A simple formula for the average gate fidelity of a quantum dynamical operation.

Charaterizing the quality of quantum channels and quantum gates is a hard task. Quantum computers are usually connected by a trace-preserving quantum operation, which can be defined by a general formula for the average gate fidelity. We also give a simplified proof of the average gate fidelity formula, which is useful for experimental determination of quantum fidelity. PACS numbers: 03.67.-a, 03.65.Yz, 89.30.+c.

Michael A. Nielsen
Centre for Quantum Computer Technology and Department of Physics
University of Queensland, Brisbane, Queensland 4072, Australia
(Dated: January 22, 2003)

This note presents a simple formula for the average gate fidelity between arbitrary quantum states and quantum channels, which is connected by a trace-preserving quantum operation. We also give a simplified proof of the average gate fidelity formula, which is useful for experimental determination of quantum fidelity. PACS numbers: 03.67.-a, 03.65.Yz, 89.30.+c.

1. Introduction

The fidelity of an arbitrary quantum operation is defined by

$$ F(\rho) = \frac{1}{2} Tr |\sqrt{\rho} S(\sqrt{\rho})|^2 $$

where $\rho$ is the density matrix of the input state and $S(\sqrt{\rho})$ is the output state. The fidelity of a quantum operation $\mathcal{E}$ is defined by

$$ F(\mathcal{E}) = \frac{1}{2} Tr |\sqrt{\rho} S(\sqrt{\rho})|^2 $$

where $\rho$ is the density matrix of the input state and $S$ is the output state. The fidelity of a quantum operation $\mathcal{E}$ is defined by

$$ F(\mathcal{E}) = \frac{1}{2} Tr |\sqrt{\rho} S(\sqrt{\rho})|^2 $$

where $\rho$ is the density matrix of the input state and $S$ is the output state.
\[ F_\omega(\mathcal{E}_\rho) = \int d\mu(\rho) F_\omega(\mathcal{E}_\rho(\rho)) \]

where \( \mathcal{E}_\rho(\rho) = \frac{1}{d} \mathcal{E}_\rho(\rho) \) is the depolarized channel.

Using Eq. (13), we can calculate the fidelity between two quantum states \( \rho \) and \( \sigma \) using the trace norm, defined as

\[ B(\rho, \sigma) = \text{tr}(\rho - \sigma) \]

For the case of two qubits, the fidelity can be calculated as

\[ F(\rho, \sigma) = \frac{1}{d^2} \text{tr}(\rho^{1/2} \sigma \rho^{1/2}) \]

where \( d = 2 \) for qubits. The fidelity reduces to the classical fidelity in the limit of \( d \to \infty \).
also suffice. Standard linear algebraic methods may be used to find co-efficients \( a_{jk} \) such that \( U_j = \sum_k a_{jk} \rho_k \), whence Eq. (17) implies

\[
\mathcal{F}(\mathcal{E}, U) = \sum_{jk} a_{jk} \text{tr} \left( U U_j \right) \mathcal{E}(\rho_k) + d^2 \sum_{jk} a_{jk} \frac{1}{d^2 (d+1)}.
\]  

(19)

Using standard state tomography (see, e.g., [13]) it is possible to determine \( \mathcal{E}(\rho_k) \), and thus to determine \( \mathcal{F}(\mathcal{E}, U) \).

In conclusion, we have obtained a simple formula for the average fidelity of a noisy quantum channel or quantum gate. This formula may be useful for experimentally characterizing quantum gates and channels. It would be interesting to generalize these results further to non-uniform starting distributions of states.

Acknowledgments

Thanks to Jennifer Dodd, Gerard Milburn, Tobias Osborne, and Lorenza Viola for their comments on the manuscript.


[14] Our definition is a special case of [2], which also considered non-maximally entangled states of \( RQ \).

[15] Note that \( U \) and \( \mathcal{E} \) act on system \( Q \) alone in these expressions, with the identity action on \( R \) implicit.