Cosmological Constraints on Tachyon Matter

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(April 29, 2002)

Abstract

We examine whether tachyon matter is a viable candidate for the cosmological dark matter. First, we demonstrate that in order for the density of tachyon matter to have an acceptable value today, the magnitude of the tachyon potential energy at the onset of rolling must be finely tuned. For a tachyon potential $V(T) \sim M_{Pl}^4 \exp(-T/\tau)$, the tachyon must start rolling at $T \simeq 60\tau$ in order for the density of tachyon matter today to satisfy $\Omega_{T,0} \sim 1$, provided that standard big bang cosmology begins at the same time as the tachyon begins to roll. In this case, the value of $\Omega_{T,0}$ is exponentially sensitive to $T/\tau$ at the onset of rolling, so smaller $T/\tau$ is unacceptable, and larger $T/\tau$ implies a tachyon density that is too small to have interesting cosmological effects. If instead the universe undergoes a second inflationary epoch after the tachyon has already rolled considerably, then the tachyon can begin with $T$ near zero, but the increase of the scale factor during inflation must still be finely tuned in order for $\Omega_{T,0} \sim 1$. Second, we show that tachyon matter, unlike quintessence, can cluster gravitationally on very small scales. If the starting value of $T/\tau$ is tuned finely enough that $\Omega_{T,0} \sim 1$, then tachyon matter clusters more or less identically to pressureless dust. Thus, if the fine-tuning problem can be explained, tachyon matter is a viable candidate for cosmological dark matter.

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I. INTRODUCTION

In brane cosmology, one often assumes that the initial condition of the universe is a non-BPS brane configuration, i.e., a brane configuration that breaks supersymmetry. For example, in the brane inflationary scenario [1], the universe starts out with branes and antibranes, as well as branes intersecting at angles (supersymmetry is broken unless branes are intersecting at specific angles [2]). The system is unstable, as is indicated by the presence of tachyon fields. One can envision that the universe began with such an unstable configuration, and evolved to a stable configuration that corresponds to the Standard Model of particle physics. Indeed some stable configurations which give rise to features resembling the Standard Model have been constructed recently [3].

Sen [4–6] has pointed out that rolling tachyons can contribute a mass density to the universe that resembles classical dust, and Gibbons [7] has considered the cosmological background equations that would result if there were a dominant tachyonic component. However, these papers leave two questions unanswered that are important for cosmology: (1) Under what conditions could the tachyon matter density be just comparable to the closure density of the universe, \( \rho_c = \frac{3H_0^2}{8\pi G} \), today? (2) Can tachyon matter cluster gravitationally?

Any viable candidate for the cosmological dark matter must have a density \( \lesssim \rho_c \) today. Moreover, in order for that component to be consistent with well-documented successes of early universe cosmology, such as the synthesis of the light elements, it must not have become dominant too early in the history of the universe. We shall see that the density associated with the rolling tachyon decays slower than \( a^{-3} \) as the universe expands, where \( a(t) \) is the cosmological scale factor (e.g. [7], and Eq. [18] below), at all times, irrespective of the precise form of \( a(t) \). Thus, if the tachyon matter density is \( \lesssim \rho_c \) today, it has only been a dominant constituent of the universe since a temperature \( \sim 10 \text{ eV} \). This means that the tachyon density at the onset of rolling must have been very small compared to the total density of the universe at that time, provided that standard big bang cosmology began at the same time. Instead, if there was a second inflationary epoch some time after the tachyon began to roll, then the tachyon density could have been large – probably even dominant – during the phase before the second inflation. Each scenario presents a fine-tuning issue that is quantified in § II B.

Since the energy density of tachyon matter decays like \( a^{-3} \) at late times (see [7] and § II B below), it is important to ask whether this component of the universe is capable of gravitational clustering. For example, quintessence fields with effective potentials \( V(\psi) \propto e^{-\psi/\psi_0} \) can also have energy density \( \propto a^{-3} \) at late times, but are incapable of clustering on any length scale smaller than the cosmological particle horizon \( H^{-1} \). As a result, the growth of cosmological density perturbations are slowed in such models for the dark energy of the universe. Consequently, only a small fraction of the mass density can be in the form of quintessence fields with exponential potentials in order for the observed large scale structure of the universe to arise from fractional density perturbations \( \sim 10^{-5} \), as are indicated by observations of temperature fluctuations in the cosmic microwave background radiation. In § II C we prove that tachyon matter can cluster under its own self-gravity, on all but very small length scales. Thus, if the fine-tuning associated with the smooth tachyon matter density can be rationalized, rolling tachyons can be a viable candidate for the cosmological
dark matter.

The calculations presented here are very similar to work completed recently and independently by Frolov, Kofman & Starobinsky [8]. One minor difference is that the computation of tachyon density fluctuation modes in § II C below is done in a different gauge than in Ref. [8]. In all other respects, the two calculations agree. In particular, we find the same minimum scale for perturbations to be self-gravitating, corresponding to a comoving wavenumber $k \simeq H a / \sqrt{1 - T_0^2}$, independent of the form of the tachyon potential, where $T_0$ is the time derivative of the tachyon field in the background and $H(t) = \dot{a}/a$ is the expansion rate of the universe. A more important difference, though, is our discussion in § II B of the fine-tuning required in order for the tachyon dark matter density to be really viable. This fine-tuning is implicit in the observation by Frolov et al. that linear theory fails to describe tachyon matter density perturbations quite early in the history of the universe, for generic initial conditions. Here, we accept the possibility that very special initial conditions are needed in order for tachyon matter to have acceptable cosmological consequences. A more explicit – but related – manifestation of the fine tuning, which we emphasize in § II B, is that unless the tachyon field begins rolling at values in a very small range, the matter density today would be either far too large or uninterestingly small.

