I. INTRODUCTION

ensemble is stable only for a part of the phase space. Black string in the canonical ensemble is locally stable, while in the grand-canonical ensemble the string is unstable. We find the thermodynamics of the asymptotically AdS black strings and perform a stability analysis both in the asymptotically AdS and fixed electric potential cases. We also study the thermodynamics of the conformal solution of the Einstein-Axion Maxwell equations with a positive cosmological constant. The grand-canonical ensemble both in the canonical and the grand-canonical ensemble. The grand-canonical ensemble has a thermal interpretation, and the Euclidean actions of the charged rotating black strings were used to calculate the thermodynamic of rotating charged black strings.
example of the AdS/CFT correspondence is the interpretation of the Hawking-Page phase transition from low-temperature confining to a high-temperature deconfining phase in the dual field theory [2].

The AdS/CFT correspondence is now a fundamental concept which furnish a means for calculating the action and thermodynamical quantities intrinsically without reliance on any reference spacetime [3, 4, 5, 6]. This conjecture has been recently extended to the case of asymptotically de Sitter spacetimes [7, 8]. Although the (A)dS/CFT correspondence applies for the case of spacially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [9]. This conjecture has also been used for the case of black objects with nonspherical horizons [10].

It is well known that the Einstein equation with positive or negative cosmological constant has black hole solutions with horizons being positive, zero, or negative constant curvature hypersurfaces [11]. The rotating solutions of the Einstein equation with a negative cosmological constant with cylindrical and toroidal horizons have been studied in Ref. [12]. The extension to include the Maxwell field has been done and the static and rotating electrically charged black string have been considered in Ref. [13]. Recently, asymptotically anti-de Sitter spacetimes generated by static and spinning magnetic string sources in general relativity have been also considered [14]. The thermodynamics of asymptotically anti-de Sitter spacetimes with nonspherical horizon has been studied by many authors [5, 15]. In this paper, we want to apply the AdS/CFT correspondence to the asymptotically anti-de Sitter charged rotating black string in four dimensions with cylindrical or toroidal event horizons, and study their thermodynamics in both the canonical and the grand-canonical ensemble. The stability conditions are investigated and a complete phase diagram is obtained. We also introduce the asymptotic de Sitter charged rotating black strings with cylindrical and toroidal horizon and calculate their conserved quantities and Euclidean actions in both the canonical and the grand-canonical ensemble.

The outline of our paper is as follows. We review the basic formalism in Sec. II. In Sec. III we consider the four-dimensional charged rotating black strings which are asymptotically anti-de Sitter and introduce the asymptotically de Sitter charged rotating strings. We also compute the conserved charges and the Euclidean actions for asymptotically AdS and dS strings. In Sec. IV, we study the thermodynamics of the string in both the canonical and the grand-canonical ensemble, and the thermal stability of the black strings is investigated.
We finish our paper with some concluding remarks.

II. THE ACTION AND CONSERVED QUANTITIES

The gravitational action of four-dimensional asymptotically (anti)-de Sitter spacetimes $\mathcal{M}$, with boundary $\delta \mathcal{M}$ in the presence of an electromagnetic field is

$$I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g}(\mathcal{R} - 2\Lambda - F_{\mu\nu}F^{\mu\nu}) + \frac{1}{8\pi} \int_{\partial \mathcal{M}} d^3x \sqrt{-\gamma} \Theta(\gamma),$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the electromagnetic tensor field and $A_{\mu}$ is the vector potential. The first term is the Einstein-Hilbert volume term with negative (AdS) or positive (dS) cosmological constant $\Lambda = \pm 3/l^2$, and the second term is the Gibbons-Hawking boundary term which is chosen such that the variational principle is well-defined. The manifold $\mathcal{M}$ has metric $g_{\mu\nu}$ and covariant derivative $\nabla_{\mu}$. $\Theta$ is the trace of the extrinsic curvature $\Theta^{\mu\nu}$ of any boundary(ies) $\partial \mathcal{M}$ of the manifold $\mathcal{M}$, with induced metric(s) $\gamma_{ij}$. In general the first and second terms of Eq. (1) are both divergent when evaluated on the solutions, as is the Hamiltonian, and other associated conserved quantities. Rather than eliminating these divergences by incorporating a reference term in the spacetime [16, 17], a new term, $I_{ct}$, is added to the action which is a functional only of the boundary curvature invariants. The counterterms for asymptotically AdS and dS spacetimes in four dimensions are [3, 8]:

