Graviton localization and Newton law for a $dS_4$ brane in a $5D$ bulk

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Abstract

We consider an $dS_4$ brane embedded in a five-dimensional bulk with a positive, vanishing or negative bulk cosmological constant and derive the localized graviton spectrum that consists of a normalizable zero-mode separated by a gap from a continuum of massive states. We estimate the massive sector contribution to the static potential at short distances and find that only in the case of a negative bulk cosmological constant there is a range, determined by the effective four-dimensional and the bulk cosmological constants, where the conventional Newton law is valid.
Models of extra dimensions, in which the standard four dimensions correspond to a brane embedded in a higher dimensional bulk have been extensively studied in the last few years as a solution to the hierarchy problem[1]. Soon it was realized that the extra dimensions could be large. Non-compact internal spaces have been employed, although with a limited success, from the early days of the KK programme [2],[3]. Their relation to gauged supergravities has also been pointed out [6],[7],[8]. Independently of how such branes are formed [9], brane models in non-compact spaces require the trapping of gravitational degrees of freedom on the brane [4],[5]. Graviton localization results from the existence of a normalizable zero mode, although less straightforward situations could arise, as in the case of a $AdS_4$ brane embedded in $AdS_5$ bulk where gravity is localized despite the fact that a zero mode is absent[10].

Although curved branes embedded in a higher dimensional bulk have been considered before [10]–[15], the case of a de Sitter brane ($dS_4$) is particularly interesting both for phenomenological as well as theoretical reasons. In this article we consider the three possible cases of embedding an $dS_4$ brane in a five dimensional bulk having positive ($\Lambda > 0$), vanishing ($\Lambda = 0$) or negative ($\Lambda < 0$) bulk cosmological constant. In all these cases the graviton spectrum consists of a zero-mode separated by a gap from a continuum of massive modes. Although the contribution of the zero-mode to the static gravitational potential is always Newtonian, the massive modes introduce a five-dimensional behaviour which dominates in all cases except the case of negative bulk cosmological constant ($AdS_5$) in which ordinary gravity is possible and five-dimensional behaviour shows up only at very short distances.

We start from the five-dimensional Einstein action with a cosmological constant $\Lambda$ and a brane of tension $\sigma$

$$S = \int d^5x \sqrt{-g} \left\{ 2M^3 R - \Lambda \right\} - \sigma \int d^4x \sqrt{-\overline{g}}$$

and introduce a general metric ansatz

$$ds^2 = e^{2A(y)}\overline{g}_{\mu\nu}(x)dx^\mu dx^\nu + dy^2$$

with a four-dimensional metric $\overline{g}_{\mu\nu}(x)$ fixed to correspond to a de Sitter space

$$\overline{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & e^{2Ht} \delta_{ij} \end{pmatrix}$$

The resulting equations of motion are

$$-3H^2 e^{-2A} + 3 \left( A'' + (A')^2 \right) = -\frac{\Lambda}{4M^3} - \frac{\sigma}{4M^3} \delta(y)$$

$$-6H^2 e^{-2A} + 6(A')^2 = -\frac{\Lambda}{4M^3}$$

These equations can easily be solved [8],[16],[17], for either sign of $\Lambda$. 

1
Positive bulk cosmological constant $\Lambda > 0$. In the case $\Lambda > 0$, we have the warp factor

$$e^{A(y)} = \frac{\sin \left(n(y_0 - |y|)\right)}{\sin(ny_0)}$$  \hspace{1cm} (6)

where, we have introduced

$$n^2 \equiv \frac{\Lambda}{24M^3}, \quad y_0 \equiv \frac{1}{n} \cot^{-1} \left(\frac{\sigma}{24M^3n}\right)$$  \hspace{1cm} (7)

The metric ansatz parameter $H$ is related to the action parameters through the relation

$$H^2 = n^2 + \left(\frac{\sigma}{24M^3}\right)^2 = \frac{n^2}{\sin^2(ny_0)} \Rightarrow (H^2 > n^2)$$  \hspace{1cm} (8)

Either expression implies that $H^2$ receives a positive contribution from the bulk cosmological constant. The Ricci scalar in this case is

$$R = \frac{2\sigma}{M^3} \delta(y) + 20n^2$$  \hspace{1cm} (9)

corresponding to a de Sitter bulk.

