Extended multiplet structure in Logarithmic Conformal Field Theories

A. Nichols\textsuperscript{1}

\textit{Theoretical Physics, Department of Physics, Oxford University}
\textit{1 Keble Road, Oxford, OX1 3NP, UK}

\textbf{Abstract}

We use the process of quantum hamiltonian reduction of $SU(2)_k$, at rational level $k$, to study explicitly the correlators of the $h_{1,s}$ fields in the $c_{p,q}$ models. We find extra, chiral and non-chiral, multiplet structure in a subset of the operators beyond the ‘minimal’ sector. At the level of the vacuum null vector $h_{1,2p-1} = (p - 1)(q - 1)$ we find that there are also two extra non-chiral fermionic fields. This extra indicial structure permeates the entire theory. We have a chiral triplet of fields at $h_{1,4p-1} = (2p - 1)(2q - 1)$. We conjecture that the triplet algebra may produce a quasi-rational extended $c_{p,q}$ model. If $p$ and $q$ are not both odd then we also find chiral fermionic doublets at $h_{1,3p-1} = (\frac{3p}{2} - 1)(\frac{3q}{2} - 1)$.

\textsuperscript{1}a.nichols1@physics.ox.ac.uk
1 Introduction

The study of conformal invariance in two dimensions has been a fascinating and productive area of research for the last twenty years [1]. There is an interesting class of conformal field theories (CFTs) called logarithmic conformal field theories (LCFTs). In these theories the irreducible primary operators do not close under fusion and indecomposable representations are inevitably generated [2]. The operators in the theory have scaling dimensions that are either degenerate or differ by integers. In these cases it is possible to have a non-trivial Jordan block structure.

LCFTs have emerged in many different areas for example: WZNW models and gravitational dressing [3–11], polymers [12–14], disordered systems and the Quantum Hall effect [15–30], string theory [31–42], 2d turbulence [43–47], multi-colour QCD at low-x [48], the Abelian sandpile model [49, 50] and the Seiberg-Witten solution of $\mathcal{N} = 2$ SUSY Yang-Mills [51, 52]. Deformed LCFTs, Renormalisation group flows and the $c$-theorem were discussed in [15, 53, 54]. The holographic relation between logarithmic operators and vacuum instability was considered in [55, 56]. There has also been much interest on LCFTs with a boundary [57–60]. For more about the general structure of LCFT see [61–64] and references therein. Introductory lecture notes on LCFT and more references can be found in [65–68]. A general approach to LCFT via deformations of the operators has been given in [69].

There has also been much work on analysing the general structure and consistency of such models in particular the $c_{p,1}$ models and the special case of $c_{2,1} = −2$ which is by far the best understood [70–73]. The key to this understanding is the fact that one may extend to Virasoro algebra by triplets of chiral $h_{3,1} = 2p − 1$ fields [74]. The resulting algebra is sufficient to create a quasi-rational LCFT, i.e. one having only a finite number of irreducible and indecomposable representations [75]. The aim of this paper is to show that this extended algebraic structure generalises in a simple way to all $c_{p,q}$ models. We shall leave the question of rationality for future work.

The WZNW model is of great importance in CFT. Correlation functions in such models obey differential, Knizhnik-Zamolodchikov (KZ), equations [76] coming from null states in the theory. The solutions to the KZ equations and correlation functions for the integrable sector of the $SU(2)_k$ model were studied by [77, 78]. In the case of the integrable representations these were previously studied in [79, 80] in which it was found that the rational solutions to the KZ equation were in one-one correspondence with the extensions of the chiral algebra. There is a simple Dotsenko-Fateev integral representation for solutions but these do not converge in many cases beyond the integrable representations. In particular in the cases in which logarithms appear we have to be very careful when analytically continuing the solutions and it is much easier, and more convincing, to solve the equations directly. We shall make use of quantum hamiltonian reduction of $SU(2)_k$ WZNW models, at rational level $k$, which gives a very efficient procedure to directly calculate differential equations for the $h_{1,s}$ fields in the $c_{p,q}$ models. By examining several examples we shall show that there is a very simple,
and elegant, structure for a certain subset of $h_{1,s}$ operators.

We find that there is a single rational solution generated by the $h_{1,2p-1} = (p-1)(q-1)$ field corresponding to the vacuum null vector of the irreducible theory. It is well known that decoupling such a null vector gives us a complete description of the ‘minimal’ $c_{p,q}$ model [81]. However at this conformal weight we find, in addition, two other primary fermionic non-chiral operators. This fermionic structure permeates the model.

We found that there are triplets of chiral bosonic fields at $h_{1,4p-1} = (2p-1)(2q-1)$. These are a natural generalisation of an algebra, generated by the $h_{1,3} = 2q - 1$ fields, that appears in the $c_{1,q}$ models. It has been previously conjectured by M. Flohr [75] that these extended $c_{p,q}$ models should be formally considered as $c_{3p,3q}$ and we conjecture that the algebra of such $h_{1,4p-1}$ fields may yield quasi-rational extended $c_{p,q}$ models. We also observed extra chiral fermionic structure in the $c_{p,q}$ model if $p, q$ were not both odd generated by the $h_{1,3p-1} = ((3p^2 - 1)(3q^2 - 1)$ fields.

Recently a particular $SU(2)_k$ theory at rational level, namely $k = -\frac{4}{3}$, was studied [8]. It was found that indecomposable representations were created in the fusion of admissible representations and that the theory was not rational. On hamiltonian reduction the discrete representations of $SU(2)$ with $2j \in \mathbb{Z}$, which are different to the admissible representations, produce $h_{1,s}$ fields in the $c_{2,3} = 0$ model. It would be interesting to see if the type of extended algebras studied in this paper could be used to construct quasi-rational models of $\widehat{SU}(2)$ at fractional level.

