A recent mathematical technique for nonlinear hyperbolic systems, maximally dissipative boundary conditions, is applied to establish a simple, well-posed version of the general relativistic initial-boundary value problem in harmonic coordinates. The method is implemented as a nonlinear evolution algorithm which, in the weak field regime, is robustly stable. A linearized version has been stably matched to a characteristic code to compute the gravitational waveform radiated to infinity.

I. INTRODUCTION

The waveform emitted in the inspiral and merger of a relativistic binary is theoretical input crucial to the success of the fledgling gravitational wave observatories. A computational approach is necessary to treat the highly nonlinear regime of a black hole or neutron star collision. Developing this computational ability has been the objective of the Binary Black Hole (BBH) Grand Challenge [1] and other worldwide efforts. The Grand Challenge built a code based on the Arnowitt-Deser-Misner [2] formulation to solve Einstein’s equations by Cauchy evolution. The numerical instabilities encountered with that code have been traced, at least in the linear regime, to the improper application of boundary conditions [3]. Other groups have encountered similar difficulties in treating boundaries (see [4] for a recent discussion) and the working practice is to forestall problems by placing the outer boundary at a large distance from the region of physical interest (see e.g. [5]).

This deficiency extends beyond numerical relativity to a lack of analytic understanding of the initial-boundary value problem for general relativity. The local-version of the initial-boundary value problem is schematically represented in Fig. 1. Given Cauchy data on a spacelike hypersurface $\mathcal{S}$ and boundary data on a timelike hypersurface $\mathcal{B}$, the problem is to determine a solution in the appropriate domain of dependence. Whereas there is considerable mathematical understanding of the gravitational initial value problem (for recent reviews see [6–8]), until recently the initial-boundary value problem has received little attention. Indeed, only relatively recently have methods been available in the mathematical literature, in particular the technique of maximally dissipative boundary conditions, which can be applied to the nonlinear initial-boundary problem of the type arising in general relativity [9–12]. Friedrich and Nagy [13] have applied these mathematical tools to give the first demonstration of a well-posed initial-boundary value formulation for Einstein’s equations. The Friedrich-Nagy work is of seminal importance for introducing the maximally dissipative technique into general relativity. However, their formulation, which uses an orthonormal tetrad, the connection and the curvature tensor as evolution variables, is quite different from the metric formulations implemented in current numerical codes designed to tackle the BBH problem. Although it is not apparent how to apply the details of the Friedrich-Nagy work to other formalisms, the general principles can be carried over provided Einstein’s equations are formulated in the symmetric hyperbolic form

$$\sum_\alpha A^\alpha(u)\partial_\alpha u = S(u) \quad (1.1)$$

with coordinates $x^\alpha = (t, x, y, z) = (t, x^i)$ and evolution variables $u = (u_1, ..., u_N)$, where $A^\alpha$ are $N \times N$ symmetric matrices and $A^t$ is positive-definite.

The simplest symmetric hyperbolic version of Einstein’s equations is based upon harmonic coordinates [14,15] satisfying $\Gamma^\alpha := -\Box x^\alpha = 0$, in which the first proofs of well-posedness were given for the initial value problem [16,17]. Well-posedness implies the existence of a unique solution with continuous dependence on the data (in a suitable function space). In the nonlinear case, existence is only guaranteed for a short time, reflecting the possibility of singularity formation. Here we describe how this approach (i) can be extended to give a well-posed version of the initial-boundary value problem, (ii) can be implemented as a robustly stable 3-dimensional nonlinear Cauchy evolution code with a cubic grid boundary and (iii) can be accurately matched, in the linearized approximation, to an exterior characteristic evolution code to provide the proper physical boundary condition for computing the waveform radiated to infinity by an isolated source. Although harmonic coordinates are restrictive, their recent use has successfully simulated the approach to a curvature singularity [18] and they can be readily adapted to include gauge source functions $\Gamma^\alpha = f^\alpha(x^\beta)$ [19] as a means of avoiding coordinate singularities. Perhaps more important, the remarkable simplicity of our results suggests that they might generalize to other formulations.