II. TACHYON COSMOLOGY

A. Field Equations and Conservation Laws

Consider tachyon matter with an energy-momentum tensor of the form [4,5,7]

$$T_{\mu \nu}^{(t)} = \frac{V(T)}{\sqrt{1 + g^{\alpha \beta} T_\alpha T_\beta}} \left[ -g_{\mu \nu} (1 + g^{\alpha \beta} T_\alpha T_\beta) + T_\mu T_\nu \right].$$

(1)

Here, $V(T)$ is a potential of arbitrary form, which we will only specify toward the end. We can put this into a more suggestive form if we define

$$U_\mu^{(t)} = -\frac{T_\mu^{(t)}}{\sqrt{-g^{\alpha \beta} T_\alpha T_\beta}} \quad \rho^{(t)} = \frac{V(T)}{\sqrt{1 + g^{\alpha \beta} T_\alpha T_\beta}} \quad p^{(t)} = -V(T) \sqrt{1 + g^{\alpha \beta} T_\alpha T_\beta};$$

(2)

then the energy momentum tensor has a perfect fluid form,

$$T_{\mu \nu}^{(t)} = p^{(t)} g_{\mu \nu} + (\rho^{(t)} + p^{(t)}) U_\mu^{(t)} U_\nu^{(t)}.$$

(3)

In terms of these definitions, the equation for the tachyon field $T$,

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{g} V(T) g^{\mu \nu} T_\nu \right) = \nabla_\mu \left( \frac{V(T) g^{\mu \nu} T_\nu}{\sqrt{1 + g^{\alpha \beta} T_\alpha T_\beta}} \right) = V'(T) \sqrt{1 + g^{\alpha \beta} T_\alpha T_\beta},$$

(4)

where $V'(T) \equiv \partial V(T)/\partial T$, becomes

$$U^{(t) \mu} \nabla_\mu \rho^{(t)} + (\rho^{(t)} + p^{(t)}) \nabla_\mu U^{(t) \mu} = 0,$$

(5)
where $\nabla_\mu$ is the covariant derivative. Conservation of energy and momentum, though, would appear to imply four equations,

$$\nabla_\mu T^{(t)\mu\nu} = 0,$$

which can be expanded out to

$$0 = U^{(t)\mu} \left[ U^{(t)\mu} \nabla_\mu \rho^{(t)} + \left( \rho^{(t)} + p^{(t)} \right) \nabla_\mu U^{(t)\mu} \right]$$

$$+ \left( g^{\mu\nu} + U^{(t)\mu} U^{(t)\nu} \right) \nabla_\mu \rho^{(t)} + \left( \rho^{(t)} + p^{(t)} \right) U^{(t)\mu} \nabla_\mu U^{(t)\nu}$$

$$= \left( g^{\mu\nu} + U^{(t)\mu} U^{(t)\nu} \right) \nabla_\mu \rho^{(t)} + \left( \rho^{(t)} + p^{(t)} \right) U^{(t)\mu} \nabla_\mu U^{(t)\nu},$$

using Eq. (5) to simplify in the final line. Eq. (7) only constitutes three independent equations, since contracting it with $U^{(t)\nu}$ yields an identity. After some algebra, the remaining equation can be written in the form

$$U^{(t)\mu} \nabla_\mu U^{(t)\nu} + \left( g^{\mu\nu} + U^{(t)\mu} U^{(t)\nu} \right) \nabla_\mu \ln s = 0,$$

where $s \equiv \sqrt{-g^{\alpha\beta} T_{\alpha\beta}}$. Eq. (8) resembles the Navier-Stokes equation of fluid mechanics, except that it is worthwhile to recall that it only depends on $\nabla_\mu T$, apart from metric factors. If we substitute for $s$ in Eq. (8), we find the equation

$$0 = \left( g^{\mu\nu} g^{\lambda\sigma} - g^{\nu\lambda} g^{\mu\sigma} \right) \nabla_\sigma T \nabla_\mu (\nabla_\lambda T) = \nabla_\sigma T \left( \nabla^\nu \nabla^\sigma - \nabla^\sigma \nabla^\nu \right) T,$$

which vanishes identically, because the covariant curl of the gradient of a scalar is zero. Thus, as expected, there is only one nontrivial equation for the dynamics of the tachyon field, Eq. (5), even though $\nabla_\mu T^{(t)\mu\nu} = 0$ appears, at first sight, to impose four different conservation laws on the single scalar field, $T(x)$.

As always, Einstein’s field equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu},$$

adopting sign conventions in Ref. [12]. This may be rewritten by using $R = 8\pi G T^\lambda_\lambda$, to find

$$R_{\mu\nu} = -8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda \right)$$

$$= -8\pi G \left[ \frac{1}{2} \left( \rho - p \right) g_{\mu\nu} + \left( \rho^{(t)} + p^{(t)} \right) U^{(t)\mu} U^{(t)\nu} + \left( \rho^{(m)} + p^{(m)} \right) U^{(m)\mu} U^{(m)\nu} \right],$$

where we have also used $T^\lambda_\lambda = -\rho + 3p$, and we include both tachyonic matter and ordinary matter, whose fluid variables are distinguished by a superscript $m$. (See [12], Eq. [15.1.14] and the equation following his Eq. [15.1.17].) With the identifications in Eq. (2), we can map conventional results for the cosmology of perfect fluids to the cosmology of tachyon matter.
B. Background Cosmology