2 Knizhnik-Zamolodchikov equation

We consider the $\widehat{SU}(2)$ theory at rational level $k$. The OPE of the affine Kac-Moody currents is given by:

$$J^3(z)J^\pm(w) \sim \pm \frac{J^\pm(w)}{z-w}$$

$$J^+(z)J^-(w) \sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}$$

$$J^3(z)J^3(w) \sim \frac{k}{2(z-w)^2}$$

We use the standard Sugawara construction for the stress tensor:

$$T = \frac{1}{k+2} \left( \frac{1}{2} J^+ J^- + \frac{1}{2} J^- J^+ + J^3 J^3 \right)$$

which yields a theory with central charge:

$$c = \frac{3k}{k+2}$$
We consider affine Kac-Moody primary operators having the simple behaviour:

\[ J^a(z)\phi_j(w) \sim \frac{t^a_j \phi_j(w)}{z - w} \]  

(4)

where \( t^a \) is a spin \( j \) matrix representation of \( SU(2) \). We also have affine Virasoro null vectors following from (2):

\[ |\chi\rangle = (L_{-1} - \frac{1}{k+2} \eta_{ab}J^a_{-1}J^b_0)|\phi\rangle \]  

(5)

Inserting these null vectors into correlation functions of affine Kac-Moody primaries one can show that they must satisfy a set of partial differential equations known as Knizhnik-Zamolodchikov (KZ) equations [76]:

\[
[k+2]\frac{\partial}{\partial z_i} + \sum_{j \neq i} \eta_{ab} \big( t^a_i \otimes t^b_j \big) \frac{1}{z_i - z_j} \langle \phi_{j_1}(z_1) \cdots \phi_{j_n}(z_n) \rangle = 0
\]  

(6)

2.1 Auxiliary variables

It will be convenient to introduce the following representation for the \( SU(2) \) generators [77]:

\[ J^+ = x^2 \frac{\partial}{\partial x} - 2jx, \quad J^- = -\frac{\partial}{\partial x}, \quad J^3 = x \frac{\partial}{\partial x} - j \]  

(7)

There is also a similar algebra in terms of \( \bar{x} \) for the antiholomorphic part. It is easily verified that these obey the global \( SU(2) \) algebra.

We introduce primary fields, \( \phi_j(x, z) \) of the affine Lie algebra. Then:

\[ J^+(z)\phi_j(x, w) \sim \frac{x^2 \frac{\partial}{\partial x} - 2jx}{z - w} \phi_j(x, w) \]  

\[ J^-(z)\phi_j(x, w) \sim -\frac{\partial}{\partial x} \phi_j(x, w) \]  

\[ J^3(z)\phi_j(x, w) \sim \frac{x \frac{\partial}{\partial x} - j}{z - w} \phi_j(x, w) \]  

(8)

The fields \( \phi_j(x, z) \) are also primary with respect to the Virasoro algebra with \( L_0 \) eigenvalue:

\[ h = \frac{j(j + 1)}{k + 2} \]  

(9)

The two point functions and three point functions are fully determined using global \( SU(2) \) and conformal transformations:

\[ \langle \phi_{j_1}(x_1, z_1)\phi_{j_2}(x_2, z_2) \rangle = A(j_1)\delta_{j_1j_2}x_{12}^{2j_1}z_{12}^{-2h} \]  

(10)
\[
\langle \phi_{j_1}(x_1, z_1) \phi_{j_2}(x_2, z_2) \phi_{j_3}(x_3, z_3) \rangle = C(j_1, j_2, j_3) x_{12}^{j_1+j_2-j_3} x_{13}^{j_1+j_3-j_2} x_{23}^{j_2+j_3-j_1}
\]

The \(C(j_1, j_2, j_3)\) are the structure constants which in principle completely determine the entire theory.

For the case of the four point correlation functions of \(SU(2)\) primaries the form is determined by global conformal and \(SU(2)\) transformations up to a function of the cross ratios. Our convention is:

\[
\langle \phi_{j_1}(x_1, z_1) \phi_{j_2}(x_2, z_2) \phi_{j_3}(x_3, z_3) \phi_{j_4}(x_4, z_4) \rangle = z_{43}^{h_2+h_1-h_3-h_4} x_{43}^{j_4-j_3+j_1+j_2} x_{34}^{j_3-j_2+j_1} x_{31}^{j_1+j_2+j_3}
\]

Here the invariant cross ratios are:

\[
x = \frac{x_{21} x_{43}}{x_{31} x_{42}} \quad z = \frac{z_{21} z_{43}}{z_{31} z_{42}}
\]

For two and three point functions the KZ equation gives us no new information. However for the four point function (12) using the representation (7) we find it becomes a partial differential equation for \(F(x, z)\):

\[
(k + 2) \frac{\partial}{\partial z} F(x, z) = \left[ \frac{\mathcal{P}}{z} + \frac{\mathcal{Q}}{z - 1} \right] F(x, z)
\]

Explicitly these are:

\[
\mathcal{P} = -x^2 (1 - x) \frac{\partial^2}{\partial x^2} + ((-j_1 - j_2 - j_3 + j_4 + 1) x^2 + 2 j_1 x + 2 j_2 x (1 - x)) \frac{\partial}{\partial x}
\]
\[
+ 2 j_2 (j_1 + j_3 + j_4) x - 2 j_1 j_2
\]
\[
\mathcal{Q} = -(1 - x)^2 x \frac{\partial^2}{\partial x^2} - ((-j_1 - j_2 - j_3 + j_4 + 1) (1 - x)^2 + 2 j_3 (1 - x) + 2 j_2 x (1 - x)) \frac{\partial}{\partial x}
\]
\[
+ 2 j_2 (j_1 + j_2 + j_3 - j_4) (1 - x) - 2 j_2 j_3
\]

For a four point correlator involving a discrete representation of spin \(2j \in \mathbb{Z}\) we can write the general solution to the KZ equation as:

\[
F(x, z) = \tilde{F}_0(z) + x \tilde{F}_1(z) + x^2 \tilde{F}_2(z) + \cdots + x^{2j} \tilde{F}_{2j}(z)
\]

This allows one to reduce the KZ equation to a linear \textit{ordinary} differential equation of order \(2j + 1\).

3 Hamiltonian reduction

When we do a quantum hamiltonian reduction of \(SU(2)_k\) theories, by imposing the constraint \(J^+ \sim 1\), it is well known \([82-86]\) that we get the \(c_{k+2,1}\) models.
The central charge of the reduced theory is precisely that of the $c_{p,q}$ model with $k + 2 = \frac{p}{q}$ (we will always take $\gcd(p, q) = 1$):

$$c_{p,q} = 1 - \frac{6(p - q)^2}{pq}$$
$$h_{r,s} = \frac{(pr - qs)^2 - (p - q)^2}{4pq}$$

(18)  
(19)

If we perform hamiltonian reduction of the discrete representations of $SU(2)$ with $2j \in \mathbb{Z}$ we get the conformal weights of the $h_{1,2j+1}$ fields in the $c_{p,q}$ model:

$$h_{1,2j+1} = \frac{j(j + 1)}{k + 2} - j$$

(20)

Here we follow an elegant realisation of this reduction that allow us to perform this at the level of the correlation functions. The motivation for such a procedure comes from the observation that setting $x_i = z_i$ in the two and three point functions (10,11) gives the expected results with the correct conformal weight (20).

