INTRODUCTION

An essential element in quantum information processing is the ability to encode and manipulate information, allowing the development of quantum computers and quantum networks. Quantum information is encoded in the state of quantum systems, which can be manipulated using quantum gates. The manipulation of quantum information is done using quantum algorithms, which can perform computations that are not possible with classical computers.

Quantum information processing involves the manipulation of quantum states to perform computations and operations. This includes quantum error correction, quantum key distribution, and quantum teleportation. Quantum information processing is also used in quantum cryptography, quantum computing, and quantum communication.

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terize the minimal interaction time required to send classical, as well as quantum, information. As a by-product, and very much in the spirit of [7] and [8], where information exchange has been used to characterize the non-local content of certain gates, we obtain a complete classification of two-qubit gates with respect to their transmission capabilities, thereby supplementing the original characterization of non-local gates.

The results we present can be summarized as follows:

- Analytical characterization of the interaction cost of any two-qubit gate by any two-qubit interaction Hamiltonian, through a new, self-contained proof (section III).
- Analytical characterization, in part, of the space of two-qubit gates, of the partial order structure based on the interaction cost (section III).
- Analytical characterization, for any two-qubit interaction, of the interaction cost of the following communication processes between two qubits (section IV):
  
  1. Transmission of one classical bit: c-bit\textsubscript{A→B}.
  2. Simultaneous, bidirectional transmission of two classical bits: c-bit\textsubscript{A→B} and c-bit\textsubscript{B→A}.
  3. Transmission of one quantum bit: q-bit\textsubscript{A→B}.
  4. Simultaneous, bidirectional transmission of one classical bit and one quantum bit: c-bit\textsubscript{A→B} and q-bit\textsubscript{B→A}.
  5. Simultaneous, bidirectional transmission of two quantum bits: q-bit\textsubscript{A→B} and q-bit\textsubscript{B→A}.

- Analytical characterization of two-qubit gates according to their capability to perform any of the above tasks (section IV).

II. DEFINITIONS AND BASIC FACTS

This section is a prelude providing the definitions and notations that will be used throughout the whole paper and reviews some facts concerning two-qubit gates which will build the basis for our further results. We shall also define the notion of majorization and collect some lemmas linked to it.

A. Two-qubit gates

Consider a system consisting of two two-dimensional subsystems (qubits), A and B. The corresponding Hilbert spaces are \( \mathcal{H}_A \cong \mathbb{C}^2 \) and \( \mathcal{H}_B \cong \mathbb{C}^2 \). The compound Hilbert space is \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \).

By a two-qubit gate \( U \) we understand a unitary operator acting on \( \mathcal{H}_{AB} \). By choosing the global phase appropriately, we can always consider such a unitary to be an element of the group \( su(4, \mathbb{C}) \). We speak of a local two-qubit gate whenever we can write \( U = U_A \otimes U_B \) where \( U_A \) and \( U_B \) are unitary operators acting only on \( \mathcal{H}_A, \mathcal{H}_B \) respectively. Again we can restrict ourselves to local unitaries being elements of \( su(2, \mathbb{C}) \otimes su(2, \mathbb{C}) \). Non-local gates are then trivially two-qubit gates which cannot be written as \( U_A \otimes U_B \).

With just the help of these two definitions we can already divide the set of non-local gates into equivalence classes. Two two-qubit gates \( U \) and \( \tilde{U} \) are said to be locally equivalent if there exist local unitaries \( U_A \otimes U_B \) and \( \tilde{U}_A \otimes \tilde{U}_B \) such that \( \tilde{U} \pm U_A \otimes U_B \tilde{U}_A \otimes \tilde{U}_B \). A useful decomposition of a general two-qubit gate developed in [3] and [5] admits to further characterize these equivalence classes enabling us to easily decide whether two gates are locally equivalent:

**Lemma 1** [3], [5] For any two-qubit gate \( U \) there exist local unitaries \( U_A \otimes U_B \) and \( \tilde{U}_A \otimes \tilde{U}_B \) and a self-adjoint operator of the form \( H = \sum_{k=1}^{4} \alpha_k \sigma_k \otimes \sigma_k \) such that \( U = \tilde{U}_A \otimes \tilde{U}_B e^{-iH} U_A \otimes U_B \).