The graviton spectrum can be obtained by performing the variation $\delta g_{MN} = \delta_M^{\mu}\delta_N^{\nu}h_{\mu\nu}(x, y)$ and introducing the ansatz $h_{\mu\nu} = \psi(y)\eta_{\mu\nu}(x)$. The resulting graviton equations are

$$-\frac{1}{2}\psi''(y) + \left(A'' + 2(A')^2\right)\psi(y) = \frac{m^2}{2}e^{-2A}\psi(y)$$ \hspace{1cm} (10)

$$-\frac{1}{2}\nabla^2\eta_{\mu\nu}(x) + H^2\eta_{\mu\nu}(x) = -\frac{m^2}{2}\eta_{\mu\nu}(x)$$ \hspace{1cm} (11)

There is a normalizable\(^3\) zero mode ($m^2 = 0$)

$$\psi_0(y) = e^{2A(y)} = \frac{\sin^2 \left(n(y_0 - |y|)\right)}{\sin^2(ny_0)}$$ \hspace{1cm} (12)

whereas the massive spectrum can be determined from

$$-\frac{1}{2}\psi''(y) + n^2\left\{-2 + \frac{(1 - m^2/2H^2)}{\sin^2 \left(n(y_0 - |y|)\right)}\right\}\psi(y) = \frac{\sigma}{12M^3} \psi(0) \delta(y)$$ \hspace{1cm} (13)

\(^1\)An equivalent representation of the metric as $ds^2 = e^{2A(z)}(g_{\mu\nu}dx^{\mu}dx^{\nu} + dz^2)$ in terms of $z = \int_0^y dy e^{-A(y)}$ gives the warp factor expression

$$e^{2A(z)} = e^{-2H|z|}\frac{(1 + \tan^2(ny_0/2))}{(1 + \tan^2(ny_0/2)e^{-2H|z|})^2}$$

\(^2\)This can be done in the harmonic gauge. The tracelessness-transversality conditions are $\eta_{\mu}^{\nu} = \nabla_\mu \eta^{\nu}_\nu = 0$.

\(^3\)Normalizability corresponds to $\int dy e^{-2A}|\psi(y)|^2 < \infty$. 

2
Acceptable solutions exist for $m^2 > 9H^2/4$. They are, up to a dimensionless multiplicative constant,

$$
\frac{\sqrt{n}}{\sin^2(n(y_0 - |y|))} \mathbf{F}_1 \left( \frac{5/4 \pm 3\delta/4, 5/4 \pm 3\delta/4; 3; \sin^2(n(|y| - y_0))} \right)
$$

(14)

where $\delta^2 = 4m^2/9H^2 - 1$.

**Vanishing bulk cosmological constant $\Lambda = 0$.** The case of vanishing bulk cosmological constant $\Lambda = 0$ is essentially a case studied long ago[18] and corresponds to the warp factor

$$
e^{A(y)} = 1 - H|y|
$$

(15)

and the relation

$$
H = \frac{\sigma}{24M^3}
$$

(16)

The four-dimensional effective Planck mass is in this case $M_P^2 = 2M^3/3H$. The graviton localization equation is

$$
-\frac{1}{2} \psi''(y) + H^2 \frac{(1 - m^2/2H^2)}{(1 - H|y|)^2} \psi(y) = 2H \psi(0) \delta(y)
$$

(17)

There is a normalizable zero-mode

$$
\psi_0(y) = \sqrt{\frac{3H}{2}} (1 - H|y|)^2
$$

(18)

and a continuum of massive delta-function normalizable states for $4m^2 > 9H^2$

$$
\psi_\delta(y) = \sqrt{\frac{3H}{8\pi}} (1 - H|y|)^{1/2} \left\{ (1 - H|y|)^{3\delta/2} + \left( \frac{i\delta - 1}{i\delta + 1} \right) (1 - H|y|)^{-3\delta/2} \right\}
$$

(19)

where $\delta$ is the previously defined parameter. This case offers the advantage of having elementary analytic expressions for the graviton wave functions in contrast to the previous case where the massive continuum states were Hypergeometric functions.

**Negative bulk cosmological constant $\Lambda < 0$.** In the case of negative bulk cosmological constant, the warp factor comes out to be

$$
e^{A(y)} = \frac{\sinh(\nu(y_0 - |y|))}{\sinh(\nu y_0)}
$$

(20)

4The metric takes the form $e^{-2H|z|} \left\{ \mathbf{g}_{\mu\nu}(x)dx^\mu dx^\nu + dz^2 \right\}$, with $\mathbf{g}_{\mu\nu} = \text{Diag}(-1, e^{2Ht} \delta_{ij})$, in terms of the variable $|z| = -\frac{1}{H} \ln(1 - H|y|)$.