What is far from obvious and is the central result of a series of papers by Petkova et al. [87–89] is that this simple procedure also extends to the four point functions. It was shown that for the discrete representation of spin $j$ in $SU(2)_k$, if instead of expanding as in (17), we expand about the point $x = z$:

$$F(x, z) = F_0(z) + (x - z)F_1(z) + (x - z)^2F_2(z) + \cdots + (x - z)^{2j}F_{2j}(z)$$

(21)

then the lowest component $F_0(z)$ obeys the correct differential equation for the field $h_{1,2j+1} = \frac{j(j+1)}{k+2} - j$. This can also obtained from the solution $F(x, z)$ by taking $x_i = z_i$ which gives $x = z$. As this is at the level of conformal blocks overall normalisations are not important, at least for individual blocks, although these can become singular in some cases [90].

This approach gives us a very efficient and practical way to study $h_{1,s}$ correlators up to a very high level. We shall discuss several examples of $h_{1,s}$ correlators the $c_{p,q}$ models in which we have found interesting sets of rational and logarithmic solutions. Everything that we say could presumably also be reinterpreted in the $\hat{SU}(2)$ theory, as solutions for $F_0(z)$ lift up to solutions for the full $F(x, z)$, but we do not attempt this here.

4 Vacuum null vector and its fermionic partners

In this section we shall comment on the vacuum null vector and we find new fermionic partner fields.
4.1 Vacuum null vector

It is known that for all $c_{p,q}$ models by studying the vacuum null vector we can learn everything about the ‘minimal’ sector with weights $h_{r,s}$ (18) with $1 \leq r \leq q-1$, $1 \leq s \leq p-1$ and identifications $h_{r,s} = h_{q-r,p-s}$ [81].

For example the Ising model at $c_{3,4} = \frac{1}{2}$ has a vacuum null vector given by:

$$\mathcal{N} = 9 \partial^4 T + 264((\partial^2 T)T) - 186(\partial T \partial T)) - 128(T(TT))$$

(22)

One can easily check by using the Virasoro algebra:

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots$$

(23)

and the normal ordering prescription:

$$(AB)(w) = \frac{1}{2\pi i} \oint_w -dz A(z)B(w)$$

(24)

that this null vector is indeed a primary field of conformal weight 6:

$$T(z)\mathcal{N}(w) \sim \frac{6\mathcal{N}(w)}{(z-w)^2} + \frac{\partial \mathcal{N}(w)}{z-w} + \cdots$$

(25)

In the irreducible theory this null vector should be set to zero in all correlation functions. In particular the zero mode of this must vanish when applied to Virasoro primary states $|h\rangle$. We know:

$$L_n |h\rangle = 0 \quad n \geq 1$$

$$L_0 |h\rangle = h |h\rangle$$

(26)

and therefore one finds:

$$\mathcal{N}_0 |h\rangle = -4h(2h-1)(16h-1) |h\rangle = 0$$

(27)

From this one easily finds the solutions $h = 0, \frac{1}{2}, \frac{1}{16}$ which are well known as the conformal weights of the irreducible operator content of the Ising model. In general imposing the zero modes of the $h = (p-1)(q-1)$ vacuum null vector gives us a polynomial of rank $r = \frac{1}{2}(p-1)(q-1)$. Solving this gives us precisely the $r$ primary operators in the ‘minimal’ $c_{p,q}$ model [81]. Furthermore all fusion rules in this theory can, in principle, be found from such a null vector. More details can be found in [67] and references therein.

In particular if we wish to go beyond the minimal $c_{p,q}$ model and consider fields outside the region with $1 \leq r \leq q-1, 1 \leq s \leq p-1$ we must not decouple this vacuum null vector. In order to achieve this we would have to introduce a logarithmic partner for this field. In the case of the well studied $c_{p,1}$ models the vacuum null vector
is at \( h = 0 \) implying, as is well known, that all these extended models must have a logarithmic partner for the vacuum itself.

Continuing with the example of the Ising model we can calculate the correlator with four \( h_{1,2p-1} = 6 \) operators and we find:

\[
F(z) = \frac{1}{z^6(1-z)^6} \left( 2090z^6 - 6270z^5 + 10869z^4 - 11288z^3 + 10869z^2 - 6270z + 2090 \right) \tag{28}
\]

This conformal block is easily seen to lead to a well behaved correlator invariant under all crossing symmetries. By analysing the leading singularity as \( z \to 0 \) we deduce that the two point function of these fields must vanish. To see that this must be true in general consider the OPE of two vacuum null vector fields of the irreducible theory having conformal weight \( h \). This must have the form (up to normalisation):

\[
\mathcal{N}(z)\mathcal{N}(w) \sim \frac{\mathcal{N}(w)}{(z-w)^h} + \cdots \tag{29}
\]

where \( \cdots \) stands for other less singular terms. There cannot be other operators in the more singular terms as these would also be vacuum null vectors, of lower conformal weight, contradicting the fact that we are considering the vacuum null vector of the irreducible theory.

This is of course confirmed by explicitly calculating the OPE of (22) with itself. However the vanishing of the two point function of \( \mathcal{N} \) immediately implies that the four point function must also vanish. In order to make the four point function non-zero and realise the conformal block (28) we must have one insertion of the logarithmic partner in order to make the correlator non-vanishing. This has been discussed in the LCFT literature many times before (see for example [73]). We found in all cases \( (c_{p,q} \text{ with } p, q \leq 6) \) that there is indeed a single rational solution generated by the \( h_{1,2p-1} = (p-1)(q-1) \) field as we expect. However as we shall see in the next section there was always two extra non-chiral states as well.

### 4.2 Non-chiral fermionic partners

In general we found that the differential equation with four \( h_{1,2p-1} = (p-1)(q-1) \) operators always admitted solutions of the form:

\[
\begin{align*}
F^{(1)}(z) &= R_1(z) \\
F^{(2)}(z) &= R_1(z) \ln z + R_2(z) \\
F^{(3)}(z) &= F^{(2)}(1-z)
\end{align*}
\tag{30}
\]

where \( R_1(z) \) and \( R_2(z) \) are rational functions. The first solution \( F^{(1)}(z) \) is the conformal block of the four point function of the vacuum null vector, with the subtleties about insertions of a logarithmic partner, that we have just discussed. We have already
commented that as this a bosonic field we expect it to be invariant under all crossing symmetries:
\[ R_1(z) = R_1(1 - z) \quad z^{2h} R_1 \left( \frac{1}{z} \right) = R_1(z) \]  (31)

The set (30) is clearly closed under monodromy transformations however in order to be closed under crossing symmetries we must have:
\[ z^{2h} R_2 \left( \frac{1}{z} \right) = -R_2(z) + \alpha R_1(z) \]  (32)

the constant \( \alpha \) is arbitrary but we shall always redefine \( F^{(2)}(z) \) by addition of \( F^{(1)}(z) \) to set \( \alpha \) to zero.