Here the \( \sigma_k \)s denote the usual Pauli spin matrices. Note that the real numbers \( \alpha_k \) are not unique as long as we do not pose further conditions on them. This is so for two reasons: Firstly operators of the type \( \pm i\sigma_k \otimes \sigma_k \) are local and commute with \( H \) so that we can always extract such a local operator from the local parts in this decomposition and include it in \( H \). This alters the corresponding coefficient \( \alpha_k \) by \( \pi/2 \). Secondly there are certain local transformations of \( H \) which conserve its form but permute the coefficients \( \alpha_k \) and change the sign of two of them. The local unitaries which cause such a transformation are of the types \( \pm i\sigma_k \otimes 1 \) and \( \pm 1 \otimes \sigma_k \). Using this it can easily be checked that it is always possible to bring \( H \) to a form where its coefficients obey the inequalities (see also [5])

\[
\frac{\pi}{4} \geq \alpha_1 \geq \alpha_2 \geq |\alpha_3|.
\]

Note that these conditions are an arbitrary choice and that it might be necessary to relax them when we are looking for optimal simulation protocols. We will come back to this point later on.

We call the decomposition of a two-qubit gate as given in lemma 1 where the coefficients \( \alpha_k \) fulfill (1) its canonical form. The purely non-local unitary \( e^{-iH} \) in this decomposition is termed the interaction content of the gate.

That the non-local characteristics of a two-qubit gate are determined by only three real parameters is a remarkable result in view of the fact that a general element of \( su(4, \mathbb{C}) \) is fixed by 15 independent parameters. It might be mentioned here that while [3] provides a profound Lie-algebraic basis for the decomposition in lemma 1, [5] gives a constructive proof which allows to determine the coefficients \( \alpha_k \) as well as the local unitaries for any given gate. Based on this method we show in appendix A how to derive the \( \alpha_k \) for a given \( U \) without constructing the local unitaries.
A necessary and sufficient criterion for two gates to be locally equivalent is now obviously that they have the same interaction content. By definition it is also clear that any two-qubit gate is locally equivalent to its own interaction content, a fact on which our results concerning simulation of gates heavily rely.

For later use we mention here that self adjoint operators of the form considered in lemma 1 are diagonal in the so called magic basis \([9]\) defined as

\[
\begin{align*}
\text{1)} & = -\frac{1}{\sqrt{3}} (|01\rangle + |10\rangle), \\
\text{2)} & = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle), \\
\text{3)} & = -\frac{1}{\sqrt{3}} (|00\rangle - |11\rangle), \\
\text{4)} & = \frac{1}{\sqrt{3}} (|01\rangle - |10\rangle). 
\end{align*}
\]

Such that we have

\[
H = \sum_{k=1}^{3} \alpha_k \sigma_k \otimes \sigma_k + \sum_{j=1}^{4} \lambda_j |j\rangle \langle j| 
\]

where the eigenvalues \(\lambda_j\) follow from the \(\alpha_k\) by

\[
\lambda_1 = \alpha_1 + \alpha_2 - \alpha_3, \\
\lambda_2 = \alpha_1 - \alpha_2 + \alpha_3, \\
\lambda_3 = -\alpha_1 + \alpha_2 - \alpha_3, \\
\lambda_4 = -\alpha_1 - \alpha_2 + \alpha_3. 
\]

In terms of the \(\lambda_j\) conditions (1) read \(3 \pi / 4 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq -3 \pi / 4\). Note also that the \(\lambda_j\)s sum up to zero (i.e. \(H\) is traceless) such that the corresponding unitary \(U = \exp(-i H)\) is an element of the special unitary group as we have required. In the following we will characterize the interaction content of non-local gates either by the three-vector \(\vec{\alpha} = (a_1, a_2, a_3)\) or by the four vector \(\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) freely switching between the representations. For operators like in (3) we write \(H_{\vec{\lambda}}\) or \(H_{\vec{\lambda}}\) and for the corresponding unitary \(U_{\vec{\alpha}}\) or \(U_{\vec{\lambda}}\).

B. Majorization

The relation of majorization emerged as a powerful tool in the issue of simulation as well as in other fields of quantum information theory. From an intuitive perspective it simply makes a precise statement out of a vague notion that the components of a vector \(\vec{x}\) are "less spread out" or "more equal" than are the components of a vector \(\vec{y}\).

**Definition 2** Let \(\vec{x} = (x_1, \ldots, x_n)\) and \(\vec{y} = (y_1, \ldots, y_n)\) be real vectors whose components are ordered non-increasingly. Then we say that "\(\vec{x}\) majorizes \(\vec{y}\)" and write \(\vec{x} \succ \vec{y}\) if

\[
\begin{align*}
\sum_{i=1}^{k} x_i & \geq \sum_{i=1}^{k} y_i, & k = 1, \ldots, n - 1, \\
\sum_{i=1}^{n} x_i & = \sum_{i=1}^{n} y_i 
\end{align*}
\]

A central result in the theory of majorization is the following:

**Lemma 3** [10] Let \(x\) and \(y\) be defined as before. Then \(\vec{x} \succ \vec{y}\) if and only if there exists a doubly stochastic [15] \(n \times n\) matrix \(Q\) such that \(\vec{y} = Q \vec{x}\).