5In terms of a $z$-variable, defined as in the $\Lambda > 0$ case, the warp factor takes the form

$$
e^{2A(z)} = \frac{(1 - \tanh^2(\nu y_0/2))(1 - \tanh^2(\nu y_0/2)e^{-2H|z|})}{e^{-2H|z|}}
$$
where, we have defined
\[ \nu^2 \equiv -\frac{\Lambda}{24 M^3}, \quad \nu y_0 = \coth^{-1}(\sigma/24 M^3 \nu) \] (21)

The relation between action and metric-ansatz parameters is
\[ H^2 = \left( \frac{\sigma}{24 M^3} \right)^2 - \nu^2 = \frac{\nu^2}{\sinh^2(\nu y_0)} \Rightarrow (H^2 < \nu^2) \] (22)

In this case the bulk is asymptotically AdS with a Ricci scalar
\[ R = \frac{2\sigma}{3M^3} \delta(y) - 20\nu^2 \] (23)

The spectrum consists of a normalizable zero mode \( \psi_0(y) = e^{2A(y)} \) and a continuum of massive states with masses \( 4m^2 > 9H^2 \) and wave-functions corresponding to the solutions of
\[ -\frac{1}{2} \psi''(y) + \nu^2 \left\{ 2 + \left( 1 - \frac{m^2}{2H^2} \right) \sinh^2(\nu(y_0 - |y|)) \right\} \psi(y) = \frac{\sigma}{12M^3} \psi(0) \delta(y) \] (24)

These solutions are, up to a dimensionless multiplicative constant,
\[ \psi(y) \sim \frac{\sqrt{\nu}}{\sinh^2(\nu(y_0 - |y|))} \, _2F_1 \left( \frac{5}{4} \pm 3i\delta/4, \frac{5}{4} \pm 3i\delta/4; 3; -\sinh^{-2}(\nu(y_0 - |y|)) \right) \] (25)

**The gravitational potential.** A most relevant question for all the above cases is the question of the gravitational potential created by a unit mass on the brane. This potential is directly related to the 00 component of the graviton, namely \( V(x,y) = \frac{1}{2} h_{00}(x,y) \). (Newton law in de-Sitter space [19] and time-depended backgrounds have been discussed in [20].) For a point particle of mass \( \mu \) at a point \( \vec{x} = 0 \) on the brane it satisfies the equation
\[ -\frac{1}{2} e^{-2A} \nabla^2 V + H^2 V - \frac{1}{2} V'' + \left( A'' + 2(A')^2 \right) V = -\lambda \delta^{(3)}(x) \delta(y) \] (26)

where \( \lambda = \mu/M^3 \). Expanding the potential in terms of the eigenfunctions \( \psi_0(y) \) and \( \psi_\delta(y) \) as
\[ V(x,y) = \phi_0(x) \psi_0(y) + \int_0^\infty d\delta \psi_\delta(x) \psi_\delta(y) \] (27)

Substituting into the equation for \( V \), we obtain for the coefficient functions \( \phi_\delta(x) \) and \( \phi_0(x) \)
\[ -\frac{1}{2} \psi_\delta'(x) + (H^2 + \frac{m^2}{2}) \phi_\delta(x) = -\lambda \psi_\delta^*(0) \delta^{(3)}(x), \]
\[ -\frac{1}{2} \psi_0'(x) + H^2 \phi_0(x) = -\lambda \psi_0(0) \delta^{(3)}(x) \] (28)