The other solutions, as we shall presently see, correspond to extra non-chiral fermionic operators. To see this it is interesting to consider the explicit example of the \( c_{3,2} = 0 \) model. This is of great importance in the field of percolation and polymers. The vacuum null vector in this case is the stress tensor \( T \) itself and imposing the vanishing of this in correlators gives us just the ‘minimal’ topological sector. Considering fields beyond this sector we must create a logarithmic partner for the stress tensor [14].

In this model we found solutions for the \( h_{1,3} = 2 \) conformal blocks:
\[
F_1(z) = \frac{z^2 - z + 1}{z^2(z - 1)^2}, \quad F_2(z) = F_1(z) \ln(z) - \frac{(5z^5 - 5z^4 + 12z^3 + 12z^2 - 5z + 5)}{24(z - 1)z^4}, \quad F_3(z) = F_2(1 - z)
\]  (33)

Before continuing to discuss these solutions we should comment on what occurs if one instead studies the correlators of the \( h_{5,1} = 2 \) field. Then one finds the same rational block \( F^{(1)}(z) \) but slightly different solutions for \( R_2(z) \) in (30). This seems to be universal and the same rational functions always appear as a subset of both solutions.

The rational solution \( F^{(1)}(z) \) forms a well behaved chiral correlator on its own and corresponds to the vacuum null vector \( T \). It is easy to see that this is the only primary \((2,0)\) operator in the theory as the other solutions in (33) on their own do not lead to single-valued correlators. It is also possible to have local \((2,2)\) operators in the theory. To see what these are we combine these conformal blocks with their antiholomorphic components into the full correlator:
\[
G(z, \bar{z}) = \sum_{a,b=1}^{3} U_{a,b} F_a(x, z) \bar{F}_b(x, \bar{z})
\]  (34)

To make this single-valued everywhere we find:
\[
G(z, \bar{z}) = U_{1,1} F_1(z) \bar{F}_1(z) + U_{1,2} \left[ F_1(z) \bar{F}_2(\bar{z}) + F_2(\bar{z}) \bar{F}_1(z) \right] \\
+ U_{1,3} \left[ F_1(z) \bar{F}_3(\bar{z}) + F_3(\bar{z}) \bar{F}_1(z) \right]
\]  (35)

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As well as the solution corresponding to the stress tensor $F^{(1)}$ we also have two other solutions which, as we have logarithms present, do not have a diagonal form.

Now consider the effect of crossing symmetries on these solutions. Under $1 \leftrightarrow 3$ we have $z \rightarrow 1 - z$ and:

$$F_1 \rightarrow F_1 \quad F_2 \rightarrow F_3 \quad F_3 \rightarrow F_2$$

(36)

Under $1 \leftrightarrow 4$ we have $z \rightarrow \frac{1}{z}$:

$$F_1 \rightarrow z^4 F_1 \quad F_2 \rightarrow -z^4 F_2 \quad F_3 \rightarrow z^4 (-i \pi F_1 - F_2 + F_3)$$

(37)

We immediately see that the other two solutions are not invariant under all crossing symmetries. To indicate these statistics we add extra labels to these non-chiral operators. From examining the behaviour under the crossing symmetries we find that these correspond to non-chiral fermionic operators $\Theta^\pm(z, \bar{z})$. To get the correct crossing symmetries we must have:

$$\langle \Theta^+(z_1, \bar{z}_1) \Theta^-(z_2, \bar{z}_2) \Theta^-(z_3, \bar{z}_3) \Theta^+(z_4, \bar{z}_4) \rangle = |z_{13}|^{-8} |z_{24}|^{-8} \left[ F_1(z) F_2(z) + F_2(z) F_1(z) \right]$$

$$\langle \Theta^+(z_1, \bar{z}_1) \Theta^+(z_2, \bar{z}_2) \Theta^-(z_3, \bar{z}_3) \Theta^-(z_4, \bar{z}_4) \rangle = |z_{13}|^{-8} |z_{24}|^{-8} \left[ F_1(z) F_3(z) + F_3(z) F_1(z) \right]$$

By expanding these we see:

$$\langle \Theta^\alpha(z_1, \bar{z}_1) \Theta^\beta(z_2, \bar{z}_2) \rangle = 0 \quad \alpha, \beta = \pm$$

(38)

It is a general fact that all fields beyond the minimal sector have vanishing two point functions. It is the non-vanishing of the four point functions that gives us a non-trivial theory.

It has been conjectured that fermionic partners to the stress tensor in $c = 0$ generate a super-algebra with $U(1|1)$ symmetry [14]. As we have seen these fields are non-chiral and so certainly cannot be generators of an affine super-algebra.

### 5 Triplet solutions

In general once one considers fusion of fields from outside the minimal region of the $c_{p,q}$ models we start to generate an infinite number of Virasoro primary states. However in the $c_{p,1}$ models this infinite number of fields become rearranged into a finite number of states with respect to a larger algebra - $W(2, 2p - 1, 2p - 1, 2p - 1)$. The $h = 2$ operator is the stress tensor $T$ and the other fields $h_{3,1} = 2p - 1$ are a triplet of fields $W^a$ with an $SO(3)$ symmetry [74].