We will use two facts related to doubly stochastic matrices:

- The first one is called Birkhoff’s theorem and states that the set of doubly stochastic matrices is the convex hull of the permutation matrices. Therefore we can write \(Q = \sum_k P_k\) (the \(P_k\geq 0\) summing up to one and \(P_k\) being permutation matrices) for any doubly stochastic matrix \(Q\).

- If we take the so called Hadamard product of a real orthogonal matrix \(O\) with itself i.e. square it componentwise (written symbolically as \(O \circ O\)) then we get a special type of doubly stochastic matrix called orthostochastic matrix.

Later on we will use this relation to compare 4-vectors \((\tilde{\lambda}, \tilde{\mu}, \tilde{\nu}, \tilde{\pi})\) of the kind introduced in the foregoing section. In related works [2,11] it has already turned out to be convenient to have at hand an equivalent relation for the corresponding 3-vectors \((\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})\) called the (special)-majorization relation. Let \(\tilde{\alpha}\) and \(\tilde{\beta}\) be two real and nonincreasingly ordered 3-vectors. Then \(\tilde{\alpha}\) s-majorizes \(\tilde{\beta}\) \((\tilde{\alpha} \succ_s \tilde{\beta})\) if

\[
\begin{align*}
\alpha_1 & \geq \beta_1 \\
\alpha_1 + \alpha_2 - \alpha_3 & \geq \beta_1 + \beta_2 - \beta_3 \\
\alpha_1 + \alpha_2 + \alpha_3 & \geq \beta_1 + \beta_2 + \beta_3 
\end{align*}
\]

Now let \(\vec{\lambda}\) and \(\vec{\mu}\) be the 4-vectors related to \(\tilde{\alpha}\) and \(\tilde{\beta}\) respectively via (4). Then it is easily verified that \(\vec{\lambda} \succ \vec{\mu}\) if \(\tilde{\alpha} \succ_s \tilde{\beta}\).

The s-majorization relation can be extended to non-ordered vectors as follows. Given a vector \(\vec{x} = (x_1, x_2, x_3)\), we construct a new "s-ordered" vector \(\vec{x}^s = (x_1^s, x_2^s, x_3^s)\) with \(x_1^s \geq x_2^s \geq x_3^s \geq |a_3|\) by first nonincreasingly reordering the modulus of the components \(a_3\), and by then giving \(a_3^s\) the sign of the product \(a_1 a_2 a_3\). Then for any pair of vectors \(\vec{x}\) and \(\vec{y}\), \(\vec{x} \succ_s \vec{y}\) denotes the set of inequalities (5) applied to \(\vec{x}^s\) and \(\vec{y}^s\). We note also that according to the above discussion a gate \(U_{\vec{\lambda}}\) \((\vec{\alpha}\) being an arbitrary 3-vector) is locally equivalent to the gate \(U_{\vec{\lambda}}^s\) corresponding to the s-ordered form of \(\vec{\alpha}\).

III. INTERACTION COSTS OF GATE SIMULATION AND PARTIAL ORDER OF GATES

The main result (theorem 1) in [4] permits to assess the interaction cost (as defined in [4]) for simulating a two-qubit gate using any given interaction Hamiltonian and fast local unitaries analytically after performing a simple optimization. The proof in [4] is based on results
developed in the areas of quantum control [3] and quantum information ([5],[2],[6]). Here we give an alternative proof relying only on the tools introduced so far. We do this by giving a necessary and sufficient condition for the existence of a simulation protocol. Before we state and prove this result we will introduce the problem of simulating a gate (see [2],[12] for a more general discussions) and describe some simplifications that can be assumed in this context.

A. Setting of gate simulation and basic assumptions

Simulating a desired two-qubit gate $U$ using a given interaction described by a Hamiltonian $H$ [16] and arbitrary local unitary transformations means to specify a series of local unitaries $\{U_1 \otimes V_1, \ldots, U_n \otimes V_n\}$ and time intervals $\{t_1, \ldots, t_n\}$ such that

\[ U = \left( U_n \otimes V_n \right) e^{-iH t_n} \left( U_{n-1} \otimes V_{n-1} \right) e^{-iH t_{n-1}} \ldots \left( U_1 \otimes V_1 \right) e^{-iH t_1} \left( U_0 \otimes V_0 \right). \]

(6)

Such a partition of a gate $U$ equals a list of instructions like: "Perform transformation $U_0$ and $V_0$ on qubit A and B respectively. Then let them interact according to $H$ for a time $t_1$. Perform $U_1$ and $V_1$. Let them interact for $t_2$. . . . . Finally perform $U_n$ and $V_n." Following this protocol one would then effectively perform the gate $U$ on the two qubits no matter what their initial state was.