The solutions to these equations can be written in terms of the de Sitter propagator \( S(x,x') \) defined as
\[ S^{-1}(x,x') \equiv \left\{ -\frac{\partial^2}{\partial \tau^2} + \vec{\nabla}^2 \right\} - \frac{H}{1 - H\tau} \frac{\partial}{\partial \tau} - \frac{m^2}{(1 - H\tau)^2} \delta^{(4)}(x - x') \] (29)
where we have introduced the conformal time
\[ \tau = \frac{1}{H} (1 - e^{-H \tau}) \, , \, -\infty < \tau < H^{-1} \]
as in [21], [22]. The coefficient functions are
\[
\phi_0(x) = 2\lambda \psi_0(0) \int \frac{d\tau'}{(1 - H \tau')^2} S_0(x, 0, \tau') \, , \, \phi_\delta(x) = 2\lambda \psi_\delta(0) \int_\tau \frac{d\tau'}{(1 - H \tau')^2} S_\delta(x, 0, \tau')
\]
The expression of the gravitational potential on the brane is
\[
V(x, 0) = 2\lambda \psi_0^2(0) \int_{-\infty}^{H^{-1}} \frac{d\tau'}{(1 - H \tau')^2} S_0(\tau, \vec{x}; \tau', 0) + 2\lambda \int_0^\infty d\delta |\psi_\delta(0)|^2 \int_{-\infty}^{H^{-1}} \frac{d\tau'}{(1 - H \tau')^2} S_\delta(\tau, \vec{x}; \tau', 0)
\]
The differential equation satisfied by the propagator can be transformed into a differential equation with respect to the geodesic distance, namely
\[
z(1 - z)S''(z) + 2(1 - 2z)S'(z) - \frac{m^2}{H^2} S(z) = 0
\]
where the geodesic distance \( z \) is defined, with \( \gamma \equiv H^{-1} - \tau \), as
\[
z \equiv \frac{(\gamma + \gamma')^2 - (\vec{x} - \vec{x}')^2}{4\gamma\gamma'}.
\]
Solving the above equation, we find that the propagator [23] is
\[
S_\delta(z) = \frac{H^2}{16\pi^2} [\Gamma(3/2 + 3i\delta/2)]^2 \, _2F_1(3/2 + 3i\delta/2, 3/2 - 3i\delta/2; 2; z)
\]
The massless propagator can be directly obtained in an elementary form from the differential equation it satisfies, namely
\[
S_0(z) = \frac{iH^2}{16\pi^2} \left( \frac{1}{1 - z - i\epsilon} - \frac{1}{z + i\epsilon} + 2 \ln \left( \frac{z + i\epsilon}{1 - z - i\epsilon} \right) \right)
\]
The contribution of the zero mode to the potential can be integrated\(^6\) and gives, in the \( R \to 0 \) limit a real part that is just Newtonian
\[
V_0(x) = \frac{2\lambda}{H^2} \psi_0^2(0) \int_\gamma \frac{d\gamma'}{\gamma'^2} S_0(z) \sim -\frac{\lambda \psi_0^2(0)}{4\pi} \frac{1}{R}
\]
In the \( \Lambda = 0 \) case, setting \( \lambda = \mu/M^3 \), we get
\[
V_0(R) \sim -\frac{3\mu H}{8\pi M^3} \frac{1}{R} = -\frac{\mu}{4\pi M_p^2} \frac{1}{R}
\]
\(^6\)The replacement of zero with \( \gamma \) as a lower integration limit is kinematical.
as we expect.

The contribution of the massive continuum is expressed as a double integral

\[
V_m(x) = \frac{2\lambda}{H^2} \int_{0}^{\infty} d\delta |\psi_\delta(0)|^2 \int_{\gamma}^{\infty} d\gamma' \frac{d\gamma'}{\gamma'^2} S_\delta(z)
\]

(36)

In the short-distance limit \( HR \ll 1 \), the propagator is expected to behave as a flat propagator. Thus, for \( R \ll H^{-1} \), but still \( m^{-1} \ll R \), we can roughly approximate the time integral with its massless value suppressed by an exponential, namely

\[
\int_{\gamma}^{\infty} d\gamma' \frac{d\gamma'}{\gamma'^2} S_\delta(z) \sim -\frac{H^2}{8\pi R} e^{-mR}
\]

(37)

Thus,

\[
V_m(R) \sim -\frac{\lambda}{4\pi R} \int_{0}^{\infty} d\delta |\psi_\delta(0)|^2 e^{-mR}
\]

(38)

In the case \( \Lambda = 0 \), we have \( |\psi_\delta(0)|^2 = \frac{3H}{2\pi} (\delta^2/(1 + \delta^2)) \). Setting \( \xi = mR > 1 \), we get \( \delta \sim 2\xi/3(HR)(1 + O(HR)^2) \) and \( |\psi_\delta(0)|^2 \sim \frac{3H}{2\pi} (1 + O(HR)^2) \). Thus, the massive states contribution to the potential can be approximated with

\[
V_m^{(\Lambda=0)}(R) \sim -\frac{3\lambda H}{8\pi R} \left( \frac{1}{6\pi HR} + O(HR) \right)
\]

(39)

where \( C_0 = \int d\xi e^{-\xi} (1 + O(HR)) \). The total potential for this case is

\[
V^{(\Lambda=0)}(R) \sim -\frac{3\lambda H}{8\pi R} \left( 1 + \frac{1}{6\pi HR} + O(HR) \right) = -\frac{\mu}{4\pi M_0^2 R} \left( 1 + \frac{C_0}{6\pi HR} + O(HR) \right)
\]

