eliminate some coefficients and to have some constraints on the remaining others. We obtain:

$$\Sigma_0 = \left( \zeta_{ij} \partial_{\mu} u^\mu + [\zeta_{ij}]_{\mu,\nu} \partial_{\mu} \left( \frac{iA_j}{4T_0^0} \right) \right) \tau_{ij}$$

(21)

$$\nu^\mu = (g^{\mu\nu} - u^\mu u^\nu) \left( \kappa \partial_{\rho} \frac{\mu^0}{T_0^0} + \kappa_{L,i,j} \partial_{\rho} \left( \frac{\mu^0 \delta^0_{ij}}{T_0^0} \right) \right) + \kappa_{R,i,j} \partial_{\rho} \left( \frac{\mu^0 \delta^0_{ij}}{T_0^0} \right) \tau_{ij}$$

(22)

$$j_{L}^\mu = (g^{\mu\nu} - u^\mu u^\nu) \left( \kappa_{L,i,j} \partial_{\rho} \frac{\mu^0}{T_0^0} + \kappa_{L \mu} \partial_{\rho} \left( \frac{\mu^0 \delta^0_{ij}}{T_0^0} \right) + [\kappa_{L}]_{ij} \partial_{\rho} \left( \frac{\mu^0 \delta_{ij}}{T_0^0} \right) \right) \tau_{ij}$$

(23)

$$j_{R}^\mu = (g^{\mu\nu} - u^\mu u^\nu) \left( \kappa_{R,i,j} \partial_{\rho} \frac{\mu^0}{T_0^0} + \kappa_{R \mu} \partial_{\rho} \left( \frac{\mu^0 \delta^0_{ij}}{T_0^0} \right) + [\kappa_{R}]_{ij} \partial_{\rho} \left( \frac{\mu^0 \delta_{ij}}{T_0^0} \right) \right) \tau_{ij}$$

(24)

$$\tau^{\mu\nu} = (g^{\mu\nu} - u^\mu u^\nu) \left( \zeta_1 \frac{2}{3} \eta \partial_{\lambda} u^\lambda + \zeta_{1,j} u^\lambda \partial_{\lambda} \left( \frac{iA_j}{4T_0^0} \right) \right) \right) + \eta \left( (g^{\rho\lambda} - u^\rho u^\lambda) \partial_{\lambda} u_{\rho} + (g^{\rho\nu} - u^\rho u^\nu) \partial_{\lambda} u_{\rho} \right) \right)$$

(25)

where $[Q]$ means that $Q$ is a $3 \times 3$ matrix. Due to Onsager reciprocity, all the matrices except $[\kappa_{LR}]$ are symmetric. $\zeta_{ij} = -(L_i^2 - L_j^2)\left[ T_0^0 \zeta_{ij} \right] / 2$ and there are actually 39 independent coefficients. If we represent the quadratic form of the entropy production by a $12 \times 12$ matrix, we can easily show that all coefficients appearing in the diagonal should be positive and that there exists inequalities between the 39 coefficients: as for the superfluid, all the principal minors have to be positive.

We see that the spontaneous breaking of the chiral symmetry implies the existence of new transport coefficients and, by consequence, the existence of new couplings between forces and fluxes. Since it is known that dissipation can affect the observables (see [12] for the modification of the temperature profile used to describe heavy ion collisions), it is of fundamental importance to determine quantitatively the influence of these new couplings.

**CONCLUSION**

We have treated in this paper the dissipation in the hydrodynamic regime for relativistic systems with spontaneously broken symmetry $U(1)$ and $SU(2)$. For the superfluid, we recovered the non-relativistic limit of [7]. For the $SU(2)$ case, we introduced new transport coefficients associated to the right and left charges and to the Goldstone modes. Since we know that transport coefficients can affect quantitatively observable, it would be interesting to express them with Kubo-type relations and then compute them explicitly from this microscopic approach. This work is under study. An other extension of this paper is to determine the relaxation time associated with all these coefficients in order to know if this implies some noticeable modifications to the conclusion concerning the typical equilibration time during heavy ions collisions.

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