Consider first the background cosmology; for homogeneous, isotropic expansion, \( U^{(m)}_\mu = U^{(t)}_\mu = (-1, 0, 0, 0) \), and denoting background matter densities and pressures with subscripts 0 we obtain the field equations

\[
H^2 = \frac{8\pi G}{3} (\rho^{(m)}_0 + \rho^{(t)}_0)
\]

\[
\dot{\rho}^{(m)}_0 = -3H (\rho^{(m)}_0 + p^{(m)}_0)
\]

\[
\dot{\rho}^{(t)}_0 = -3H (\rho^{(t)}_0 + p^{(t)}_0),
\]

where we have adopted the spatially flat FRW metric

\[
d^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,
\]

and dots denote differentiation with respect to time; as usual, \( H \equiv \dot{a}/a \). In the cosmological background, the tachyon field is \( T_0(t) \), and we have

\[
\rho^{(t)}_0 = \frac{V(T_0)}{\sqrt{1 - T_0^2}}, \quad p^{(t)}_0 = -V(T_0) \sqrt{1 - T_0^2}, \quad \rho^{(t)}_0 + p^{(t)}_0 = \dot{T}_0^2 \rho^{(t)}_0.
\]

As has been noted in Refs. [4,5,7], as the tachyon field evolves, it moves toward \( V(T_0) \to 0 \), but retains finite energy density, \( \rho^{(t)}_0 \), so that \( \dot{T}_0^2 \to 1 \); thus, at late times, \( \dot{\rho}^{(t)}_0 \to -3H \rho^{(t)}_0 \), so \( \rho^{(t)}_0 a^3 \to \text{constant} \). [7] Thus, insofar as the background solution is concerned, the tachyon behaves like a nonrelativistic gas of dust asymptotically. In this sense, the tachyon field would appear to be a good candidate for the dark matter.

However, there are constraints on the tachyon field, namely that the density today must be a fraction \( \Omega_{T,0} \leq 1 \) of the total mass density of the universe, and also that the tachyon density was subdominant at early times, particularly during the nucleosynthesis epoch. To examine these constraints roughly, suppose that \( \dot{T}_0(t) = 1 - \epsilon(t) \) at all times since well before nucleosynthesis (perhaps since inflation, for example). Then the density of tachyons at time \( t \) is \( \rho^{(t)}_0 \simeq V(T_0(t))/\sqrt{2\epsilon(t)} \), and \( T_0(t) \simeq t \). To examine the constraint imposed by the present day mass density, suppose that tachyon matter comprises a fraction \( \Omega_{T,0} \) of the total mass density of the universe today, and that the universe is flat (as indicated by CMB observations), so the total mass density is \( 3H_0^2 M_{Pl}^2/8\pi \), where \( H_0 \) is the Hubble constant, and \( M_{Pl} \) the Planck mass. Then the value of \( \epsilon \) today must be

\[
\epsilon_{\text{today}} \simeq \frac{32\pi^2 V^2_{\text{today}}}{9\Omega_{T,0}^2 H_0^2 M_{Pl}^4 H_0^4};
\]

clearly, we want \( \epsilon_{\text{today}} \ll 1 \), so we must have \( V_{\text{today}} \ll 3\Omega_{T,0} H_0^2 M_{Pl}^2/4\pi \sqrt{2} \). Assuming that a natural scale for \( V(T) \) is \( M_{Pl} \), we can recast this bound as \( V_{\text{today}}/M_{Pl}^2 \ll 3\Omega_{T,0} H_0^2/4\pi \sqrt{2} M_{Pl}^2 \simeq 2.5 \times 10^{-123} \Omega_{T,0} \), since \( H_0/M_{Pl} \simeq 1.2 \times 10^{-61} \) (i.e. \( H_0 \simeq 70 \text{ km s}^{-1} \text{Mpc}^{-1} \) in Planck units). Suppose, as a specific example, that \( V(T) = M^4 \exp(-T/\tau) \), where \( M \sim M_{Pl} \) and \( \tau \) is a characteristic timescale, which might also be
expected to be the Planck scale. Since \( T_0(t) \approx t \), the inequality can be satisfied as long as
the universe is older than about 280\( \tau \) – which is true with plenty of room to spare for any reasonable \( \tau \) based on mass scales of elementary particle physics.

Although it is easy to arrange for \( \epsilon_{\text{today}} \ll 1 \), considerable fine-tuning is necessary to achieve \( \Omega_{T,0} \sim 1 \). Suppose that \( \epsilon \sim 1 \) at some time in the past, when the temperature of the universe was \( T_{\text{start}} \) and the tachyon potential was \( V_{\text{start}} \). Then the tachyon matter density at this time, which represents the onset of the rolling of the tachyon field, is \( \rho_{\text{today}} \frac{T_{\text{start}}^3}{\rho_{\text{today}}} \) where \( T_{\text{today}} = 2.73 \text{ K} \) is the temperature of the universe today, provided that the big bang phase of cosmological evolution begins at \( T_{\text{start}} \).\(^1\) As a result,

\[
\frac{V_{\text{start}}}{T_{\text{start}}^4} \sim \frac{\rho_{\text{today}}}{T_{\text{start}}} \sim \frac{100 \text{ eV} \Omega_{T,0}}{T_{\text{start}}},
\]

which implies that \( V_{\text{start}}/T_{\text{start}}^4 \ll 1 \) in order for \( T_{\text{start}} \) to exceed \( 0.1 \sim 1 \text{ MeV} \), the temperature at the nucleosynthesis epoch. This means that if \( T_{\text{start}} \sim M_{\text{Pl}} \), for example, then \( V_{\text{start}}/M_{\text{Pl}}^4 \sim 10^{-26} \). The requisite tuning is less severe if the fundamental scale of the theory
is smaller, but even for \( T_{\text{start}} \sim \text{TeV} \), we must require that \( V_{\text{start}}/T_{\text{start}}^4 \sim 10^{-10} \) or so. Larger values of \( V_{\text{start}}/T_{\text{start}}^4 \) would imply larger values of \( \rho_{\text{today}}/T_{\text{today}}^3 \) than in the observed universe. Smaller values are consistent with all observed properties of the universe, but imply very small \( \Omega_{T,0} \), so that tachyon matter is not important.