Although these were originally found by different methods it is interesting to see how they arise from the rational solutions for the conformal blocks. For example in the second member of this series, the well known $c_{1,2} = -2$ model, we find exactly three
rational four point functions for the $h_{1,3}$ or the $h_{7,1}$ fields both with $h = 3$. They are:

\[
F_{3333}^{(1)}(z) = \frac{1}{(z-1)^6} z^4 \left( 6 - 6z + z^2 \right) \\
F_{3333}^{(2)}(z) = \frac{1}{z^6(z-1)^6} \left( 2 - 12z + 12z^2 + 50z^3 - 225z^4 + 468z^5 - 588z^6 + 468z^7 - 225z^8 \\
+ 50z^9 + 12z^{10} - 12z^{11} + 2z^{12} \right) \\
F_{3333}^{(3)}(z) = F_{3333}^{(1)}(1-z) = \frac{1}{z^6} \left( 1 - 9z^2 + 16z^3 - 9z^4 + z^6 \right)
\]  

Note that $F_{3333}^{(2)}(z)$ is the unique solution satisfying:

\[
F_{3333}^{(2)}(1-z) = F_{3333}^{(2)}(z) \\
F_{3333}^{(3)}(z) = z^{2h} F_{3333}^{(2)} \left( \frac{1}{z} \right)
\]  

It therefore leads to a correlator that is invariant under all exchanges of operators. $F^{(1)}(z)$ is the unique solution with no poles as $z \to 0$ whereas $F^{(3)}(z)$ is the unique solution with no poles as $z \to 1$. It is easily seen that these requirements can be met by assuming that the fields are actually a bosonic triplet of fields $W^a$ with the following correlators:

\[
\langle W^+(0)W^+(z)W^-(1)W^-(\infty) \rangle = F_{3333}^{(1)}(z) \\
\langle W^3(0)W^3(z)W^3(1)W^3(\infty) \rangle = F_{3333}^{(2)}(z) \\
\langle W^+(0)W^-(z)W^-(1)W^+(\infty) \rangle = F_{3333}^{(3)}(z)
\]  

These fields are well known in $c = -2$ and are indeed a bosonic triplet as can be verified from a simple free field construction [70]. However we see that the arguments leading us to this relied only on the existence of the three rational solutions with the stated pole structure and behaviour under crossing symmetry. We shall always write our functions $F^{(i)}$ in this notation allowing one to immediately write the correlators.

One therefore suspects that the same is true in general and that a triplet of rational solutions, that are closed under crossing symmetry, will lead to a triplet algebra. One can check that there is indeed a triplet of rational solutions in all the other cases of $c_{p,q}$, we have done this explicitly for $p \leq 9$.

In generalising this discussion it will be useful to note that in $c_{p,1}$ we have $h_{3,1} = h_{1,4p-1}$. We will find that it is the fields $h_{1,4p-1} = (2p - 1)(2q - 1)$ that become the triplet fields in the general $c_{p,q}$ models.

### 5.1 Correlators in the $c_{1,1}$ model

The $c_{1,1} = 1$ model is a rather peculiar case and so we shall discuss it separately in this section. The $h_{1,s}$ fields come from the hamiltonian reduction of the $SU(2)_{-1}$ theory.
The fields have weights:

\[ h_{1,2j+1} = h_{2j+1,1} = j(j+1) - j = j^2 \]

In this case we find the first few fields have dimensions: \(0, \frac{1}{2}, 1, \ldots\). The \(j \in \mathbb{Z}\) fields have integer dimensions and all correlators of these fields that we studied were found to be rational functions.

In particular we found that the \(h_{1,3} = 1\) fields have three rational solutions behaving exactly as before and so we deduce:

\[
\begin{align*}
\langle W^+(0)W^+(z)W^-(1)W^-(\infty) \rangle &= \frac{z^2}{(1-z)^2} \\
\langle W^3(0)W^3(z)W^3(1)W^3(\infty) \rangle &= \frac{(1-z+z^2)^2}{z^2(1-z)^2} \\
\langle W^+(0)W^-(z)W^-(1)W^+(\infty) \rangle &= \frac{(1-z)^2}{z^2}
\end{align*}
\]

These correlators (43) are exactly those corresponding to four point functions of affine currents \(J^a\) which generate a \(SU(2)_1\) Kac-Moody algebra in the extended \(c_{1,1} = 1\) model.

It is interesting to examine this from the point of view of hamiltonian reduction. We start with the \(SU(2)_{-1}\) theory with three Kac-Moody currents and the triplet of \(j = 1\) fields. Note in this case there is potential confusion as the extended fields are triplets of the \(SU(2)_{-1}\) algebra (as the have \(j = 1\)) and also have an extended \(SO(3)\) triplet index. After hamiltonian reduction the \(SU(2)_{-1}\) structure is lost but the extended one remains. What is remarkable, in this example, is that the extended structure after reduction is in fact itself an \(SU(2)\) affine Kac-Moody algebra, this time at level \(k = 1\). As the \(SU(2)_1\) model is one of the very simplest rational CFTs one may hope by considering the extended triplet algebra in \(SU(2)_{-1}\) that this model should be a relativity simple example of a rational non-unitary CFT. It is not clear if this theory involves indecomposable representations or not. The four point correlators for the irreducible representations were all rational functions but further fusions may yield other representations.

### 5.2 Correlators in the \(c_{p,q}\) models

We found for every \(c_{p,q}\) model (we tested \(p \leq 5, q \leq 5\)) that there was always exactly three rational solutions for the \(h_{1,4p-1} = (2p - 1)(2q - 1)\) fields. Rather more non-trivially if one exchanges \(p\) and \(q\) the differential equations are of a different order but the same set of three rational solutions solves both of them. These triplets appeared to always have a bosonic nature under crossing symmetry.

As we have discussed the case \(c_{2,1} = c_{1,2}\) in the previous section we shall begin with the first new example: the \(c_{2,3} = 0\) theory. In the \(c_{2,3} = 0\) model the solutions are
given by:

\[
F^{(1)} = \frac{1}{(1-z)^{28}} \left( (357106464 - 2856851712 z + 10509841628 z^2 - 2357398636 z^3 \\
+36044249670 z^4 - 39790427248 z^5 + 32773983814 z^6 - 20529517008 z^7 \\
+988047186 z^8 - 3667147120 z^9 + 1048374600 z^{10} - 229634210 z^{11} \\
+38248769 z^{12} - 4810728 z^{13} + 452625 z^{14} - 30294 z^{15} + 1122 z^{16} ) z^{10} \right)
\]

\[
F^{(2)} = \frac{1}{z^{28} (1-z)^{28}} \left( 2244 - 60588 z + 905250 z^2 - 9621456 z^3 + 76497538 z^4 \\
-459268420 z^5 + 2096749200 z^6 - 7334294240 z^7 + 19760294372 z^8 \\
-41059034016 z^9 + 65547967628 z^{10} - 79580854496 z^{11} + 7208849340 z^{12} \\
-36330724836 z^{13} - 200733901482 z^{14} + 2212292459088 z^{15} - 14422439940116 z^{16} \\
+6856249363130 z^{17} - 254028569259777 z^{18} + 763908934818536 z^{19} \\
-1917517271406737 z^{20} + 4101816418782654 z^{21} - 7599053781520630 z^{22} \\
+12352604911298080 z^{23} - 17809256023135980 z^{24} + 22972890487011504 z^{25} \\
-2668957867423868 z^{26} + 28044134317298400 z^{27} - 2668957867423868 z^{28} \\
+22972890487011504 z^{29} - 17809256023135980 z^{30} + 12352604911298080 z^{31} \\
-7599053781520630 z^{32} + 4101816418782654 z^{33} - 1917517271406737 z^{34} \\
+763908934818536 z^{35} - 254028569259777 z^{36} + 6856249363130 z^{37} \\
-14422439940116 z^{38} + 2212292459088 z^{39} - 200733901482 z^{40} \\
-36330724836 z^{41} + 7208849340 z^{42} - 79580854496 z^{43} + 65547967628 z^{44} \\
-41059034016 z^{45} + 19760294372 z^{46} - 7334294240 z^{47} + 2096749200 z^{48} \\
-459268420 z^{49} + 76497538 z^{50} - 9621456 z^{51} + 905250 z^{52} - 60588 z^{53} + 2244 z^{54} \right)
\]

