R-matrices and the Tensor Product Graph Method

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A systematic method for constructing trigonometric R-matrices corresponding to the (multiplicity-free) tensor product of any two affinizable representations of a quantum algebra or superalgebra has been developed by the Brisbane group and its collaborators. This method has been referred to as the Tensor Product Graph Method. Here we describe applications of this method to untwisted and twisted quantum affine superalgebras.

1. Introduction

The (graded) Yang-Baxter equation (YBE) plays a central role in the theory of (supersymmetric) quantum integrable systems. Solutions to the YBE are usually called R-matrices. The knowledge of R-matrices has many physical applications. In one-dimensional lattice models, R-matrices yield the Hamiltonians of quantum spin chains\(^1\). In statistical mechanics, R-matrices define the Boltzmann weights of exactly soluble models\(^2\) and in integrable quantum field theory they give rise to exact factorizable scattering S-matrices\(^3\). So the construction of R-matrices is fundamental in the study of integrable systems.

Mathematical structures underlying the YBE and therefore R-matrices and integrable models are quantum affine (super)algebras. A systematic method for the construction of trigonometric R-matrices arising from untwisted and twisted quantum affine (super)algebras has been developed in\(^4,5,6,7,8,9\) (see also\(^10\) for rational cases). This method is called the Tensor Product Graph (TPG) method. The method enables one to construct spectral dependent R-matrices corresponding to the (multiplicity-free) tensor product of any two affinizable representations of a quantum algebra or superalgebra.

In this contribution, we describe the TPG method in the context of untwisted and twisted quantum affine superalgebras. Quantum superalgebras are interesting since the tensor product decomposition often has indecomposables and integrable
models associated with them may in some instances be interpreted as describing strongly correlated fermion systems \textsuperscript{11,12}.

2. Quantum Affine Superalgebras and Jimbo Equation

Let us first of all recall some facts about the affine superalgebra \( G^{(k)} \), \( k = 1, 2 \). Let \( \mathcal{G}_0 \) be the fixed point subalgebra under the diagram automorphism of \( \mathcal{G} \) of order \( k \). In the case of \( k = 1 \), we have \( \mathcal{G}_0 \equiv \mathcal{G} \). For \( k = 2 \) we may decompose \( \mathcal{G} \) as \( \mathcal{G}_0 \oplus \mathcal{G}_1 \), where \( [\mathcal{G}_0, \mathcal{G}_1] \subset \mathcal{G}_1 \). Let \( \psi \) be the highest root of \( \mathcal{G}_0 \equiv \mathcal{G} \) for \( k = 1 \) and \( \theta \) be the highest weight of the \( \mathcal{G}_0 \)-representation \( \mathcal{G}_1 \) for \( k = 2 \).

Quantum affine superalgebras \( U_q[G^{(k)}] \) are \( q \)-deformations of the universal enveloping algebras \( U[G^{(k)}] \) of \( G^{(k)} \). We shall not give the defining relations for \( U_q[G^{(k)}] \), but mention that the action of the coproduct on its generators \( \{h_i, e_i, f_i, 0 \leq i \leq r \} \) is given by

\[
\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \\
\Delta(e_i) = e_i \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes e_i, \\
\Delta(f_i) = f_i \otimes q^{\frac{h_i}{2}} + q^{-\frac{h_i}{2}} \otimes f_i.
\]

Define an automorphism \( D_z \) of \( U_q[G^{(k)}] \) by

\[
D_z(e_i) = z^{k\delta_{i\omega}} e_i, \\
D_z(f_i) = z^{-k\delta_{i\omega}} f_i, \\
D_z(h_i) = h_i.
\]