We base the evolution on the metric density $\gamma^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$, with $\gamma = \det(\gamma^{\alpha\beta}) = \det(g_{\alpha\beta}) = g$. Following
and where $S^{\alpha\beta}$ contains no second derivatives of the metric. When the harmonic conditions $\Gamma^\alpha = 0$ are satisfied, Einstein’s equations reduce to $E^{\alpha\beta} = 0$, whose principle part is governed by the nonlinear wave operator $\gamma^{\mu\nu} \partial_\mu \partial_\nu \gamma_{\alpha\beta} + S^{\alpha\beta}$ (1.2)

Here $S^{\alpha\beta}$ is the standard energy flux for the sum of 10 independent scalar fields. The requirement that the reduced system be maximally dissipative requires that $u$ satisfy a homogeneous boundary condition $M u = 0$, where $M$ is a matrix independent of $u$. This requirement can be satisfied in many ways, e.g. by combinations of the homogeneous Dirichlet boundary condition $\partial_\nu \gamma_{\alpha\beta} = T^{\alpha\beta} = 0$, the homogeneous Neumann condition $\partial_\nu \gamma_{\alpha\beta} = Z^{\alpha\beta} = 0$, and the homogeneous Sommerfeld condition $(\partial_\nu + \partial_\tau) \gamma_{\alpha\beta} = T^{\alpha\beta} + Z^{\alpha\beta} = 0$ on the various field components. The inequality (1.3) strengthens the customary requirement that boundary conditions only be applied to variables propagating along characteristics entering the evolution region from the exterior [21]. For instance, assignment of a homogeneous boundary condition to the variable $T^{\alpha\beta} - Z^{\alpha\beta}$, which propagates from the interior toward the boundary, would violate (1.3).

Whereas establishment of the well-posedness of the initial-boundary problem for the reduced equations is easy and can be accomplished in a variety of ways, the extension to Einstein’s equations is more subtle. The Bianchi identities imply [16,17,22]

$$\gamma^{\mu\nu} \partial_\mu \partial_\nu \Gamma^\alpha + A^\mu_{\nu} \partial_\mu \Gamma^\nu = 0, \quad \text{(1.4)}$$

where $A^\mu_{\nu}(u)$ depends algebraically on the evolution variables $u$. Thus $\Gamma^\alpha$ obeys a symmetric hyperbolic equation of the form (1.1). Along with constraints on the initial Cauchy data guaranteeing that $\Gamma$ and $\partial_\Gamma$ vanish on $S$, uniqueness of the solution guarantees that $\Gamma = 0$ in the domain of dependence $D_1$ (see Fig. 1) and hence the well-posedness of the initial value problem for the full Einstein equations. To extend the result to the initial boundary problem, i.e. to include region $D_2$ in Fig. 1, we apply a maximally dissipative boundary condition to $\Gamma^\alpha$ that is consistent with the maximally dissipative boundary data for the reduced problem.

We accomplish this by adopting homogeneous Neumann boundary conditions $Z^{ab} = \partial_z z^{ab} = 0$. Along with the earlier boundary condition that $T^{za} = 0$, this implies that $\Gamma^z$ satisfies the homogeneous Dirichlet boundary condition $\Gamma^z = 0$. However, a homogeneous Dirichlet boundary condition on the tangential components $\Gamma^a$ would be inconsistent with the maximally dissipative property of the reduced system. Instead, we proceed differently. $\Gamma^a$ takes the form

$$\Gamma^a = \partial_a \gamma^{ab} + \partial_\tau \gamma^{az}, \quad \text{(1.5)}$$

We differentiate $\Gamma^a$ to obtain at the boundary

$$\partial_\tau \Gamma^a = \partial_\tau^2 \gamma^{az}, \quad \text{(1.6)}$$

where we have applied the Neumann boundary conditions $Z^{ab} = 0$.

Remarkably, subject to the above boundary conditions, the reduced evolution equation $E^{za} = 0$ reduces to $\partial_\tau^2 \gamma^{az} = 0$ at the boundary! This is the crucial result needed to establish that homogeneous boundary conditions $\Gamma^z = \partial_\tau \Gamma^a = 0$ provide a consistent maximally dissipative boundary condition for the full Einstein equations consisting of the reduced Einstein equations and Eq. (1.4) for the evolution of the harmonic conditions. The remaining ingredient necessary for a well-posed initial-boundary problem is the consistency between the initial Cauchy data and the boundary data. This consistency depends upon the way the Hamiltonian constraint is solved. In the simplest case to analyze, the initial data are prescribed to have local reflection symmetry at the boundary. Then such data are automatically consistent with the homogeneous boundary conditions given above.