$$I_{ct} = -\frac{1}{4\pi l} \int_{\partial \mathcal{M}_{\infty}} d^3x \sqrt{-\gamma} \left( 1 \pm \frac{\ell}{4R} \right),$$

where $R$ is the Ricci scalar of the boundary metric $\gamma_{ab}$. The $+$ and $-$ sign correspond to asymptotically AdS and dS spacetimes, respectively. These counterterms have been used by many authors for a wide variety of the spacetimes, including Schwarzschild-(A)dS, topological Schwarzschild-(A)dS, Kerr-(A)dS, Taub-NUT (Newmann-Uni-Tamborino)-AdS, Taub-bolt-AdS, and Taub-bolt-Kerr-AdS [5, 6, 8]. The total action can be written as a linear combination of the gravity term (1) and the counterterms (2) as

$$I = I_G + I_{ct}.$$ 

In order to obtain the Einstein-Maxwell field equations by the variation of the volume integral with respect to the fields, one should impose the boundary condition $\delta A_{\mu} = 0$ on $\delta \mathcal{M}$. Thus the action (3) is appropriate to study the grand-canonical ensemble with fixed electric potential [18].
To study the canonical ensemble with fixed electric charge, one should impose the boundary condition \( \delta(n^a F_{ab}) = 0 \), and therefore the total action is [20]

\[
\tilde{I} = I - \frac{1}{4\pi} \int_{\partial M} d^3x \sqrt{-\gamma} n_a F^{ab} A_b. \tag{4}
\]

Using the Brown and York [16] definition one can construct a divergence-free stress-energy tensor from the total action (3) as

\[
T^{ab} = \frac{1}{8\pi} \left\{ (\Theta^{ab} - \Theta \gamma^{ab}) - \frac{2}{7} \left[ \gamma^{ab} \pm \frac{l^2}{2} \left( R^{ab} - \frac{1}{2} R \gamma^{ab} \right) \right] \right\}, \tag{5}
\]

Again the + and − sign correspond to asymptotically AdS and dS spacetimes, respectively.

To compute the conserved charges of the spacetime, one should choose a spacelike surface \( \mathcal{B} \) in \( \partial M \) with metric \( \sigma_{ij} \), and write the boundary metric in ADM form:

\[
\gamma_{ab} dx^a dx^b = -N^2 dt^2 + \sigma_{ij} \left( d\phi^i + V^i dt \right) \left( d\phi^j + V^j dt \right),
\]

where the coordinates \( \phi^i \) are the angular variables parametrizing the hypersurface of constant \( r \) around the origin. When there is a Killing vector field \( \xi^a \) on the boundary, then the conserved quantities associated with the stress tensors of Eq. (5) can be written as

\[
Q(\xi) = \int_{\mathcal{B}_\infty} d^{n-1}\phi \sqrt{\sigma} T_{ab} n^a \xi^b, \tag{6}
\]

where \( \sigma \) is the determinant of the metric \( \sigma_{ij} \), \( n^a \) are the Killing vector field and the unit normal vector on the boundary \( \mathcal{B} \). For boundaries with timelike \( (\xi = \partial/\partial t) \) and rotational Killing vector fields \( (\zeta = \partial/\partial \phi) \), we obtain

\[
M = \int_{\mathcal{B}_\infty} d^2\phi \sqrt{\sigma} T_{ab} n^a \xi^b, \tag{7}
\]

\[
J = \int_{\mathcal{B}_\infty} d^{n-1}\phi \sqrt{\sigma} T_{ab} n^a \zeta^b, \tag{8}
\]

provided the surface \( \mathcal{B} \) contains the orbits of \( \zeta \). These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary. Note that they will both be dependent on the location of the boundary \( \mathcal{B} \) in the spacetime, although each is independent of the particular choice of foliation \( \mathcal{B} \) within the surface \( \partial M \).