Posing the problem of finding such a simulation protocol naturally evokes other questions: Is there always a solution? How much time will it take to perform a possible simulation protocol? What is the minimal time of simulation? Do we have to allow for infinitesimal time steps? In case we can restrict on taking finite time steps, how many of them will suffice? In the following we will give an answer to all of them.

To do so we adopt two simplifications. At first we employ a physical idealization namely the fast control limit which is well justified in most of the proposed settings for quantum information processing. It states that the control operations - in our case the local unitary transformations - can be executed in times where the natural evolution - here the interaction of the qubits - has no considerable effect on the system's state. In other words local manipulations and interactions have to take place on significantly different times scales. That is what we assume and what allows us to define the simulation time simply as $t_S = \sum_{i=1}^n t_i$ implying that the local transformations in (6) take effectively no time. We term the minimal time $t_S$ such that we can find a simulation protocol its "interaction cost" ($C_I(U)$) because it actually measures the time of interaction required to perform the gate.

The second simplification is of pure mathematical nature and concerns the system's Hamiltonian $H$. Based on results of [2],[6] we use that although a general two-qubit Hamiltonian has the form $H = c_0 1 \otimes 1 + \sum_{i=1}^3 a_i \sigma_i \otimes 1 + \sum_{j=1}^3 b_j \sigma_j \otimes 1 + \sum_{i,j=1}^3 c_{ij} \sigma_i \otimes \sigma_j$ we can restrict ourselves to much simpler Hamiltonians $H_B$ (or equivalently $H_A$) as given in (3). This is due to the fact that for any general Hamiltonian there exists a Hamiltonian $H_B$, called its canonical form, and efficient protocols for simulating the evolution according to the latter in terms of the first. By an efficient simulation protocol we mean that we can obtain the evolution $e^{-iH_B t_B}$ for any time $t_B$ by using $H$ for the same period of time $t$. (Note that such a simulation involves infinitesimal time steps, see [2].) For the purpose of simulation these Hamiltonians are equivalent in the sense that both are equally effective in simulating other Hamiltonians or gates.

B. Necessary and sufficient condition for gate simulation

We are now ready to give a necessary and sufficient condition for the existence of a simulation protocol.

Result 1. Given a two-qubit gate $U$ having $H_A$ interaction content $H_A$ and a Hamiltonian $H$ having a canonical form $H_B$ there exists a simulation protocol of type (6) consuming a total time $t_S \geq 0$ of a vector \( \vec{n} = (n_1, n_2, n_3) \) of integers exists such that $\vec{\beta} \(B_{A} - B_{B} + \pi/2\vec{n} )$ satisfies

\[ \vec{\beta} \approx \beta + \pi / 2 \vec{n} \]

(7)

Proof. We first show that this is a necessary condition. According to the above discussion a simulation protocol for $U$ using $H$ for a time $t$ is equivalent to a protocol for $U_B$ using $H_B$ for the same time $t$. Moreover we can assume that the protocol we have consists entirely of infinitesimal time steps $\delta t$ since any finite time step can be decomposed into infinitesimal ones. Then (6) reads as:

\[ U_B = (U_n \otimes V_n) e^{-iH_B \delta t} (U_{n-1} \otimes V_{n-1}) \ldots (U_1 \otimes V_1) e^{-iH_B \delta t} (U_0 \otimes V_0) \]

Let us assume that at a time $0 \leq t \leq t_S$ we perform the $k^{th}$ intermediate local transformation having then attained an effective transformation $U_k = (U_k \otimes V_k) e^{-iH_B \delta t} \ldots (U_1 \otimes V_1) e^{-iH_B \delta t} (U_0 \otimes V_0)$. Since $U_k$ is itself a gate, we can decompose it as $U_k = U_k \otimes U_k \otimes V_k \otimes V_k$ where $U_k = e^{-iH_B \delta t}$ is the interaction content of $U_k$. The index $k$ indicates the time dependence of all these unitaries.

To determine how $\vec{\beta}$ varies with $t$ we take the next infinitesimal time step $e^{-iH_B \delta t}$ in the protocol and get:

\[ e^{-iH_B \delta t} U_k = e^{-iH_B \delta t} U_k \otimes U_k \otimes V_k \otimes V_k \]

\[ = U_{k+\delta t} \otimes U_{k+\delta t} \otimes U_{k+\delta t} \otimes U_{k+\delta t}. \]
For convenience we change here to the 4-vector representation [as defined in (4)]. Denote by $\mathcal{L}, \mathcal{V}, \tilde{x}$ the vectors corresponding to $\gamma, \tilde{n}^a, \tilde{a}^a, t$ respectively. After local transformations the last identity can be written as

$$e^{-iH_\lambda \delta t} U \otimes \mathcal{V} \mathcal{U}_\lambda = W \otimes X \mathcal{U}_\lambda Y \otimes Z$$

(8)

where $W \otimes X$ and $Y \otimes Z$ are appropriately defined local unitaries and all time indices are omitted. The right hand side of (8) is a decomposition of the left hand side, but we do not require this to be the canonical form as defined in section IIIA. We therefore have the possibility to put further conditions on the unitaries in this decomposition.