Given any two minimal irreducible representations \( \pi_\lambda \) and \( \pi_\mu \) of \( U_q[\mathcal{G}_0] \) and their affinizations to irreducible representations of \( U_q[G^{(k)}] \), we obtain a one-parameter family of representations \( \Delta^z_{\lambda\mu} \) of \( U_q[G^{(k)}] \) on \( V(\lambda) \otimes V(\mu) \) defined by

\[
\Delta^z_{\lambda\mu}(a) = \pi_\lambda \otimes \pi_\mu \left( (D_z \otimes 1)\Delta(a) \right), \quad \forall a \in U_q[G^{(k)}],
\]

where \( z \) is the spectral parameter. Let \( R^{\lambda\mu}(z) \) be the spectral dependent R-matrices associated with \( \pi_\lambda \) and \( \pi_\mu \), which satisfies the YBE. Moreover it obeys the intertwining properties:

\[
R^{\lambda\mu}(z) \Delta^z_{\lambda\mu}(a) = (\Delta^T)^z_{\lambda\mu}(a) R^{\lambda\mu}(z)
\]

which, according to Jimbo \textsuperscript{13}, uniquely determine \( R^{\lambda\mu}(z) \) up to a scalar function of \( z \). We normalize \( R^{\lambda\mu}(z) \) such that \( R^{\mu\lambda}(z) R^{\mu\lambda}(z^{-1}) = I \), where \( R^{\lambda\mu}(z) = P R^{\lambda\mu}(z) \) with \( P : V(\lambda) \otimes V(\mu) \rightarrow V(\mu) \otimes V(\lambda) \) the usual graded permutation operator.

In order for the equation (4) to hold for all \( a \in U_q[G^{(k)}] \) it is sufficient that it holds for all \( a \in U_q(\hat{L}_0) \) and in addition for the extra generator \( e_0 \). The relation for \( e_0 \) reads explicitly

\[
\hat{R}^{\lambda\mu}(z) \left( z \pi_\lambda(e_0) \otimes \pi_\mu(q^{h_0/2}) + \pi_\lambda(q^{-h_0/2}) \otimes \pi_\mu(e_0) \right) \\
= \left( \pi_\mu(e_0) \otimes \pi_\lambda(q^{h_0/2}) + z \pi_\mu(q^{-h_0/2}) \otimes \pi_\lambda(e_0) \right) \hat{R}^{\lambda\mu}(z).
\]

Eq.(5) is the Jimbo equation for \( U_q[G^{(k)}] \).

3. Solutions to Jimbo Equation and Tensor Product Graph Method
Let $V(\lambda)$ and $V(\mu)$ denote any two minimal irreducible representations of $U_q[\mathcal{G}^{(k)}]$. Assume the tensor product module $V(\lambda) \otimes V(\mu)$ is completely reducible into irreducible $U_q[\mathcal{G}_0]$-modules as

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} V(\nu) \quad (6)$$

and there are no multiplicities in this decomposition. We denote by $P_{\nu}^{\lambda\mu}$ the projection operator of $V(\nu)$ on $V(\lambda) \otimes V(\mu)$ and set $R_{\nu}^{\lambda\mu} = P_{\nu}^{\lambda\mu} P_{\nu}^{\mu\lambda} = P_{\nu}^{\mu\lambda} R_{\nu}^{\lambda\mu} = 1$.

We may thus write

$$\hat{R}_{\nu}^{\lambda\mu}(z) = \sum_{\nu'} \rho_{\nu'}(z) P_{\nu'}^{\lambda\mu}, \quad \rho_{\nu'}(1) = 1. \quad (7)$$

Following our previous approach $^5$, the coefficients $\rho_{\nu}(z)$ may be determined according to the recursion relation

$$\rho_{\nu}(z) = \frac{q^{C(\nu)/2} + \epsilon_{\nu} \epsilon_{\nu'} z q^{C(\nu')/2}}{z q^{C(\nu)/2} + \epsilon_{\nu} \epsilon_{\nu'} q^{C(\nu')/2}} \rho_{\nu'}(z), \quad (8)$$

which holds for any $\nu \neq \nu'$ for which

$$P_{\nu}^{\lambda\mu} \left( \pi_{\lambda}(e_0) \otimes \pi_{\mu}(q^h/2) \right) P_{\nu'}^{\lambda\mu} \neq 0. \quad (9)$$

Here $C(\nu)$ is the eigenvalue of the universal Casimir element of $\mathcal{G}_0$ on $V(\nu)$ and $\epsilon_{\nu}$ denotes the parity of $V(\nu) \subseteq V(\lambda) \otimes V(\mu)$.