In the context of the (A)dS/CFT correspondence, the limit in which the boundary \( \mathcal{B} \) becomes infinite \( (\mathcal{B}_\infty) \) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface \( \mathcal{B} \) into a reference spacetime is required and the quantities which are computed are intrinsic to the spacetimes.
III. ROTATING CHARGED BLACK STRINGS IN FOUR DIMENSIONS

In this section we consider the solutions of the Einstein-Maxwell equations with negative and positive cosmological constants which possess cylindrical symmetry. We see that for a suitable choice of the parameters, these solutions describe stationary charged rotating black strings with cylindrical or toroidal horizons.

A. Asymptotically anti-de Sitter case

The asymptotically AdS solution of the Einstein-Maxwell equations with cylindrical symmetry can be written as [13]

$$
ds^2 = -\Xi^2 \left( f(r) - \frac{a^2 r^2}{\Xi^2 l^4} \right) dt^2 + \frac{1}{f(r)} dr^2 - 2 \frac{a}{r} \frac{\Xi l}{r} \left( b - \frac{l^2}{r^2} \phi \right) dt d\phi \\
+ \left[ \Xi^2 r^2 - a^2 f(r) \right] d\phi^2 + \frac{r^2}{l^2} dz^2, \\
A_\mu = -\Xi \frac{l}{r} (\delta_\mu^0 - \frac{a}{\Xi} \delta_\mu^2),
$$

where

$$
f(r) = \frac{r^2}{l^2} - \frac{b l}{r} + \frac{\lambda^2 l^2}{r^2}, \\
\Xi^2 = 1 + \frac{a^2}{l^2}.
$$

$a$, $b$, and $\lambda$ are the constant parameters of the metric. It is worthwhile to mention that for the case of $-\infty < z < \infty$, Eqs. (9)-(11) describe a stationary black string with cylindrical horizon, and if one compactifies the $z$ coordinate ($0 \leq z < 2\pi l$) one has a closed black string (or black hole) with toroidal horizon. As we will show in the next section the angular momentum is proportional to the parameter $a$, and therefore $a$ is the rotational parameter of the spacetime. It is easy to show that in the nonrotating case ($a = 0$), $\lambda/2$ and $b/4$ are the linear charge and mass densities of the $z$-line as we will see in the next section.

The metric of Eqs. (9)-(11) has two inner and outer horizons located at $r_-$ and $r_+$, provided the parameter $b$ is greater than $b_{\text{crit}}$ given as

$$
b_{\text{crit}} = 4 \times 3^{-3/4} \lambda^{3/2}.
$$

In the case that $b = b_{\text{crit}}$, we will have an extreme black string. The horizon area per unit length of the string for the case of a cylindrical horizon is $2\pi \Xi r_+^2 / l$. Since the area law of
entropy is universal, and applies to all kinds of black holes and black strings [6, 19, 20, 21],
the entropy per unit length is
\[
S = \frac{\pi^2 r_+^4}{2l}.
\]  
(13)
For the case of a toroidal horizon, the horizon area of the string is $2\pi l S$.

Analytical continuation of the Lorentzian metric by $t \rightarrow i\tau$ and $a \rightarrow ia$ yields the
Euclidean section, whose regularity at $r = r_+$ requires that we should identify $\tau \sim \tau + \beta_+$
and $\phi \sim \phi + i\beta_+ \Omega_+$, where $\beta_+$ and $\Omega_+$ are the inverse Hawking temperature and the angular
velocity of the outer event horizon. It is a matter of calculation to show that
\[
\beta_+ = \frac{4\pi \Xi}{f'(r_+)} = \frac{4\pi \Xi F_1 r_+^3}{3r_+^4 - \lambda^2 l^4},
\]
(14)
\[
\Omega_+ = \frac{a}{\Xi l^2}.
\]
(15)
Using Eqs. (1)-(4) the Euclidean action per unit length of the black string with a cylin-
drical horizon in the grand-canonical and the canonical ensemble can be calculated as
\[
\mathcal{I} = -\frac{\beta_+ b_s}{8},
\]
(16)
\[
\overline{\mathcal{I}} = \frac{\beta_+}{8} \left(3b - 4\frac{r_+^3}{f^3}\right),
\]
(17)
valid for a fixed potential and a fixed charge, respectively. For the case of toroidal black
strings the total action is $2\pi l \mathcal{I}$ and $2\pi l \overline{\mathcal{I}}$, respectively.