In practice, homogeneous boundary conditions do not correspond to a given physical problem, e.g. homogeneous Dirichlet data at the end of a string lead to a fixed endpoint whereas the the endpoint might be undergoing a forced oscillation requiring inhomogeneous boundary data. This flexibility is supplied within the maximally dissipative formalism by the ability to extend the homogeneous boundary condition $Mu = 0$ to the inhomogeneous form $Mu = q(x^a)$ [13]. The proper choice of boundary data $q(x^a)$ for a given physical problem is a separate and difficult problem, especially for nonlinear systems.
One approach is to supply \( q(x^a) \) by matching the interior Cauchy solution to an exterior solution. A possible strategy for the simulation of an isolated source is to evolve the interior region by Cauchy evolution with a cubic outer boundary and match the solution to a characteristic evolution which evolves on a sequence of outgoing null cones extending to infinity (for a review see [24]). In the case of nonlinear scalar waves, such Cauchy-characteristic matching (CCM) is the most accurate and efficient way to apply an outer boundary condition and obtain the waveform radiated to infinity. Previous attempts at CCM in the gravitational case were plagued by instabilities growing on a time scale of 10 to 20 grid crossing times. Tests carried out with a linearized harmonic Cauchy code matched to a linearized characteristic code show no instabilities when the Cauchy boundary is treated in accord with the above methods.

In constructing the code to demonstrate these results, we have taken considerable liberty with the symmetric hyperbolic formalism. In particular, we base, the code on a second differential order form of the differential equations with the 10 variables \( \gamma^{\alpha\beta} \) rather than the 50 first order variables \( u \). The tests reported here were carried out using finite difference techniques described in [23] where robust stability of an analogous linearized harmonic evolution code was demonstrated. All codes were tested to be second order accurate in grid size.

We test for robust stability by initializing the evolution with random, constraint violating initial data \( \gamma^{\alpha\beta} = \eta^{\alpha\beta} + \epsilon^{\alpha\beta} \) and by assigning random boundary data \( q(x^a) = \epsilon \) at each point of the cubic grid boundary, where the \( \epsilon \)'s are random numbers in the range \((10^{-12}, 10^{+12})\). Under these conditions, we have confirmed that the nonlinear code remains stable when run for 2000 crossing times on a \( 21^3 \) grid, and for over 100 crossing times on a \( 50^3 \) grid. In addition we have tested the ability of the nonlinear code to propagate a physical pulse by choosing initial data representing a perturbation of amplitude \( 10^{-7} \) that satisfies the linearized constraints and boundary data corresponding to the exact linearized solution. Figure 2 shows a sequence of profiles of the metric component \( \gamma^{xy} \) propagating cleanly through the grid boundary.

The tests of matching to an exterior characteristic code were carried out using linearized versions of the evolution codes. In these tests the Cauchy boundary data \( q \) is supplied by the exterior characteristic evolution. In turn, data on an interior spherical boundary for the characteristic evolution is supplied by the Cauchy code. Robust stability was tested for 2000 crossing times using a Cauchy grid of \( 45^3 \). Again choosing initial data for a linearized wave pulse, Figure 3 shows a sequence of profiles of the metric component \( \gamma^{xy} \) propagating cleanly through the spherical boundary as they pass to the characteristic grid, where they are propagated to infinity. Further details of the numerical implementation and tests will be reported elsewhere.

We thank H. Friedrich and B. Schmidt for educating us in the intricacy and subtlety of the initial-boundary problem. The codes were parallelized using Cactus with help from the Cactus development team of the AEI. The work was supported by NSF grant PHY 9988663.

FIG. 1. Schematic representation of the domain of dependence $D_1$ of the initial value problem and the domain of dependence $D_1 \cup D_2$ of the initial-boundary value problem.

FIG. 2. Sequence of $z = 0$ slices of the metric component $\gamma^{xy}$, evolved for one crossing time, with the non-linear Cauchy code. Negligible back-reflection from the Cauchy boundary is evident from the last snapshot.

FIG. 3. Sequence of $z = 0$ slices of the metric component $\gamma^{xy}$, evolved for one crossing time, with the linear matched Cauchy-characteristic code. In the last snapshot, the wave has propagated cleanly onto the characteristic grid.