If we multiply from the left by $U^\dagger \otimes V^\dagger$ and sandwich this equation between $|k\rangle$, one of the magic states, we find

$$\langle \psi_k | e^{-iH_\lambda \delta t} | \psi_k \rangle e^{-i\omega_k} = \langle \psi_k | W \otimes X \mathcal{U}_\lambda Y \otimes Z | k \rangle$$

(9)

where $| \psi_k \rangle := U \otimes V | k \rangle$. In order to have equality for $\delta t = 0$ we make use of the above mentioned freedom and require for this case $W \otimes X = U \otimes V, Y \otimes Z = 1 \otimes 1$ and $\xi = \tilde{\nu}$.

For infinitesimal $\delta t$ we can thus expand

$$\langle \psi_k | W \otimes X = | k \rangle + [\delta k^L]$$

$$Y \otimes Z | k \rangle = | k \rangle + [\delta \tilde{k}^L]$$

$$\xi = \tilde{\nu} + \delta \tilde{\nu}.$$

where we may assume $\langle \delta k^L | k \rangle = 0$. Combining everything in (9) and collecting terms up to first order we find

$$\langle \psi_k | H_\lambda | \psi_k \rangle \delta t = \delta \omega_k$$

which has to hold for all $k$.

Let us now take a closer look at the diagonal elements $\langle \psi_k | H_\lambda^2 | \psi_k \rangle$. With regard to the definition $| \psi_\lambda \rangle$ and now again including the time dependence of $U_t \otimes V_t$ we have

$$\langle \psi_\lambda | H_\lambda^2 | \psi_\lambda \rangle = | k \rangle \langle k | (U_t \otimes V_t)^\dagger H_\lambda (U_t \otimes V_t) | k \rangle.$$

In the magic basis local unitaries take on the form of real orthogonal matrices $[U_t \otimes V_t]^\dagger \rightarrow O(t)$ and the Hamiltonian gets diagonal $[H_\lambda \rightarrow D_\lambda := \text{diag}(\lambda)]$. Therefore $\delta \omega_k = \delta t [D_\lambda O(t)] k \rangle = \delta t [(O \circ O) \tilde{x}]_k$ where $(O \circ O)$ denotes the Hadamard product of the real orthogonal matrix $O(t)$ with itself. Defining $Q(t) := (O \circ O) t) we can write compactly

$$\frac{\delta \omega_k}{\delta t} = Q(t) \tilde{x}.$$

(10)

Recall that $\tilde{x}^a$ is the variation of the interaction content at some intermediate time $0 \leq t \leq t_s$ in our simulation protocol. The overall interaction content $\tilde{v}(t_s)$ is found by integrating (10) from $0$ to $t_s$. As initial condition we have $\tilde{v}(0) = 0$ since our simulation protocol starts from the identity having no interaction content. We then find

$$\tilde{v}(t_s) = \int_0^{t_s} Q(t) dt \tilde{x} = S \tilde{x} t_s$$

where $S := 1/t_s \int_0^{t_s} Q(t) dt$ is again a doubly stochastic matrix. To see this observe

$$\sum_{j=1}^n S_{jk} = 1/t_s \int_0^{t_s} \sum_{j=1}^n Q(t)_{jk} dt = 1/t_s \int_0^{t_s} dt = 1.$$

The same holds for summation over $k$.

With lemma 3 we can state that $\tilde{v}(t_s) \sim \tilde{\alpha} t_s$ or switching again to the 3-vector representation $\gamma(t_s) \sim \alpha t_s$ [see the definitions preceding equation (8)]. Remember that our basic assumption was that we have a simulation protocol for a gate $U = U \otimes \mathcal{V} \mathcal{U}_\lambda \otimes \tilde{V}$. However, by means of $\gamma(t_s)$ we can add $\alpha$ - possibly different - decomposition since $U = U_{ts} \otimes U_{t_s}$. From the discussion in section IIIA we know that the vectors $\tilde{v}$ and $\tilde{\alpha}_s$ have to be related via the local operations specified there. There are two operations that can be done to alter $\tilde{\beta}$: (i) add multiples of $\pi/2$ to its components, i.e. build $\tilde{\beta}' = \tilde{\beta} + \pi/2 \tilde{\alpha}$ for a vector $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$; and (ii) permute and simultaneously change the sign of two components, which can be expressed easily by multiplication with an appropriate matrix $P$. Therefore we must have $\tilde{\gamma}(t_s) = P \tilde{\beta}' \sim \alpha t_s$ for some $P$ and $\tilde{\alpha}$. Recalling the definition of s-ordering of vectors [see (5) and the remarks there] we find $(P \tilde{\beta}')_s = (\tilde{\beta}'_s)$ and therefore $\tilde{\alpha}_s \sim \alpha t_s$.