The conserved mass and angular momentum per unit length of the string with a cylin-
derical horizon calculated on the boundary $\mathcal{B}$ at infinity can be calculated through the use
of Eqs. (7) and (8),
\[
\mathcal{M} = \frac{1}{8}(3\Xi^2 - 1)b_s, \quad \mathcal{J} = \frac{3}{8} \Xi b a.
\]
(18)
For $a = 0 \ (\Xi = 1)$, the angular momentum and mass per unit length are 0 and $b/4$, respectively. Thus $a$ is the rotational parameter and $b/4$ is associated to the mass density
per unit length. The total mass and angular momentum of the string with toroidal horizon
are $2\pi l \mathcal{M}$ and $2\pi l \mathcal{J}$, respectively.

The charge per unit length, $\mathcal{Q}$, can be found by calculating the flux of the electromagnetic
field at infinity, yielding
\[
\mathcal{Q} = \frac{\Xi \lambda}{2}.
\]
(19)
The electric potential $\Phi$, measured at infinity with respect to the horizon, is defined by [18]
\[
\Phi = A_\mu \chi^\mu \big|_{r \to \infty} - A_\mu \chi^\mu \big|_{r = r_+},
\]
(20)
where $\chi = \partial_t + \Omega_4 \partial_\phi$ is the null generator of the horizon. One finds

$$\Phi = \frac{\lambda l}{\Xi r_+}. \tag{21}$$

B. Asymptotically de Sitter case

The solution of the Einstein-Maxwell equations with a positive cosmological constant which has cylindrical symmetry can be written as

$$ds^2 = -\Gamma^2 \left( h(r) - \frac{a^2 r^4}{\Gamma^2 l^4} \right) dt^2 + \frac{1}{h(r)} dr^2 - 2 \frac{a \Gamma l}{r} \left( b - \frac{l}{r} \lambda^2 \right) dt d\phi$$

$$+ \left[ \Gamma^2 r^2 - a^2 h(r) \right] d\phi^2 + \frac{r^2}{l^2} dz^2, \tag{22}$$

$$A_\mu = -i \Gamma \frac{l \lambda}{r} \left( \delta_\mu^0 + \frac{a}{\Gamma} \delta_\mu^2 \right), \tag{23}$$

where

$$h(r) = - \left( \frac{r^2}{l^2} - \frac{bl}{r} + \frac{\lambda^2 l^2}{r^2} \right),$$

$$\Gamma^2 = 1 - \frac{a^2}{l^2}. \tag{24}$$

Again $a$, $b$, and $\lambda$ are the constant parameters of the metric. For a suitable choice of the parameters $a$, $b$, and $\lambda$, the solution (22)-(24) describes a stationary black string with cylindrical or toroidal horizons. In the case of $-\infty < z < \infty$, one has a black string with a cylindrical horizon, and if one compactifies the $z$ coordinate ($0 \leq z < 2\pi l$) one has a closed black string (or black hole) with a toroidal horizon. Again, as in the case of asymptotically anti-de Sitter spacetimes, $a$, $b/4$, and $\lambda/2$ can be interpreted as the rotational parameter, the mass, and charge densities of the $z$-line.

The metric of Eqs. (22)-(24) has two outer and cosmological horizons located at $r_+$ and $r_c$ provided the parameter $b$ is greater than $b_{crit}$ given by Eq. (12). In the case in which $b = b_{crit}$, we will have an extreme black string. The area law of the entropy also applies to the cosmological event horizon of the asymptotic de Sitter black holes [22], and therefore the entropy of the string for the case of cylindrical and toroidal horizons is $S_{ds} = \pi \Gamma r_+^3/(2l)$ and $2\pi l S_{ds}$, respectively.