Small values of \( V_{\text{start}}/T_{\text{start}}^4 \) can arise if the tachyon begins to roll from a relatively large value of \( T/\tau \).\(^9\) For example, if \( V(T) \sim M_{\text{Pl}}^4 \exp(-T/\tau) \), then the tachyon field must begin to roll at \( T/\tau \approx 60 \) if \( \Omega_{T,0} \sim 1 \); for a fundamental mass scale \( \sim \text{TeV} \), the roll begins at \( T/\tau \approx 23 \). However, if we adopt \( V_{\text{start}}/T_{\text{start}}^4 \sim \exp(-T_{\text{start}}/\tau) \), then Eq. (16) implies that\(^2\)

\[
\Omega_{T,0} \sim \frac{T_{\text{start}} \exp(-T_{\text{start}}/\tau)}{100 \text{ eV}} \cdot
\]

Thus, the value of \( \Omega_{T,0} \) is exponentially sensitive to the value of \( T_{\text{start}}/\tau \), given the values of \( T_{\text{start}} \) and the known value of \( T_{\text{today}} \); only a very restricted range of \( T_{\text{start}}/\tau \) is consistent with \( \Omega_{T,0} \sim 1 \). A smaller value of \( T_{\text{start}}/\tau \) would imply \( \Omega_{T,0} > 1 \), which would be inconsistent

\(^1\)This estimate presumes that \( \rho_{\text{start}}^{(t)} a^3 \) is constant. Actually, for a short period at the start of rolling, \( \rho_{\text{start}}^{(t)} \) decreases considerably more slowly than \( a^{-3} \). Thus, the estimate of \( \rho_{\text{start}}^{(t)} \) we are using is an overestimate, and the fine tuning problem is actually somewhat more severe. We note, though, that particle annihilations after \( T_{\text{start}} \) increase the value of \( aT \), and this would alleviate the fine tuning, but only by a factor of at most of order 100 in the value of \( V_{\text{start}} \).

\(^2\)More generally, if \( V_{\text{start}}/T_{\text{start}}^4 = v(T_{\text{start}}/\tau) \), we would find

\[
\Omega_{T,0} \sim \frac{T_{\text{start}} v(T_{\text{start}}/\tau)}{100 \text{ eV}}
\]

and assuming that \( v(x) \to 1 \) for \( x \ll 1 \), and \( v(x) \to 0 \) for \( x \to \infty \), then we still must fine tune \( T_{\text{start}}/\tau \) to a large value in order to get \( \Omega_{T,0} < 1 \). If \( v(x) \) declines slower than \( e^{-x} \), the values of \( T_{\text{start}}/\tau \) for which \( \Omega_{T,0} \sim 1 \) are larger, but the value of \( \Omega_{T,0} \) is not as sensitive to \( T_{\text{start}}/\tau \).
with the observed flatness of the universe (or else inconsistent with the observed $T_{\text{today}}$ if we insist on $\Omega_{T,0} \sim 1$, which would be unreasonable). A larger value of $T_{\text{start}}/\tau$ would imply a very small value of $\Omega_{T,0}$, which would not pose any inconsistencies, but would be uninteresting, since it would mean that tachyon matter is only an inconsequential component of the universe.

We have presented this fine tuning from the point of view of integrating backward from the present day to the time when the tachyon starts to roll toward $V = 0$. We can also state the same result from the point of view of integrating forward. Suppose that the tachyon starts rolling at $T_{\text{start}}$, when the potential is $V_{\text{start}} = \lambda T_{\text{start}}^3$; generalize the above discussion by also assuming that $\sqrt{1 - \dot{T}_0^2} = 2\epsilon_{\text{start}}$ at this time. Then the tachyon matter density is $\rho_{\text{start}} = \lambda T_{\text{start}}^3 / 2\epsilon_{\text{start}}$ when the tachyon begins to roll. As long as $\dot{T}_0$ is not too close to zero initially, a rough estimate of the present tachyon matter density is $\rho_{\text{today}} \sim \rho_{\text{start}} T_{\text{today}}^3 / T_{\text{start}}^3$.

Thus, we find that $\rho_{\text{today}} \sim \lambda T_{\text{start}}^3 T_{\text{today}}^3 / 2\epsilon_{\text{start}}$, and since $\rho_{\text{today}} / T_{\text{today}}^3 \sim 100\Omega_{T,0} eV / T_{\text{start}}$, we conclude that $\lambda / 2\epsilon_{\text{start}} \sim 100\Omega_{T,0} eV / T_{\text{start}}$. For any reasonable value of $T_{\text{start}}$, we see that $\lambda / 2\epsilon_{\text{start}} \ll 1$ is required in order for $\Omega_{T,0}$ to be of order one.