We now turn to the second part of our proof and show sufficiency. Since this has already been proven in [11] we will just sketch this proof. Let $\tilde{\mu}$ and $\tilde{\lambda}$ be the 4-vectors corresponding to $\tilde{\beta}$ and $\tilde{\alpha}$. Then (7) reads as $\tilde{\mu} = \lambda t_s$ and it follows by Birkoff's theorem (see section IIIB) that we can write $\tilde{\mu} = \sum_{i=1}^n U_{i} \tilde{\lambda}_i t_s$ and correspondingly $\tilde{\alpha}_s \sim \alpha t_s$. Then each of the $4^l = 24$ permutations $P_t$ of the magic states $\{\psi_\lambda\}$ can be performed through appropriate local unitaries $U_t \otimes V_t$ we have

$$U_{t_s} = e^{-iH_\lambda} = \exp \left( -i \sum_{i=1}^n H_{P_{i} \lambda} \right)$$

$$= \exp \left( -i \sum_{i=1}^n U_{i} \otimes V_{i} H_{X} U_{i}^\dagger \otimes V_{i}^\dagger t_{i} \right)$$

$$= \prod_{i=1}^n U_{i} \otimes V_{i} e^{-iH_{P_{i} \lambda} t_{i}} U_{i}^\dagger \otimes V_{i}^\dagger.$$

For the last line we took into account that $[U_{i} \otimes V_{i} H_{X} U_{i}^\dagger \otimes V_{i}^\dagger, U_{j} \otimes V_{j} H_{X} U_{j}^\dagger \otimes V_{j}^\dagger] = 0 \ \forall i, j$ since the local transformations involved only permute the eigenvectors of $H_{X}$. The last line provides clearly a proper simulation protocol for $U_{t_s}$ and - by applying appropriate local unitaries $U_t \otimes V_t$ at the beginning and at the end - for all locally equivalent gates (including $U_{t_s}$).

We remark here that [11] shows how to find explicitly the probability distribution $\{P_t\}$ and permutations $\{P_t\}$
which determine the time steps \( \{ t_i \} \) and the local unitaries \( \{ U_i \otimes V_i \} \). There also the maximal number \( n \) of evolution steps sufficient in any simulation protocol was determined. It turned out to be three for time optimal protocols.

This condition for the simulation of gates is an analogue to the one established in \([2]\) for efficient Hamiltonian simulation. Such a correspondence was, in principle, only expected for infinitesimal gates. It is remarkable that it extends in such a tight analogy to finite gates. The main difference is that here we have to include all different decompositions of the gate under consideration by allowing for variations \( \tilde{\beta}_\theta = \beta + \pi/2 \tilde{n} \). There is no analog to this in the case of Hamiltonian simulation. The reason for this is that here we have to accommodate the periodicity properties of unitary operators while in the setting of Hamiltonian simulation we deal with a linear space of Hermitian operators.

C. Interaction costs

To finally assess the interaction cost \( C_H(U) \) - i.e. the minimal time to simulate \( U \) using \( H \) and local unitaries as defined in \([4]\) - we just have to optimize condition (7) with respect to both \( t_S \) and \( \tilde{n} \). Doing so we reproduce the main result of \([4]\):

**Result 2** The interaction cost \( C_H(U) \) is the minimal value of \( t_S \geq 0 \) such that either \( \tilde{\beta}_{(0,0,0)} \prec_s \tilde{\alpha}_{t_S} \) or \( \tilde{\beta}_{(-1,0,0)} \prec_s \tilde{\alpha}_{t_S} \) holds.

**Proof** This is equivalent to result 1 under the restriction that it suffices to look at \( \tilde{n} \) being \((0,0,0)\) or \((-1,0,0)\) to find the smallest \( t_S \). This is because in case \( \tilde{n} \) is not one of these two vectors we can show that either \( \tilde{\beta}_{(0,0,0)} \prec_s \tilde{\beta}_0 \) or \( \tilde{\beta}_{(-1,0,0)} \prec_s \tilde{\beta}_0 \). For the minimal time \( t_S \) such that \( \tilde{\beta}_0 \prec_s \tilde{\alpha}_{t_S} \) for a given \( \tilde{n} \) we therefore essentially have either \( \tilde{\beta}_{(0,0,0)} \prec_s \tilde{\alpha}_{t_S} \) or \( \tilde{\beta}_{(-1,0,0)} \prec_s \tilde{\alpha}_{t_S} \) for the same time \( t_S \). Obviously letting \( \tilde{n} \) be \((0,0,0)\) or \((-1,0,0)\) will make for at least the same minimal time. The optimization for \((0,0,0)\) or \((-1,0,0)\) cannot be avoided since in general \( \tilde{\beta}_{(0,0,0)} \) and \( \tilde{\beta}_{(-1,0,0)} \) are incomparable according to the s-majorization relation. To show that either \( \tilde{\beta}_{(0,0,0)} \prec_s \tilde{\beta}_0 \) or \( \tilde{\beta}_{(-1,0,0)} \prec_s \tilde{\beta}_0 \) for all \( \tilde{n} \) different than \((0,0,0)\) or \((-1,0,0)\) we distinguish two cases.