\[
F^{(3)} = \frac{1}{z^{28}} \left( (1122 + 12342 z + 132855 z^2 + 1026528 z^3 + 5156450 z^4 \\
+17580680 z^5 + 42038555 z^6 + 70854550 z^7 + 83500300 z^8 + 70854550 z^9 \\
+42038555 z^{10} + 17580680 z^{11} + 5156450 z^{12} + 1026528 z^{13} \\
+132855 z^{14} + 12342 z^{15} + 1122 z^{16} ) (1-z)^{10} \right)
\]

where we have again used the same conventions as before in the labelling of the $F^{(i)}$.

Although the detailed form of the solutions is extremely complicated the structure is very simple. In an exactly analogous way to the arguments used for the extra indicial nature of the triplet fields in $c = -2$ we deduce that the triplet of dimension $h_{1,7} = 15$ fields in $c_{2,3} = 0$ behave as:

\[
\begin{align*}
\langle W^+(0) W^+(z) W^-(1) W^-(\infty) \rangle &= F^{(1)} \\
\langle W^3(0) W^3(z) W^3(1) W^3(\infty) \rangle &= F^{(2)} \\
\langle W^+(0) W^-(z) W^-(1) W^+(\infty) \rangle &= F^{(3)}
\end{align*}
\]
It is not yet possible to conclude that all these algebras are closed in the sense of $W$-algebras as it is extremely difficult to read off the operator content from the rational correlation functions. We shall see that they do indeed appear closed as the other $h_{1,s}$ operators that could potentially contribute in the singular terms of the OPE obey Fermi statistics.

6 Chiral Fermions

The triplet algebra as we have found, but certainly not proved, appears in all the $c_{p,q}$ models. However we also found that if $p$ and $q$ were not both odd then there was also a doublet of rational solutions with fermionic behaviour generated by the $h_{1,3p-1} = (\frac{3p}{2} - 1)(\frac{3q}{2} - 1)$ fields. If $p$ and $q$ are both odd then this field does not have $2h \in \mathbb{Z}$ and so cannot be a local chiral field.

The classic example is the $h_{1,2}$ or $h_{5,1}$ fields at $h = 1$ in the $c_{2,1} = -2$ model where we have the rational solutions:

\[
F^1(z) = 1 - \frac{1}{z^2} \quad (48)
\]

\[
F^2(z) = 1 - \frac{1}{(1 - z)^2} \quad (49)
\]

$F^{(1)}$ is now unique solution with no poles as $z \to 1$ and we have $F^{(2)}(z) = F^{(1)}(1 - z)$. By using similar crossing symmetry arguments as before we see that there are two chiral fermionic states:

\[
\langle \Psi^+(0) \Psi^-(z) \Psi^-(1) \Psi^+(\infty) \rangle = F^{(1)}(z) \quad (50)
\]

\[
\langle \Psi^+(0) \Psi^+(z) \Psi^-(1) \Psi^-(\infty) \rangle = F^{(2)}(z)
\]

These are precisely the $h = 1$ symplectic fermion fields $\Psi^{\pm}(z)$ from the $c = -2$ theory:

\[
\Psi^+(z) \Psi^-(w) \sim \frac{1}{(z - w)^2} \quad (51)
\]

In $c_{2,3} = 0$ we also find similar solutions for $h_{1,5} = 7$ operators:

\[
F^1(z) = \frac{1}{z^{12}} \left( (-22z^9 - 44z^8 - 323z^7 - 859z^6 - 1302z^5 - 1302z^4 - 859z^3 - 323z^2 - 44z - 22)(1 - z) \right) \quad (52)
\]

\[
F^2(z) = \frac{1}{(1 - z)^{12}} \left( (22z^9 - 242z^8 + 1467z^7 - 6200z^6 + 18475z^5 - 37854z^4 + 51884z^3 - 45424z^2 + 22950z - 5100)z \right) \quad (53)
\]

We have checked many other $c_{p,q}$ models, with $pq \in 2\mathbb{Z}$, and always found 2 rational solutions with a fermionic symmetry for the $h_{1,3p-1} = (\frac{3p}{2} - 1)(\frac{3q}{2} - 1)$ fields.
7 General structure

The structure of rational solutions and Bose/Fermi assignments to operators is very suggestive. It seems that an odd number of rational solutions corresponds to bosonic operators and an even one to fermionic ones. The cases we studied all fit into the sequence $h_{1,2np-1} = (np-1)(nq-1)$ having $2n-1$ rational solutions where $n = 1, 2, 3, \cdots$ for bosonic fields and $n = \frac{3}{2}, \frac{5}{2}, \cdots$ for fermionic fields.

If these are indeed the only chiral $h_{1,s}$ operators in the theory then we immediately see in the singular terms of the OPE of two $h_{1,4p-1}$ triplet fields we can only create $h_{1,2p-1}$ and $h_{1,4p-1}$ fields. The next possible bosonic field is at $h_{1,6p-1} > 2h_{1,4p-1}$ and so lies beyond the singular terms in the chiral OPE. Therefore the triplet algebra should close as a $W$-algebra with a schematic OPE:

$$h_{1,4p-1}^a \otimes h_{1,4p-1}^b = \delta^{ab} [h_{1,2p-1}] + f_{c}^{ab} [h_{1,4p-1}]$$

where $\delta^{ab}$ and $f_{c}^{ab}$ are the metric and structure constants of SU(2) and $[h]$ denotes an operator and all its descendents. Recall that the $h_{1,2p-1}$ field is the vacuum null vector of the irreducible theory and so $[h_{1,2p-1}] = [1]$.