The fine-tuning problem derived above holds if standard big bang cosmology begins at the same time as the tachyon begins its roll toward $V = 0$. Alternatively, we might imagine that the universe undergoes a second inflationary period of exponential expansion after the tachyon begins to roll. In that case, the universe could begin with $V_{\text{start}} \sim T_{\text{start}}^4$, and would become tachyon dominated soon afterward. The density of tachyonic matter today would be $\rho_{\text{today}} \sim T_{\text{start}}^4 a_{\text{start}}^3$, but $a_{\text{start}} \ll T_{\text{today}} / T_{\text{start}}$ as a result of the intervening inflation. In order to get a particular value of $\Omega_{T,0}$, we must have $T_{\text{start}} a_{\text{start}}^3 / 0.003\Omega_{T,0}^{1/4} eV$; for $T_{\text{start}} \sim M_{\text{Pl}}$, we find $a_{\text{start}} \sim 10^{-41} eV$, which is $\sim 10^{-9}\Omega_{T,0}^{1/3} eV / M_{\text{Pl}}$. Although such a model allows $V_{\text{start}} \sim T_{\text{start}}^4$, it still presents a fine-tuning problem, since the value of $T_{\text{start}} a_{\text{start}}^3$ must be adjusted precisely to yield $\Omega_{T,0} \sim 1$. This means that the expansion factor during the second inflation must be just right; for $T_{\text{start}} \sim M_{\text{Pl}}$, the expansion factor must be $\sim 10^9 \Omega_{T,0}^{-1/3}$. Too much second inflation would dilute the tachyon density to an uninteresting level, whereas too little second inflation would leave an unacceptably high tachyon density.\(^3\)

This fine-tuning is an unavoidable property of the tachyon dark matter model. The tachyon matter density $\rho_0^{(t)}$ evolves according to $\rho_0^{(t)} = -3H\dot{T}_0^2 \rho_0^{(t)}$ (see Eqs. [12] and [14], as well as Eq. [18] below; see [7]). Since $\dot{T}_0 \leq 1$, we see that $\rho_0^{(t)}$ decreases slower than $a^{-3}$ at all times as the universe expands, irrespective of the time dependence of the cosmological scale factor $a(t)$. Thus, once the standard big bang phase of early universe cosmology starts, it is unacceptable for $\rho_0^{(t)}$ to be comparable to the radiation density in the universe – if it were, then tachyons would dominate right away, and there would never be a radiation dominated phase of expansion. (Subsequent particle annihilations bump up the density of radiation by numerical factors, but unless there are enough annihilations, and they are timed propitiously, premature tachyon domination is still inevitable.) If the standard big bang phase of the universe begins at the same time as tachyons start to roll toward $V(T) \rightarrow 0$, then in order

\(^3\)We also note that density fluctuations in the tachyon matter could grow during the pre-inflation epoch in such a model (see [8], and § II C below).
that tachyons only become dominant at a temperature \( \sim 10 \text{ eV} \) the starting value of \( V(T) \) must be very small compared to its “natural scale” (perhaps \( \sim M_{\text{Pl}}^4 \)). Whatever the tachyon potential, this requires that \( T \) started to roll from a relatively large value. Even if there was a subsequent, second epoch of inflation, before which tachyons were dominant, it must lower the tachyon density enough that they do not dominate again before a temperature \( \sim 10 \text{ eV} \).

Either scenario presents a fine tuning issue irrespective of the precise form of the tachyon potential, \( V(T) \), if tachyons are the cosmological dark matter. Tachyon matter could be present at a negligible density today, in which case the dark matter must be something else, if either the tachyon began rolling from a large enough value of \( T \), or the tachyon matter density inflated to a minute level subsequent to the onset of rolling.

C. Do Tachyons Cluster Gravitationally?

It is well known, from studies of cosmological perturbation theory, that a non-particle species of dark energy with \( \rho_0 a^3 \) is only an acceptable component of the universe provided that it can cluster gravitationally. Otherwise, the growth rate of cosmological density perturbations is truncated: instead of growth proportional to the scale factor \( a(t) \), one find growth proportional to \([a(t)]^\sigma\), with \( \sigma = \frac{1}{4} \left( \sqrt{1 + 2 f_c^4} - 1 \right) \), where \( f_c \) is the fraction of the density of the universe (all of which is presumed to diminish as \( a^{-3} \)) that can cluster gravitationally. Thus, the key question is whether tachyon matter clusters gravitationally.

To see whether tachyon matter can cluster gravitationally, let us anticipate that it does, and consider only a universe that is actually dominated by it. (We can insert a cosmological constant, or quintessence field, and ordinary matter later – the issue is whether or not the tachyon field clusters under gravity, irrespective of whether the gravity is due to the tachyonic matter itself, or to other types of matter.) In this case the background equations Eq. (12) simplify to [7]

\[
H^2 = \frac{8\pi G \rho_0 \rho_0^{(t)}}{3} = \frac{8\pi G V(T_0)}{3\sqrt{1 - T_0^2}}
\]

\[
\dot{\rho}_0^{(t)} = \frac{d}{dt} \left( \frac{V(T_0)}{\sqrt{1 - T_0^2}} \right) = -3H \left( \rho_0^{(t)} + p_0^{(t)} \right) = -\frac{3H T_0^2 V(T_0)}{\sqrt{1 - T_0^2}} = -3H T_0^2 \rho_0^{(t)} .
\]

We can find the perturbation equations easily using results given in \( \S \) 15.10 in [12]; these results are valid in synchronous gauge, and assume, as in Weinberg [12], a metric of the form

\[
ds^2 = -dt^2 + \left[ a^2(t) \delta_{ij} + h_{ij}^{(W)}(x, t) \right] dx^i dx^j = -dt^2 + a^2(t) \left[ \delta_{ij} + h_{ij}(x, t) \right] dx^i dx^j ,
\]

where the superscript \( W \) denotes Weinberg’s definition of the perturbed metric, which we alter slightly. It doesn’t matter that Weinberg’s equations are not written in terms of gauge invariant forms; we can take gauge invariant combinations later. The proof that tachyon matter can cluster should not depend on whether we use gauge independent or gauge dependent variables, although one has to be careful about modes that appear to grow, but are merely gauge artifacts.