Again the inverse Hawking temperature and the angular velocity of the cosmological event horizon can be calculated as

$$\beta_c = \frac{4\pi \Gamma}{h'(r_c)} = - \frac{4\pi \Gamma l^2 r_c^3}{3r_c^4 - \lambda^2 l^4}, \tag{25}$$
\[ \Omega_c = \frac{a}{\Gamma^2}. \]  

Using Eqs. (1)-(4) for the de Sitter case, the Euclidean action per unit length of the black string with a cylindrical horizon in the grand-canonical and the canonical ensemble can be calculated as

\[
\mathcal{I}_{ds} = \frac{\beta_\omega}{8} b, \\
\overline{\mathcal{I}}_{ds} = -\frac{\beta_\omega}{8} (3b - 4\gamma \beta^3).
\]

For the case of a toroidal black string the total action in the grand-canonical and the canonical ensemble is \(2\pi l \mathcal{I}_{ds}\) and \(2\pi l \overline{\mathcal{I}}_{ds}\).

The conserved mass and angular momentum per unit length of the string with a cylindrical horizon calculated on the boundary \(\mathcal{B}\) at infinity can be calculated through the use of Eqs. (7) and (8),

\[
\mathcal{M}_{ds} = -\frac{1}{8} (3\Gamma^2 - 1)b, \quad \mathcal{J}_{ds} = \frac{3}{8} \Gamma ba. 
\]

For \(a = 0\), the angular momentum and mass per unit length are 0 and \(b/4\), respectively. Thus the parameters \(a\) and \(b/4\) are associated to the angular momentum and mass density per unit length. The total mass and angular momentum of the string with toroidal horizon are \(2\pi l \mathcal{M}_{ds}\) and \(2\pi l \mathcal{J}_{ds}\), respectively.

Here the charge per unit length is \(Q_{ds} = \Gamma \lambda / 2\), and the electric potential \(\Phi\) can be calculated as

\[
\Phi_{ds} = \frac{\lambda l}{\Gamma r_c}. 
\]

### IV. THERMODYNAMICS OF BLACK STRINGS

#### A. Energy as a function of entropy, angular momentum, and charge

We first obtain the mass per unit length as a function of \(S\), \(\mathcal{J}\), and \(Q\). Using the expression (18) for the mass and angular momentum per unit length, Eq. (13) for the entropy, and the fact that \(f(r_+) = 0\), one obtains by simple algebraic manipulation

\[
\mathcal{M} = (8\pi^3 b^3 S)^{-1/2} \left\{ \mathcal{Y} - 2T \left( \frac{S^2 + l^2 \pi^2 Q^2}{\mathcal{Y}} \right)^4 \right\}, \\
\mathcal{Y} = 4\pi^3 lS \mathcal{J}^2 + \sqrt{16\pi^6 l^2 S^2 \mathcal{J}^4 + 81(S^2 + l^2 \pi^2 Q^2)^4}. 
\]
One may then regard the parameters $\mathcal{S}$, $\mathcal{J}$, and $\mathcal{Q}$ as a complete set of energetic extensive parameters for the mass per unit length $\mathcal{M} = \mathcal{M}(\mathcal{S}, \mathcal{J}, \mathcal{Q})$ and define the quantities conjugate to $\mathcal{S}$, $\mathcal{J}$, and $\mathcal{Q}$. These quantities are the temperature, the angular velocity, and the electric potential

$$T = \left( \frac{\partial \mathcal{M}}{\partial \mathcal{S}} \right)_{\mathcal{J}\mathcal{Q}}, \quad \Omega = \left( \frac{\partial \mathcal{M}}{\partial \mathcal{J}} \right)_{\mathcal{S}\mathcal{Q}}, \quad \Phi = \left( \frac{\partial \mathcal{M}}{\partial \mathcal{Q}} \right)_{\mathcal{J}\mathcal{S}}. \quad (32)$$

It is a matter of straightforward calculation to show that the quantities calculated by Eq. (32) for the temperature, the angular velocity, and the electric potential coincide with Eqs. (14), (15), and (21) found in Sec. III. Thus, the thermodynamical quantities calculated in Sec. III satisfy the first law of thermodynamics,

$$d\mathcal{M} = T d\mathcal{S} + \Omega d\mathcal{J} + \Phi d\mathcal{Q}. \quad (33)$$