We note that $e_0 \otimes q^h/2$ transforms under the adjoint action of $U_q[\mathcal{G}_0]$ as the lowest weight of $\mathcal{G}_0$-module $V(\psi)$ [resp. $V(\theta)$] for $k = 1$ (resp. $k = 2$) (i.e. as the lowest component of a tensor operator). Throughout we adopt the notation

$$< a >_\pm = \frac{1 \pm z q^a}{z \pm q^a}, \quad (10)$$

so that the relation (8) may be expressed as

$$\rho_{\nu}(z) = \left\langle \frac{C(\nu') - C(\nu)}{2} \right\rangle \epsilon_{\nu} \epsilon_{\nu'} \rho_{\nu'}(z). \quad (11)$$

To graphically encode the recursion relations between different $\rho_{\nu}$ we introduce the **Extended TPG** for $U_q[\mathcal{G}^{(1)}]$ and **Extended Twisted TPG** for $U_q[\mathcal{G}^{(2)}]$.

**Definition 1:** The **Extended TPG** associated to the tensor product $V(\lambda) \otimes V(\mu)$ is a graph whose vertices are the irreducible modules $V(\nu)$ appearing in the decomposition (6) of $V(\lambda) \otimes V(\mu)$. There is an edge between two vertices $V(\nu)$ and $V(\nu')$ iff

$$V(\nu') \subset V_{\text{adj}} \otimes V(\nu) \quad \text{and} \quad \epsilon(\nu) \epsilon(\nu') = -1. \quad (12)$$

The condition (12) is a necessary condition for (9) corresponding to $U_q[\mathcal{G}^{(1)}]$ to hold.
Definition 2: The Extended Twisted TPG which has the same set of nodes as the twisted TPG but has an edge between two vertices $\nu \neq \nu'$ whenever

$$V(\nu') \subseteq V(\theta) \otimes V(\nu) \quad (13)$$

and

$$\epsilon_{\nu}\epsilon_{\nu'} = \begin{cases} +1 & \text{if } V(\nu) \text{ and } V(\nu') \text{ are in the same irreducible representation of } G, \\ -1 & \text{if } V(\nu) \text{ and } V(\nu') \text{ are in different irreducible representations of } G. \end{cases} \quad (14)$$

The conditions (13) and (14) are necessary conditions for (9) corresponding to $U_q[G^{(2)}]$ to hold.

We will impose a relation (8) for every edge in the extended (twisted) TPG but we will be imposing too many relations in general. These relations may be inconsistent and we are therefore not guaranteed a solution. If however a solution to the recursion relations exists, then it must give the unique correct solution to the Jimbo’s equation.

4. Examples of $R$-matrices for $U_q[gl(m|n)^{(1)}]$ Throughout we introduce $\{\epsilon_i\}_{i=1}^m$ and $\{\delta_j\}_{j=1}^n$ which satisfy $(\epsilon_i, \epsilon_j) = \delta_{ij}$, $(\delta_i, \delta_j) = -\delta_{ij}$ and $(\epsilon_i, \delta_j) = 0$. As is well known, every irreducible representation of $U_q[gl(m|n)]$ provides also an irreducible representation for $U_q[gl(m|n)^{(1)}]$. Here, as examples, we will construct the $R$-matrices corresponding to the following tensor product: rank $a$ antisymmetric tensor with rank $b$ antisymmetric tensor of the same type. Without loss of generality, we assume $m \geq a \geq b$ and the antisymmetric tensors to be contravariant. The tensor product decomposition is