(i) First we look at vectors \( \tilde{n} \) having at least one component \( |n_j| > 1 \).

Since the components of \( \tilde{\beta} \) have to fulfill (1) the maximal component of the reordered form of \( \tilde{\beta}_0 \) (see section II A) is at least \( 3 \pi/4 \). We then have \( \tilde{\beta}_0 \prec_s (3\pi/4, 0, 0) \) and this last vector clearly s-majorizes both \( \tilde{\beta}_{(0,0,0)} \) and \( \tilde{\beta}_{(-1,0,0)} \). (ii) The vectors \( \tilde{n} \) satisfying \( |n_j| \leq 1 \) have to be checked case by case. We find \( \tilde{\beta}_{(-1,0,0)} \prec_s \tilde{\beta}_0 \) for \( \tilde{n} \in \{ (-1, -1, -1), (0, 1, 0), (0, 0, -1), (0, 0, 1) \} \) and \( \tilde{\beta}_{(0,0,0)} \prec_s \tilde{\beta}_0 \) for the remaining \( \tilde{n} \).

Let us formulate result 2 as a kind of recipe. In order to time optimally perform a gate \( U \) using an interaction described by a Hamiltonian \( H \) together with arbitrary local unitaries proceed as follows:

1. Determine \( \tilde{\beta} \) characterizing the interaction content of \( U \) following \([5]\) (see also appendix A). Using \([2],[6]\) compute the canonical form of \( H \) to get \( \tilde{\alpha} \).

2. Test whether \( \tilde{\beta} \) or \( \tilde{\beta}_{(-1,0,0)} \) is s-majorized by \( \tilde{\alpha}_{t_S} \) for a smaller time \( t_S \).

3. For the vector yielding the better result as well as for \( \tilde{\alpha} \) compute the corresponding 4-vectors \( \tilde{\mu} \) and \( \lambda \) respectively. Following \([11]\) find the permutations \( P_i \) and probabilities \( p_i \) \((i = 1, 2, 3)\) such that

\[
\tilde{\mu} = \sum_{i=1}^{3} p_i P_i \tilde{\lambda}.
\]

4. The \( p_i \) determine the time steps \( t_i \) and the \( P_i \) give the local unitaries to be applied in between. This provides a simulation protocol for \( U_\beta \) using \( H_\lambda \) for at most 3 finite time steps.

5. Simulate the evolutions according to \( H_\lambda \) by using the Hamiltonian \( H \) for the same period of time following \([2]\). Apply appropriate local unitaries (determined using \([5]\)) in the beginning and at the end of the overall simulation to effectively perform \( U \).

We now discuss certain special cases for which some of the above points can be dropped or get simpler.

- In case the Hamiltonian we use describes solely pure interaction, that is to say is of the form

\[
H = \sum_{i\neq j} \alpha_{ij} \sigma_i \otimes \sigma_j
\]

without any local parts, we can attain its canonical form by a local transformation

\[
H_\lambda = U \otimes V H U^\dagger \otimes V^\dagger
\]

(see \([2]\)). Since \( e^{-iH}\lambda_{t_S} = U \otimes V e^{-iM t^\dagger} U^\dagger \otimes V^\dagger \) we do not have to employ infinitesimal simulations (as required in step 5.) and the simulation protocol will only contain 3 finite time steps.

- In case the interaction content of the desired gate is characterized by a vector \( \tilde{\beta} = (\beta_1, \beta_2, \beta_3) \) satisfying \( \tilde{\beta} \prec_s \tilde{\beta}_{(-1,0,0)} \) we can skip the optimization (step 2.) and state directly: The interaction cost \( C_H(U) \) is the minimal value of \( t_S \) such that \( \tilde{\beta} \prec_s \tilde{\alpha}_{t_S} \).