B. Thermodynamic potentials

We now consider the thermodynamic potentials in the grand-canonical and the canonical ensemble. For the grand-canonical ensemble using the definition of the Gibbs potential $G(T, \Omega, \Phi) = \mathcal{M}/\beta$, and the expression (17) for the action, Eq. (14) for the inverse Hawking temperature, Eq. (15) for the angular velocity, and Eq. (21) for the electric potential of asymptotically AdS spacetime, we obtain

$$G(T, \Omega, \Phi) = \mathcal{M} - \Omega \mathcal{J} - T \mathcal{S} - \Phi \mathcal{Q}, \quad (34)$$

which means that $G(T, \Omega, \Phi)$ is indeed the Legendre transformation of the energy $\mathcal{M}(\mathcal{S}, \mathcal{J}, \mathcal{Q})$ with respect to $\mathcal{S}$, $\mathcal{J}$, and $\mathcal{Q}$. It is a matter of straightforward calculations to show that the extensive quantities

$$\mathcal{S} = -\left( \frac{\partial G}{\partial T} \right)_{\Omega\Phi}, \quad \mathcal{J} = -\left( \frac{\partial G}{\partial \Omega} \right)_{T\Phi}, \quad \mathcal{Q} = -\left( \frac{\partial G}{\partial \Phi} \right)_{T\Omega},$$

turn out to coincide precisely with the expressions (13), (18), and (19).

For the canonical ensemble, the Helmholtz free energy $F(T, \mathcal{J}, \mathcal{Q})$ is defined as

$$F(T, \mathcal{J}, \mathcal{Q}) = \frac{T}{\beta} + \Omega \mathcal{J}, \quad (35)$$

where $\bar{T}$ is given by Eq. (17). One can verify that the conjugate quantities

$$\mathcal{S} = -\left( \frac{\partial F}{\partial T} \right)_{\mathcal{J}\mathcal{Q}}, \quad \Omega = \left( \frac{\partial F}{\partial \mathcal{J}} \right)_{T\mathcal{Q}}, \quad \Phi = \left( \frac{\partial F}{\partial \mathcal{Q}} \right)_{T\mathcal{J}},$$
agree with expressions (13), (15), and (21) and one has also

$$F(T, \mathcal{J}, \mathcal{Q}) = \mathcal{M} - T\mathcal{S}. \quad (36)$$

Thus, $F$ is the Legendre transform of $\mathcal{M}(\mathcal{S}, \mathcal{J}, \mathcal{Q})$ with respect to $\mathcal{S}$.

C. Stability in the canonical and the grand-canonical ensemble

The local stability analysis in any ensemble can in principle be carried out by finding the determinant of the Hessian matrix $[\partial^2 S / \partial X_i \partial X_j]$, where $X_i$'s are the thermodynamic variables of the system [23]. In our case the entropy $S$ is a function of the mass, the angular momentum, and the charge per unit length. But, the number of the thermodynamic variables depends on the ensemble which is used. The more $X_i$ we regard as variable parameters, the smaller is the region of stability. In the canonical ensemble, the charge and angular momentum are fixed parameters, and for this reason the positivity of the thermal capacity $C_{\mathcal{J}, \mathcal{Q}}$ is sufficient to assure the local stability. The thermal capacity $C_{\mathcal{J}, \mathcal{Q}}$ at constant charge and angular momentum is

$$C_{\mathcal{J}, \mathcal{Q}} = T \frac{\partial \mathcal{S}}{\partial T}, \quad (37)$$

where $T$ is the inverse of $\beta_+$ given by Eq. (14) and $\partial \mathcal{S} / \partial T$ can be calculated as

$$\frac{\partial \mathcal{S}}{\partial T} = \frac{2\pi^2 \Xi^2 r_+^4 (\Xi^2 + 1)(r_+^4 + q^2 l^4)}{\beta[4r_+^8(\Xi^2 - 1)^2 + \Xi^2(q^2 l^4 - r_+^4)^2 + 4q^2 r_+^4 l^4]}. \quad (38)$$

As one can see from Eq. (36), $\partial \mathcal{S} / \partial T$ is positive for all the allowed values of the metric parameters discussed in Sec. III, and therefore the asymptotically AdS charged rotating black string in the canonical ensemble is locally stable.