To solve the problem, we introduce a perturbation \( T_1 \) to the tachyon field, relative to its background value; associated with it are perturbations
\[ \rho_1^{(t)} = \frac{V'(T_0)T_1}{\sqrt{1 - \dot{T}_0^2}} + \frac{V(T_0)\dot{T}_0 T_1}{(1 - T_0^2)^{3/2}} \]
\[ p_1^{(t)} = -V'(T_0)T_1 \sqrt{1 - \dot{T}_0^2} + \frac{V(T_0)\dot{T}_0 T_1}{\sqrt{1 - T_0^2}} \]
\[ U_1^{(t)} = -\frac{\nabla T_1}{a^2 \dot{T}_0}. \] (20)

where \( \nabla \) is an ordinary gradient with respect to the spatial coordinates, computed as if space is flat. Note that although the perturbations (like the background) do not obey any simple equation of state, nevertheless \( p_1^{(t)}/\rho_1^{(t)} \sim 1 - \dot{T}_0^2 \), which we expect to be very small.

The perturbed Einstein field equations follow from Eqs. (15.10.29)-(15.10.31) in [12]:
\[ -8\pi G(\rho_1^{(t)} - p_1^{(t)}) \delta_{ij} = \frac{\nabla^2 h_{ij}}{a^2} - \frac{h_{ij,jk}}{a^2} - \frac{h_{jk,ik}}{a^2} + \frac{h_{ij}}{a^2} - \ddot{h}_{ij} - 3H\dot{h}_{ij} - \delta_{ij}H \dot{h} \]
\[ -16\pi G(\rho_0^{(t)} + p_0^{(t)}) \frac{T_{1,i}}{T_0} = \frac{\partial(h_{i,j} - h_{j,i})}{\partial t} \]
\[ -8\pi G(\rho_1^{(t)} + 3p_1^{(t)}) = \dot{h} + 2H\dot{h}, \] (21)

where \( h = \text{Tr}(h_{ij}) \). The perturbed equation of motion for the tachyon field is (from Eq. [5] or Eq. [15.10.33] in Weinberg)
\[ \dot{\rho}_1^{(t)} + 3H(\rho_1^{(t)} + p_1^{(t)}) = - (\rho_0^{(t)} + p_0^{(t)}) \left( \frac{\dot{h}}{2} - \frac{T_{1,ii}}{a^2 T_0} \right). \] (22)

Note that we should be able to derive Eq. (22) from appropriate manipulation of Eqs. (21), since the (linearized) Bianchi identities imply that the (linearized) field equations follow from the divergence of the (linearized) Einstein equations.

We are interested in scalar modes, so the metric perturbations ought to reduce to scalar functions; let us assume that (e.g. [10])
\[ h_{ij}(x,t) = A(x,t) \delta_{ij} + [B(x,t)]_{ij} \Rightarrow h = 3A(x,t) + \nabla^2 B(x,t). \] (23)

so that Einstein’s equations may be rewritten as
\[ -8\pi G(\rho_1^{(t)} - p_1^{(t)}) \delta_{ij} = \left( \frac{\nabla^2 A}{a^2} - \ddot{A} - 6H\dot{A} - H\nabla^2 \dot{B} \right) \delta_{ij} + \left( \frac{A}{a^2} - \ddot{B} - 3H\dot{B} \right) \]
\[ -16\pi G(\rho_0^{(t)} + p_0^{(t)}) \frac{T_{1,i}}{T_0} = (2\ddot{A})_i \]
\[ -8\pi G(\rho_1^{(t)} + 3p_1^{(t)}) = 3\ddot{A} + 6H\dot{A} + \nabla^2 \dot{B} + 2H\nabla^2 \dot{B} \]
\[ = \frac{\partial^2(3A + \nabla^2 B)}{\partial t^2} + 2H \frac{\partial(3A + \nabla^2 B)}{\partial t}, \] (24)

and we may also rewrite Eq. (22) as
\[ \dot{\rho}_1^{(t)} + 3H(\rho_1^{(t)} + p_1^{(t)}) = - (\rho_0^{(t)} + p_0^{(t)}) \left[ \frac{1}{2} \left( 3\ddot{A} + \nabla^2 \dot{B} \right) - \frac{T_{1,ii}}{a^2 T_0} \right]. \] (25)
Take the trace of the first of Eqs. (24) to find
\[-24\pi G (\rho_1^{(t)} - p_1^{(t)}) = \frac{4\nabla^2 A}{a^2} - 3\ddot{A} - 18H \dot{A} - \nabla^2 \ddot{B} - 6H \nabla^2 \dot{B} \]
\[= \frac{4\nabla^2 A}{a^2} - \frac{\partial^2 (3A + \nabla^2 B)}{\partial t^2} - 6H \frac{\partial (3A + \nabla^2 B)}{\partial t}, \tag{26} \]
then take \(\nabla_i \nabla_j\) of the first of Eqs. (24) to get
\[-8\pi G \nabla^2 (\rho_1^{(t)} - p_1^{(t)}) = \nabla^2 \left( \frac{2\nabla^2 A}{a^2} - \ddot{A} - 6H \dot{A} - \nabla^2 \ddot{B} - 4H \nabla^2 \dot{B} \right), \tag{27} \]
and finally subtract \(\frac{1}{3} \nabla^2\) of Eq. (26) to find the homogeneous equation
\[0 = \nabla^4 \left( \frac{2A}{3a^2} - \frac{2B}{3} - 2H \dot{B} \right); \tag{28} \]
conclude that (for modes \(\propto e^{ik \cdot x}\) with \(|k| \neq 0\))
\[\frac{A}{a^2} = \ddot{B} + 3H \dot{B}. \tag{29} \]
Add the last of Eqs. (24) to Eq. (26), and divide the result by four:
\[-8\pi G \rho_1^{(t)} = \frac{\nabla^2 A}{a^2} - 3H \dot{A} - H \nabla^2 \dot{B}; \tag{30} \]
subtract the result from the last of Eqs. (24), and use Eq. (29) to find
\[\ddot{A} + 3H \dot{A} = -8\pi G \rho_1^{(t)}. \tag{31} \]
Following Ref. [10], we can shall use the last three equations, supplemented by the middle of Eqs. (24), to derive gauge invariant equations describing the evolution of cosmological density perturbations.