Throughout this paper we have only considered $c_{p,q}$ with $p, q \in \mathbb{Z}^+$. The case of $c_{-p, q}$ with $c > 25$ can also be obtained from hamiltonian reduction of $SU(2)_k$ with $k + 2 < 0$. In this case all discrete representations have negative dimensions however one also observes rational functions for certain correlators.

It is clear that these structures deserve much closer investigation.

8 The moduli space of CFTs

In this section we shall analyse the approach to local logarithmic CFTs in two particular cases. The first is the well known appearance of an indecomposable representation and the second is a situation in which operators may have extended indices, We shall analyse these in the well known $c_{2,1} = -2$ model as the operator content is particularly well known.

The first correlator that we shall analyse is the original one studied by Gurarie [2] for the $h_{1,2} = -\frac{1}{8}$ operators:

$$\langle \mu(z_1, \bar{z}_1) \mu(z_2, \bar{z}_2) \mu(z_3, \bar{z}_3) \mu(z_4, \bar{z}_4) \rangle = |z_{13}z_{24}|^{1/2}|z(1-z)|^{1/2}G(z, \bar{z})$$

We can easily find the conformal blocks and the unique single-valued combination is:

$$G(z, \bar{z}) = F(z)\overline{F(1-z)} + \overline{F(z)}F(1-z)$$

where $F(z)$ is the hypergeometric function: $\frac{1}{2} \frac{1}{2} 1; z$. This leads to a correlator invariant under all exchanges of operators. It is well known that this correlator has logarithmic singularities and it is interesting to see how these emerge as $c \rightarrow -2$. 

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To analyse this we examine the \( c_{k+2,1} \) model where \( k \) is taken to be small. The central charge is:

\[
c = 13 - 6 \left( k + 2 + \frac{1}{k+2} \right) = -2 - \frac{9k}{2} + O(k^2)
\]  

(57)

The first few operators in the Kac-table have dimensions:

\[
h_{1,1} = 0 \\
h_{1,2} = \frac{3}{4(k+2)} - \frac{1}{2} = -\frac{1}{8} - \frac{3k}{16} + O(k^2) \\
h_{1,3} = \frac{2}{k+2} - 1 = -\frac{k}{2} + O(k^2)
\]  

(58)

At the point \( k = 0 \) we have \( h_{1,3} = h_{1,1} = 0 \) but for generic values of \( k \) there is no degeneracy in the levels. We can find the general solutions for the four point function of \( h_{1,2} \) operators. The full correlator is given by:

\[
\langle h_{1,2}(z_1, \bar{z}_1) h_{1,2}(z_2, \bar{z}_2) h_{1,2}(z_3, \bar{z}_3) h_{1,2}(z_4, \bar{z}_4) \rangle = |z_1 z_2 z_3 z_4|^{-4h} |z|^{\frac{2k+1}{k+2}} |1 - z|^{\frac{1}{k+2}} G(z, \bar{z})
\]

where:

\[
G(z, \bar{z}) = \sum_{i,j=1}^{2} U_{i,j} F_i(z) F_j(\bar{z})
\]  

(59)

and the conformal blocks \( F_i(z) \) are found by solving the differential equations or via the Coulomb gas approach. They are:

\[
F_1(z) = \frac{\Gamma\left(\frac{k+1}{k+2}\right) \Gamma\left(\frac{k+1}{k+2}\right)}{\Gamma\left(\frac{2k+2}{k+2}\right)} \, _2F_1\left(\frac{1}{k+2}, \frac{k+1}{k+2}; \frac{2k+2}{k+2}; z\right)
\]

(60)

\[
F_2(z) = z^{-\frac{k}{k+2}} \frac{\Gamma\left(\frac{1-k}{k+2}\right) \Gamma\left(\frac{k+1}{k+2}\right)}{\Gamma\left(\frac{2}{k+2}\right)} \, _2F_1\left(\frac{1}{k+2}, \frac{1-k}{k+2}; \frac{2}{k+2}; z\right)
\]

(61)

We have included the normalisations so that we can use standard results. The solutions \( F_1(z) \) and \( F_2(z) \) are respectively the conformal blocks for the contributions from the \( h_{1,1} \) and \( h_{1,3} \) operators respectively as can be seen from the leading powers of \( z \). However we immediately see that these two solutions become identical in the limit as \( k \to 0 \).

The full correlator must of course be single-valued everywhere. Monodromy around \( z = 0 \) leads to the requirement that:

\[
U_{1,2} e^{2\pi i \frac{k}{k+2}} = U_{1,2} \\
U_{2,1} e^{-2\pi i \frac{k}{k+2}} = U_{2,1}
\]  

(62)
Now for the case of generic values of $k$ we have $\frac{k}{k+2} \notin \mathbb{Z}$ and we conclude $U_{1,2} = U_{2,1} = 0$ and the correlator must be diagonal:

$$G(z, \bar{z}) = U_{1,1}|F_1(z)|^2 + U_{2,2}|F_2(z)|^2$$  \hspace{1cm} (63)

Now imposing the monodromy around $z = 1$ leads to the condition [91]:

$$\frac{U_{1,1}}{U_{2,2}} = \frac{\sin \pi(a + b + c) \sin \pi b}{\sin \pi a \sin \pi c}$$ \hspace{1cm} (64)

where $a = \frac{-2k-1}{k+2} \ , \ b = c = \frac{-1}{k+2}$. Expanding this in the limit $k \to 0$ we get:

$$\frac{U_{1,1}}{U_{2,2}} = -1 + O(k^2)$$ \hspace{1cm} (65)

Therefore:

$$G(z, \bar{z}) = U_{1,1} \left(|F_1(z)|^2 - |F_2(z)|^2\right)$$ \hspace{1cm} (66)

The minus sign is absolutely crucial. It signifies that we have negative norm states. Logarithms can occur when these are cancelled to leading order by the positive norm states. Expanding $F_1$ and $F_2$ gives:

$$F_1(z) = \pi \mathcal{F}(z) + kC(z)$$ \hspace{1cm} (67)
$$F_2(z) = \pi \mathcal{F}(z) + kD(z)$$ \hspace{1cm} (68)

where:

$$C - D = \frac{\pi^2}{2} \mathcal{F}(1 - z)$$ \hspace{1cm} (69)

and $\mathcal{F}(z)$ is again the hypergeometric function: $_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; z \right)$. In order to make the full correlator non-vanishing in the limit $k \to 0$ we will have to choose the overall rescaling of the four point function (66) to be $U_{1,1} \sim \frac{1}{k}$. With this choice we find that we have a smooth limit as $k \to 0$.