In the grand-canonical ensemble, the thermodynamic variables are the mass, the charge and the angular momentum per unit length. Direct computation of the elements of the Hessian matrix of $S(\mathcal{M}, \mathcal{J}, \mathcal{Q})$ with respect to $\mathcal{M}$, $\mathcal{J}$, and $\mathcal{Q}$ is a burdensome task. We find it more efficient to work with the thermodynamic potential, $G(T, \Omega, \Phi)$. It is a matter of calculation to show that the black string is locally stable in the grand-canonical ensemble if:

$$(\Xi^2 - 2)(3r_+^4 - q^2 l^4) - 8(\Xi^2 + 1)q^2 l^4 r_+^4 \geq 0. \quad (39)$$

As one may note the condition (39) is not satisfied for $\Xi \leq \sqrt{2}$ [or using Eq. (11) for $a \leq l$]. Thus, the black string is not locally stable for $a \leq l$. For $\Xi > \sqrt{2}$, we consider the stability
condition for a fixed potential in the \((\Xi, r_+)\) plane. The stability condition (39) in terms of the electric potential can be written as

\[
9(\Xi^2 - 2)r_+^4 - 2l^4\Xi^4(7\Xi^2 - 2)r_+^2 + l^4\Phi^4\Xi^4((\Xi^2 - 2)^2) \geq 0. \tag{40}
\]

The condition (40) is satisfied if the radius of the horizon \(r_+\) is less than \(r_{+1}\) or greater than \(r_{+2}\), where

\[
r_{+1} = \frac{l\Xi\Phi}{3\sqrt{\Xi^2 - 2}}\left[7\Xi^2 - 2 + 2\sqrt{10\Xi^4 + 2\Xi^2 - 8}\right]^{1/2}, \tag{41}
\]

\[
r_{+2} = \frac{l\Xi\Phi}{3\sqrt{\Xi^2 - 2}}\left[7\Xi^2 - 2 - 2\sqrt{10\Xi^4 + 2\Xi^2 - 8}\right]^{1/2}. \tag{42}
\]

Figures 1 and 2 show the phase diagrams in the \((\Xi, r_+)\) plane. The black string is locally stable in the regions [I] and [II] of Figs 1 and 2.

V. CLOSING REMARKS

In this paper, we used the AdS/CFT correspondence to calculate the conserved quantities and the Euclidean actions for the two cases of fixed charge and fixed electric potential of asymptotically anti-de Sitter rotating charged black string. The asymptotically de Sitter charged rotating black string was introduced and the conserved quantities and the Euclidean actions were calculated. We obtained the mass per unit length as a function of the extensive
parameters $\mathcal{S}$, $\mathcal{J}$, and $\mathcal{Q}$, calculated the temperature, the angular velocity, and the electric potential, and showed that these quantities satisfy the first law of thermodynamics. Using the conserved quantities and the Euclidean actions, the thermodynamic potentials of the system in the canonical and the grand-canonical ensemble were calculated. We found that the Helmholtz free energy, $F(T, \mathcal{J}, \mathcal{Q})$, is a Legendre transformation of the mass per unit length $\mathcal{M}(\mathcal{S}, \mathcal{J}, \mathcal{Q})$ with respect to $\mathcal{S}$, and in the grand-canonical ensemble, the Gibbs potential is a Legendre transformation of the mass per unit length with respect to the extensive parameters $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{Q}$.

The local stability of the asymptotically charged rotating black string in both the canonical ensemble and the grand-canonical ensemble was investigated through the use of the Hessian matrix of the entropy with respect to its thermodynamic variables. We showed that for the canonical ensemble, where the only thermodynamic variable was the mass per unit length, the string is locally stable for the whole phase space. In the grand-canonical ensemble, where the thermodynamic variables were $\mathcal{S}$, $\mathcal{J}$, and $\mathcal{Q}$, we showed that the region for which the string is stable is smaller than for the canonical ensemble. We found that the black string is unstable for $\Xi \leq \sqrt{2}$ ($a \leq l$), and for $a > l$ there exist two locally stable phases. Indeed we studied the thermodynamics of the string for a fixed electric potential in the $(\Xi, r_+)$ plane and found that the string is locally stable for $r_+ < r_{+1}$ and $r > r_{+2}$, where $r_{+1}$ and $r_{+2}$ are given by Eqs. (41) and (42).
The thermodynamics of the asymptotically de Sitter case may be investigated by the conserved quantities and the Euclidean actions obtained in Sec. III B. But, because of the cosmological horizon, a more efficient way of its consideration is through the use of quasilocal thermodynamics, which we give elsewhere.