Consider the gauge invariant combination (for proof, see [10])
\[\phi_H = \frac{1}{2} \left( A - a^2 H \dot{B} \right) = \frac{1}{2} a^2 (\ddot{B} + 2H \dot{B}) \equiv \frac{1}{2} \psi_H, \tag{32} \]
where \(\psi_H = a^2 \dot{B}\). We can rewrite Eq. (31) as
\[\ddot{\psi}_H + 4H \dot{\psi}_H + (2\dot{H} + 3H^2) \dot{\psi}_H + (\ddot{H} + 3H \dot{H}) \psi_H = -8\pi G \rho_1^{(t)}, \tag{33} \]
or, in terms of \(\phi_H\),
\[2\ddot{\phi}_H + 8H \dot{\phi}_H + 2(2\dot{H} + 3H^2) \dot{\phi}_H + (\ddot{H} + 3H \dot{H}) \psi_H = -8\pi G \rho_1^{(t)}. \tag{34} \]
To eliminate \(\psi_H\) from this equation, use the middle of Eqs. (24), which can be integrated once spatially to yield

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\[ \dot{A} = -8\pi G(\rho_0^{(t)} + p_0^{(t)}) \frac{T_1}{T_0}; \]  

(35)

Eq. (29) implies that \( A = \psi_H + H\psi, \) and therefore

\[ \ddot{\psi}_H + H\dot{\psi}_H + H\dot{\psi}_H = -8\pi G(\rho_0^{(t)} + p_0^{(t)}) \frac{T_1}{T_0}, \]  

(36)

which we can solve for \( \psi_H: \)

\[ \psi_H = -\frac{8\pi G(\rho_0^{(t)} + p_0^{(t)}) T_1}{HT_0} - \frac{H}{H} \frac{\psi_H}{H} - \frac{\ddot{\psi}_H}{H} = -\frac{8\pi G(\rho_0^{(t)} + p_0^{(t)}) T_1}{HT_0} \frac{2H\phi_H}{H} - \frac{2\dot{\phi}_H}{H}. \]  

(37)

Substituting for \( \psi_H \) results in the equation

\[ \ddot{\phi}_H + \left( H - \frac{\dot{H}}{H} \right) \dot{\phi}_H + \left( 2\dot{H} - \frac{H\dot{H}}{H} \right) \phi_H = -4\pi G \left[ \rho_1^{(t)} - \frac{(\rho_0^{(t)} + p_0^{(t)})(\ddot{H} + 3H\dot{H}) T_1}{HT_0} \right]. \]  

(38)

We can get a second equation for \( \phi_H \) from Eqs. (30), which may be rewritten in the form

\[ \frac{\nabla^2 \phi_H}{a^2} = -4\pi G \rho_1^{(t)} + \frac{3H\dot{A}}{2} = -4\pi G \left[ \rho_1^{(t)} + \frac{3H(\rho_0^{(t)} + p_0^{(t)}) T_1}{T_0} \right], \]  

(39)

where we have used Eq. (29) to substitute for \( \dot{A}. \) \(^4\)

To go further, we need to return to the equations for the cosmological background and compute various derivatives of the expansion rate, \( H, \) starting from Eqs. (12). We then find

\[ \dot{H} = -4\pi G(\rho_0^{(t)} + p_0^{(t)}), \]  

\[ \ddot{H} = 12\pi G H(\rho_0^{(t)} + p_0^{(t)})(1 + c_s^2), \]  

\[ \frac{\ddot{H} + 3H\dot{H}}{H} = -3Hc_s^2, \]  

(40)

where \( c_s^2 \equiv \rho_0^{(t)}/\rho_0^{(t)}. \) For rolling tachyons we have

\[ \dot{p}_0^{(t)} = -\rho_0^{(t)}(1 - 2\ddot{T}_0^2) + 2\rho_0^{(t)}\dot{T}_0\ddot{T}_0 = (1 - 2\ddot{T}_0^2) \left[ \rho_0^{(t)} - \frac{2\rho_0^{(t)} V'(T_0) \dot{T}_0}{V(T_0)} \right] \]  

\[ c_s^2 = (1 - 2\ddot{T}_0^2) \left[ 1 - \frac{2\rho_0^{(t)} V'(T_0) \dot{T}_0}{V(T_0) \rho_0^{(t)}} \right] = (1 - \dot{T}_0^2) \left[ 1 + \frac{2V'(T_0)}{3T_0 V(T_0) H} \right]; \]  

(41)