$$V(\lambda_a) \otimes V(\lambda_b) = \bigoplus_c V(\Lambda_c) \quad (15)$$

where, when $a + b \leq m$,

$$\lambda_b = \sum_{i=1}^b \epsilon_i, \quad \Lambda_c = \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i, \quad c = 0, 1, \ldots, b \quad (16)$$

and when $a + b > m$,

$$\Lambda_c = \sum_{i=1}^{a+c} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i, \quad c = 0, 1, \ldots, m - a$$

$$\Lambda_c = \sum_{i=1}^{m} \epsilon_i + \sum_{i=1}^{b-c} \epsilon_i + (a + c - m)\delta_1, \quad c = m - a + 1, \ldots, b \quad (17)$$

The corresponding TPG is

$$V(\lambda_a) \otimes V(\lambda_b) = \begin{array}{c}
\Lambda_0 \\
\Lambda_1 \\
\vdots \\
\Lambda_{b-1} \\
\Lambda_b
\end{array} \quad (18)$$
which is consistent; such is always the case when a graph is a tree (i.e. contains no closed loops). From the graph we obtain

\[ \hat{R}^{\lambda_a,\lambda_b}(x) = \sum_{a=0}^{b-c} \prod_{i=1}^{c} (2i + a - b) \cdot P_{\lambda_a,\lambda_b}^{\lambda_a,\lambda_b} \]  \hspace{1cm} (19)

The \( a = b = 1 \) case had been worked out before, which is known to give rise to the

Perk-Schultz model R-matrices 14,15.

5. Examples of R-matrices for \( U_q[gl(n|n)^{(2)}] \)

To begin with, we introduce the concept of minimal representations. By minimal irreducible representations of \( G \), we mean those irreducible representations which are also irreducible under the fixed subalgebra \( G_0 \). We can determine R-matrices for any tensor product \( V(\lambda_a) \otimes V(\lambda_b) \) of two minimal representations \( V(\lambda_a), V(\lambda_b) \) of \( U_q[gl(m|n)^{(2)}] \), where \( V(\lambda_a) \) is also irreducible model under \( U_q[osp(m|n)] \) with the corresponding \( U_q[osp(m|n)] \) highest weight \( \lambda_a = (\hat{0}|a,\hat{0}) \). Recall that for our case \( G_0 \equiv osp(m|n) \) and \( \theta = \delta_1 + \delta_2 \). Below we shall illustrate the method for the interesting case of \( a = b, m = n > 2 \), where an indecomposable appears in the tensor product decomposition.

The decomposition of the tensor product of two minimal irreducible representations of \( U_q[osp(m|n)] \): 9

\[ V(\lambda_a) \otimes V(\lambda_b) = \bigoplus_{c=0}^{a} \bigoplus_{k=0}^{c} V(k, a + b - 2c); \]  \hspace{1cm} (20)

here and throughout \( V(a, b) \) denotes an irreducible \( U_q[osp(m|n)] \) module with highest weight \( \lambda_{a,b} = (\hat{0}|a + b, a, \hat{0}) \). Note that one can only get an indecomposable in (20) when \( m = n > 2 \) and \( a + b - 2c = 0 \). Since \( a \leq b, c \leq a \), this can only occur when \( a = b \) and \( c = a \). In that case the \( U_q[osp(m|n)] \)-modules \( V(k,0), k = 0,1 \), will form an indecomposable. From now on we denote by \( V \) this indecomposable module, and write the \( U_q[osp(n|n)] \) module decomposition (20) as

\[ V(\lambda_a) \otimes V(\lambda_a) = \bigoplus_{\nu} V(\nu) \bigoplus V, \]  \hspace{1cm} (21)