$$G(z, \bar{z}) = \frac{1}{k} \left( (\pi \mathcal{F} + kC)(\pi \mathcal{F} + kC) - (\pi \mathcal{F} + kD)(\pi \mathcal{F} + kD) \right)$$ \hspace{1cm} (70)

$$\to [\mathcal{F}(z)\mathcal{F}(1 - z) + \mathcal{F}(1 - z)\mathcal{F}(z)]$$

In this case we were able to get a smooth approach to a logarithmic correlator from a non-logarithmic one. One might be therefore tempted to think that LCFT is merely some continuous limit of ordinary CFT. However we shall soon see that this is not always the case.
To illustrate this we shall examine the correlator:

\[
\langle h_{1,2}(z_1, \bar{z}_1) h_{1,2}(z_2, \bar{z}_2) h_{1,3}(z_3, \bar{z}_3) h_{1,3}(z_4, \bar{z}_4) \rangle = |z_{34}|^{4h_{1,2} - 4h_{1,3}} |z_{24}|^{-4h_{1,2}} |z_{13}|^{-4h_{1,2}} |z|^{2k+1 \over 8+2} [1 - z^{4 \over 2+k+2}] G(z, \bar{z})
\] (71)

Evaluating this correlator for the \( c = -2 \) theory we get two solutions:

\[
\mathcal{F}_1(z) = (1 - z)^{-1/2}
\]

\[
\mathcal{F}_2(z) = (1 - z)^{-1/2} \arctan(\sqrt{z - 1})
\]

The function \( \arctan(\sqrt{z - 1}) \) has the following behaviour near \( z = 0 \):

\[
\arctan(\sqrt{z - 1}) \sim -i \over 2 \ln z + \text{regular}
\] (73)

Making \( G(z, \bar{z}) \) single valued requires no logarithmic branch cuts and therefore we have two possible single-valued correlators:

\[
G(z, \bar{z}) = U_{1,1} \mathcal{F}_1 \bar{\mathcal{F}}_1 + U_{1,2} (\mathcal{F}_1 \bar{\mathcal{F}}_2 + \mathcal{F}_2 \bar{\mathcal{F}}_1)
\] (74)

The solution with \( U_{1,2} = 0 \) corresponds to the correlator:

\[
\langle \mu(z_1, \bar{z}_1) \mu(z_2, \bar{z}_2) \Omega(z_3, \bar{z}_3) \Omega(z_4, \bar{z}_4) \rangle = |z_{12}|^{1/2}
\]

The other solution with logarithmic terms corresponds to the correlator:

\[
\langle \mu(z_1, \bar{z}_1) \mu(z_2, \bar{z}_2) \Theta^+ (z_3, \bar{z}_3) \Theta^- (z_4, \bar{z}_4) \rangle = |z_{12}|^{1/2} \left( \arctan(\sqrt{z - 1}) + \arctan(\sqrt{z - 1}) \right)
\]

where \( \Omega(z, \bar{z}) \) is the normal vacuum and \( \Theta^\pm \) are the non-chiral fermionic \( h = 0 \) operators that we have already discussed in general.

For any value of \( k \) we can again solve to find the conformal blocks. They are:

\[
F_1(z) = _2F_1 \left( \begin{array}{cc} 2 & k + 1 \\ k + 2 & k + 2 \end{array} \right) \left( {2k + 2 \over k + 2} ; z \right)
\]

\[
F_2(z) = z^{-k \over k + 2} _2F_1 \left( \begin{array}{cc} 1 & 2 - k \\ k + 2 & k + 2 \end{array} \right) \left( {2 \over k + 2} ; z \right)
\]

---

\( \star \)This is most easily seen using:

\[
\arctan(\sqrt{z - 1}) = \int {1 \over 2z \sqrt{z - 1}} \, dz = \frac{1}{2i} \int \left[ {1 \over z} + {1 \over 2} + {3 \over 8} z + \cdots \right] \, dz \]

\[
= \frac{1}{2i} \ln z + \text{regular}
\]
Again for generic values of $k$ we must have the diagonal correlator:

$$G(z, \bar{z}) = U_{2,2} \left\{ \frac{U_{1,1}}{U_{2,2}} |F_1|^2 + |F_2|^2 \right\}$$  \hspace{0.5cm} (77)

Now imposing monodromy around $z = 1$ we find:

$$\frac{U_{1,1}}{U_{2,2}} = -\frac{2}{\pi^2 k^2} - \frac{2}{\pi^2 k} + O(1)$$  \hspace{0.5cm} (78)

We therefore see that, in order to have a well defined limit in (77), we must take $U_{2,2} \sim k^2$ and we then find:

$$G(z, \bar{z}) \rightarrow |F_1(z)|^2$$

Therefore we see that in the limit of the correlators we do not find the second solution $F_2(z)$ corresponding to operators $\Theta^\pm$. We now have a rather interesting puzzle. For $k \neq 0$ we have no degeneracy and get a unique correlator. However at the point $k = 0$ we have a choice of two different correlators coming from the extra indicial structure of $\Theta^\pm(z, \bar{z})$. The fundamental reason for this is that the moduli space of solutions to the monodromy constraints:

$$U_{1,2} e^{2\pi i \frac{k}{k+2}} = U_{1,2}$$

is not smooth as a function of $k$. We see that the condition is trivial if $\frac{k}{k+2} \in Z \Leftrightarrow h_{1,3} - h_{1,1} \in Z$. It is exactly in the cases in which conformal dimensions differ by integers, and we may get logarithms, that the monodromy constraints break down.

This conclusion is applicable to any conformal field theory in which one has an extended multiplet structure at a certain point. The limit of the correlators is not the same as solving the theory at the limiting point. It would be particularly interesting to analyse this in the context of disordered systems which can be studied in the replica limit or using the supersymmetric approach [27]

### 9 Conclusion

We have investigated the structure of the $h_{1,s}$ fields in the $c_{p,q}$ models by directly studying their correlation functions. We found that the vacuum null vector of the irreducible theory is accompanied by two extra primary non-chiral fermionic fields. We also found a chiral triplet algebra generated by $h_{1,4p-1}$ fields. For $pq \in 2Z$ we also found extra chiral fermionic structure.

We were not able to understand the appearance of this structure but in the case of the non-chiral partners to the vacuum null vector it naively comes from the fact:

$$\ln \left| \frac{1}{z} \right| = -\ln |z|$$
It is the minus sign which indicates that operators behave in a fermionic manner under crossing symmetries. We have seen that such an extended symmetry of particular fields does not arise in the limit of correlation functions.

It would be extremely interesting to understand many of these points more deeply.

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References


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