\(^4\)The source term in Eq. (39) is easily shown to be the gauge invariant density perturbation.
although the second factor in the above can be large, we still expect very small \( c_s^2 \) (e.g. \( 1 - \dot{T}_0^2 \) is exponentially small for \( V(T) = V_0 \exp(-T/\tau) \) at late times).\(^5\) We can combine Eqs. (38) and (39) into a single differential equation for \( \phi_H \) as follows. First, multiply Eq. (39) by \( 1 - \dot{T}_0^2 \) to find
\[
(1 - \dot{T}_0^2) \frac{\nabla^2 \phi_H}{a^2} = -4\pi G (1 - \dot{T}_0^2) \left[ \rho_1^{(t)} - \rho_0^{(t)} \frac{T_1}{T_0} \right]
\]
\[
= -4\pi G \left( p_1^{(t)} + \frac{2V'(T_0)\dot{T}_0}{1 - \dot{T}_0^2} - \rho_0^{(t)} (1 - \dot{T}_0^2) \right) \frac{T_1}{T_0}
\]
\[
= -4\pi G \left( p_1^{(t)} - \frac{c_s^2 \rho_0^{(t)} T_1}{T_0} \right)
\]
\[
= -4\pi G \left( p_1^{(t)} - \frac{\dot{\rho}_0^{(t)} T_1}{T_0} \right),
\]
where, to get the final version of this equation, we used Eq. (20) and Eq. (41). But Eqs. (40) can be used to rewrite Eq. (38) as
\[
\ddot{\phi}_H + \left( H - \frac{\dot{H}}{H} \right) \dot{\phi}_H + \left( 2\dot{H} - \frac{H\ddot{H}}{H^2} \right) \phi_H = -4\pi G \left( p_1^{(t)} - \frac{\dot{\rho}_0^{(t)} T_1}{T_0} \right);
\]
combining the last two results we find the homogeneous equation
\[
\ddot{\phi}_H + \left( H - \frac{\dot{H}}{H} \right) \dot{\phi}_H + \left( 2\dot{H} - \frac{H\ddot{H}}{H^2} \right) \phi_H - (1 - \dot{T}_0^2) \frac{\nabla^2 \phi_H}{a^2} = 0.
\]

The last term is very small because \( 1 - \dot{T}_0^2 \ll 1 \), so that the effective Jeans length for density perturbations is only of order \( L_{J, t} = H^{-1} \sqrt{1 - \dot{T}_0^2} \). Thus, tachyon matter can cluster on all but very small length scales, once \( \dot{T}_0 \to 1 \). At earlier times, when \( \dot{T}_0 < 1 \), clustering is prevented on scales smaller than \( \sim H^{-1} \); if the tachyon field rolls slowly, the growth of subhorizon tachyon matter perturbations is suppressed, just as for quintessence.

For perturbations with proper sizes larger than \( L_{J, t} \), Eq. (44) implies that
\[
\phi_H = C_0 + C_1 \frac{t}{a^{5/3}};
\]
during epochs when the tachyon density dominates and decreases \( \sim a^{-3} \) to a very good approximation. Then from Eq. (39), the gauge-invariant tachyon matter density contrast
\[
\Delta^{(t)} \equiv \frac{\rho_1^{(t)} - \dot{\rho}_0^{(t)} T_1/\dot{T}_0}{\rho_0^{(t)}} \propto \frac{\phi_H}{a^2 \rho_0^{(t)}} = C_0 t^{2/3} + C_1 t,
\]
which is a linear combination of the well-known growing and shrinking modes for perturbations of nonrelativistic matter. (See e.g. Weinberg [12], § 15.9, 15.10, or Peebles [11], § 11.)

\(^5\)With these results it is easy to show that the source term in Eq. (38) is just the gauge invariant pressure perturbation.
III. CONCLUSIONS

We have shown that tachyon matter is an acceptable candidate for the dark matter of the universe, provided the initial conditions for the rolling tachyon field are fine tuned. In particular, the mass density in tachyonic form decreases $\propto a^{-3}$ at late times, as has been observed by others [7,8], but only has an acceptable value today provided that the initial value of the tachyon field is in a rather small, and special range (e.g. [6] and § II B), assuming that standard big bang cosmology proceeds uninterrupted after the tachyon begins to roll. If the tachyon field begins rolling at too small a value, then the density today would turn out to be unacceptably large, given the temperature of the universe today – a well-measured cosmological parameter – and a large range of reasonable values for the temperature of the universe when the tachyon starts to roll. If the tachyon field begins rolling at too large a value, then its density today is inconsequential (i.e. $\Omega_{T,0} \ll 1$), so it plays a relatively harmless, but uninteresting role in cosmology. Since the required starting value of $T_0$ estimated in § II B is rather large (about $60\tau$ if the fundamental scale is the Planck mass, or about $23\tau$ if it is TeV, for an exponential potential $V(T) \propto \exp(-T/\tau)$ and $\Omega_{T,0} \sim 1$), small fractional differences have substantial consequences for the subsequent evolution of the universe. (This fine-tuning avoids the over-growth of density fluctuations noted in Ref. [8], at the price of possibly unnatural initial conditions.) Why the tachyon field begins rolling from a small range of initial values far from unity must be explained if the rolling tachyon is to be regarded as a natural candidate for the dark matter of the universe.

Alternatively, the tachyon field could have started rolling from “generic” initial conditions provided that a second period of inflation occurred after the tachyon field had rolled for awhile. In that case, there is still a fine tuning issue, since the value of $T_{start}^{3/4}$ must be adjusted precisely (to a value of about $0.003\Omega_{T,0}^{1/4}$ eV; see § II B) in order for $\Omega_{T,0} \sim 1$. For a given value of $T_{start}$ (e.g. the Planck scale, or whatever the fundamental mass scale of string theory might be), this implies a very small range of values for the expansion factor during the second inflationary phase. For a smaller expansion factor, the density of tachyons could not satisfy $\Omega_{T,0} < 1$, whereas for a larger factor, the tachyon density today would be too low to be of any importance to cosmology.

ACKNOWLEDGMENTS

We would like to thank Lam Hui, Bhuvnesh Jain, Ashoke Sen, and Max Tegmark for discussions. The work of GS was supported in part by the DOE grants DE-FG02-95ER40893, DE-EY-76-02-3071 and the University of Pennsylvania School of Arts and Sciences Dean’s funds. G.S. also thanks the Michigan Center for Theoretical Physics for hospitality while this paper was written.
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