where the sum on \( \nu \) is over the irreducible highest weights. Note that \( V \) contains a unique submodule \( \bar{V}(\delta_1 + \delta_2) \) which is maximal, indecomposable and cyclically generated by a maximal vector of weight \( \delta_1 + \delta_2 \) such that \( V / \bar{V}(\delta_1 + \delta_2) \cong V(\hat{0}|\hat{0}) \) (the trivial \( U_q[osp(n|n)] \)-module). Moreover \( V \) contains a unique irreducible submodule \( V(\hat{0}|\hat{0}) \subset \bar{V}(\delta_1 + \delta_2) \). The usual form of Schur’s lemma applies to \( \bar{V}(\delta_1 + \delta_2) \) and so the space of \( U_q[osp(n|n)] \) invariants in \( \text{End}(V) \) has dimension 2. It is spanned by the identity operator \( I \) together with an invariant \( N \) (unique up to scalar multiples) satisfying

\[ NV = V(\hat{0}|\hat{0}) \subset \bar{V}(\delta_1 + \delta_2), \quad N \bar{V}(\delta_1 + \delta_2) = (0). \]  \hspace{1cm} (22)
It follows that $N$ is nilpotent, i.e. $N^2 = 0$.

We can show that the minimal irreducible $U_q[osp(n|n)]$ modules, $V(\lambda_a)$, with highest weight $\lambda_a$, are affinizable to carry irreducible representations of $U_q[gl(n|n)^{(2)}]$. We now determine the extended twisted TPG for the decomposition given by (21). We note that $V$ can only be connected to two nodes corresponding to highest weights

$$\nu = \begin{cases} 2\delta_1 \text{ (opposite parity), } & (c,k) = (a-1,0) \\ 2(\delta_1 + \delta_2) \text{ (same parity), } & (c,k) = (a,2). \end{cases}$$

We thus arrive at the extended twisted TPG for (21), given by Figure 1.

It can be shown that the extended twisted TPG is consistent, i.e. the recursion relations (8) give the same result independent of the path along which one recurses. To prove this it suffices to show for each closed loop of four vertices in the graph, that the difference in Casimir eigenvalues for $osp(n|n)$ along one edge equals the difference along the opposite edge.

Let $P_V \equiv P^{\lambda_a,\lambda_a}_V$ be the projection operator from $V(\lambda_a) \otimes V(\lambda_a)$ onto $V$ and $P_\nu \equiv P^{\lambda_a,\lambda_{\nu}}_\nu$ the projector onto $V(\nu)$. Then the R-matrix $\hat{R}(z) \equiv \hat{R}^{\lambda_a,\lambda_{\nu}}(z)$ from the extended twisted TPG can be expanded in terms of the operators $N$, $P_V$ and
\( P_{\nu} : \)

\[
\hat{R}(z) = \rho_N(z)N + \rho_V(z)P_V + \sum_{\nu} \rho_{\nu}(z)P_{\nu}.
\]  (24)

The coefficients \( \rho_{\nu}(z) \) can be obtained recursively from the extended twisted TPG. However, the coefficients \( \rho_N(z) \) and \( \rho_V(z) \) cannot be read off from the extended twisted TPG since the corresponding vertex refers to an indecomposable module. Rather they are determined by the approach \(^{16}\) to \( U_q[gl(2|2)^{(2)}] \). The result is \(^9\)

\[
\hat{R}(z) = \rho_N(z)N + \rho_V(z)P_V + \sum_{c=0}^{a} \sum_{k=0}^{c-k} \prod_{j=1}^{(2j-2a)} \prod_{i=1}^{(i-2a-1)} P_{(2a-2c+k)d_1+kd_2},
\]  (25)

where the primes in the sums signify that terms corresponding to \( c = a \) with \( k = 0,1 \) are omitted from the sums, and \( \rho_V(z), \rho_N(z) \) are given by

\[
\rho_V(z) = \frac{z - q^2}{1 - zq^2} \prod_{j=1}^{a-1} (2j - 2a) + \prod_{i=1}^{a-1} (i - 2a - 1),
\]

\[
\rho_N(z) = (-1)^a q^{-a^2} \frac{1 - z}{1 + z} \rho_V(z).
\]  (26)

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**References**