(40)

In the case \( \Lambda > 0 \), in order to proceed further in the expression for \( V_m \) we note that, up to a dimensionless constant,

\[
|\psi_\delta(0)|^2 = \frac{n}{\sin^4(ny_0)} \left| 2F_1 \left( \frac{5}{4} \mp 3i\delta/4, \frac{5}{4} \pm 3i\delta/4; 3; \sin^{-2}(ny_0) \right) \right|^2
\]

\[
= H \left( \frac{H}{n} \right)^3 \left| F(\delta, H^2/n^2) \right|^2
\]

Thus,

\[
V_m(R) \sim -\frac{\lambda H}{4\pi R} \left( \frac{H}{n} \right)^3 \int_{0}^{\infty} d\delta \left| F(\delta, H^2/n^2) \right|^2 e^{-mR}
\]

(41)

Setting

\[
\int d\delta \left| F(\delta, H^2/n^2) \right|^2 e^{-mR} \sim \frac{2}{3(HR)} \int d\xi \left| F(\xi/(HR), H^2/n^2) \right|^2 e^{-\xi} \equiv \frac{3C_+ (HR, H/n)}{2(HR)}
\]

we can write the full potential as

\[
V^{(\Lambda>0)}(R) \sim -\frac{3\lambda H}{8\pi R} \left( 1 + \left( \frac{H}{n} \right)^3 \frac{C_+}{HR} \right)
\]

(42)
Unfortunately, in both of the above cases ($\Lambda \geq 0$) there can be no argument in favour of the suppression of the “corrections” due to the massive states. Taking into account the increasing behaviour on $H/n$, these corrections dominate for $R < H^{-1}$ and the five dimensional behaviour is dominant in the potential which is nowhere Newtonian. It should be noted that this behaviour is expected since in both the above cases the effective size of the fifth dimension is $y_0 \sim H^{-1}$.

This is in sharp contrast to the case of negative bulk cosmological constant ($\Lambda < 0$). In that case, we have, up to a dimensionless multiplicative constant,

$$|\psi_\delta(0)|^2 = \frac{\nu}{\sinh^2(\nu y_0)} \left| \frac{\nu}{\sinh(\frac{H \nu}{2})} \right|^2$$

Thus,

$$V_m(R) \sim -\frac{\lambda H}{4\pi R} \left( \frac{H \nu}{2} \right)^3 \int_0^\infty d\delta \left| F(\delta, -H^2/\nu^2) \right|^2 e^{-mR}$$

Setting

$$\int d\delta \left| F(\delta, -H^2/\nu^2) \right|^2 e^{-mR} \sim \frac{2}{3(HR)} \int d\xi \left| F(\xi/(HR), -H^2/\nu^2) \right|^2 e^{-\xi} \equiv \frac{3C_-(HR, H/\nu)}{2(HR)}$$

we can write the full potential as

$$V^{(\Lambda < 0)}(R) \sim -\frac{3\lambda H}{8\pi R} \left( 1 + \left( \frac{H}{\nu} \right)^3 \frac{C_-}{HR} \right)$$

Here, since the ratio $H/\nu$ can be chosen to be as small as desired, taking into account the decreasing behaviour of $C_-$ as a function of it, we can easily argue that there is a range

$$\nu^{-1} << R < H^{-1}$$

where the Newtonian term due to the zero mode dominates and gravity is ordinary. Five-dimensional behaviour comes in at shorter distances $R << \nu^{-1}$. Note that this behaviour goes along with the small size of the extra dimension in the case $H < \nu$ which is $y_0 \sim (1/\nu) \log(\nu/H)$.

In conclusion, we have considered curved $dS_4$ branes in a 5D bulk and in particular, graviton localization and the existence of a Newtonian limit. We looked at the static gravitational potential at short distances $HR << 1$, given approximately by the equations (40), (42) and (44) for zero, positive and negative bulk cosmological constant, respectively. In all cases, there are corrections to the 4D Newton law due to the massless graviton as a result of massive gravitational KK states. Consistency with observations requires the suppression of these corrections. However, this is met only the case of negative bulk cosmological constant, where we have the conventional Newton law for distances greater than $\nu^{-1}$ and smaller than $H^{-1}$. In all other cases, the 5D nature of the background dominates leading to a 5D gravitational interaction on the brane.
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