The condition \( \tilde{\beta} \prec_s \beta_{(-1,0,0)} \) to be true is \( \beta_1 + |\beta_3| \leq \pi/4 \). To see this we have to apply the inequalities (5) defining the s-majorization to \( \tilde{\beta} \) and \( \beta_{(-1,0,0)} \) to be true is \( \beta_1 + |\beta_3| \leq \pi/4 \). The equality \( \tilde{\beta} \prec_s \beta_{(-1,0,0)} \) to be true is \( \beta_1 + |\beta_3| = \pi/4 \). The first inequality is fulfilled trivially since \( \beta_1 \leq \pi/4 \) in any case. The last two inequalities are equivalent to \( \beta_1 + |\beta_3| \leq \pi/4 \) and this is what we claimed.
The reverse $\bar{\beta}_{(-1,0,0)} <_s \bar{\tilde{\beta}}$ is never true because the first inequality is violated for any $\bar{\tilde{\beta}}$. In all the other cases where $\bar{\tilde{\beta}}$ and $\bar{\beta}_{(-1,0,0)}$ are incomparable it will depend on the Hamiltonian which of the two vectors yields the optimal time.

D. Interaction costs of basic gates

As an illustration we shall give here explicitly the interaction costs for three specific gates (CNOT, D(ouble)CNOT, SWAP) and for the whole class of controlled-$U$ gates. We choose these ones not only because they play a prominent role in quantum information but also due to their role as “landmarks” in the set of two-qubit gates as we will show in the next section. Let us list them here by first giving their definition in terms of their action on the computational basis $\{(i, j)\}_{i, j = 0, 1}$, then characterizing their interaction content by the corresponding vector $\bar{\beta}$ and finally assessing the interaction costs pursuant to a general Hamiltonian $H$ with canonical form $H_{\beta}$.

1. CNOT gate and controlled-$U$ gates

The C(ontrolled)NOT gate is the prototypical two-qubit quantum logic gate. Its action is defined compactly as $|i\rangle_A \otimes |j\rangle_B \rightarrow |i\rangle_A \otimes |i + j\rangle_B$ where $\otimes$ denotes addition modulo 2. That is, it flips the second (target) qubit iff the first (control) qubit is in state $|1\rangle$. Let us denote the CNOT gate by $U_{\text{CNOT}}^{AB}$ where the first superscript indicates the control and the second the target qubit. In appendix A we show that the interaction content of this gate is given by $\bar{\beta} = \pi/4(|0\rangle, 0, 0)$. Therefore the CNOT belongs to the special class of gates where we can skip the optimization in result 2 and go straight ahead to majorization in order to determine the interaction cost. Requiring $\bar{\beta} <_s \bar{\tilde{\beta}}$ is equivalent to:

$$\pi/4 \leq \alpha_1 t_S$$
$$\pi/4 \leq (\alpha_1 + \alpha_2 \pm \alpha_3) t_S$$

Clearly the first inequality yields the tighter bound. The interaction cost for simulating a CNOT is $C_H(\text{CNOT}) = \frac{\pi}{\frac{\pi}{4} + \alpha_1}$. The CNOT is a representative of the general class of controlled-$U$ gates. These gates apply a unitary operation on the target qubit iff the control qubit is in state $|1\rangle$. Thus they have the form

$$U_{\text{ctrl-}U} = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes U.$$
\[ V \leq U \equiv C_H(V) \leq C_H(U) \quad \forall H. \]

**Result 3** Let \( U \) and \( V \) be two two-qubit gates with corresponding ordered vectors \( \vec{\beta}_1 \) and \( \vec{\beta}_2 \) such that in both cases the restriction \( \beta_1 + |\beta_2| \leq \pi/4 \) holds. Then gate \( U \) is more non-local than gate \( V \) if and only if \( \vec{\beta}_1 < \vec{\beta}_2 \).

**Proof.** Since both vectors \( \vec{\beta} \) satisfy \( \beta_1 + |\beta_2| \leq \pi/4 \) the interaction costs \( C_H(V) \) and \( C_H(U) \) are given, respectively, by the smallest \( t_V, t_U \geq 0 \) such that

\[ \vec{\beta}_1 < \vec{\beta}_2, \quad \vec{\beta}_2 < \vec{\beta}_1. \]

Suppose first \( V \leq U \), that is, for any Hamiltonian \( H \) we have \( C_H(V) \leq C_H(U) \) and in particular \( \vec{\beta}_V <_\beta \vec{\beta}_U \). If we rewrite this relation for the particular Hamiltonian where \( \vec{\alpha} = \vec{\beta}_1 \) and use that in this case \( C_H(U) = 1 \) we find \( \vec{\beta}_V <_\beta \vec{\beta}_U \). This proves the direct implication. The inverse follows right away by using the partial order property of majorization. \( C_H(U) <_\beta C_H(V) \) directly implies \( C_H(U) \geq C_H(V) \) (see the proof of result 2).

Once more coming back to the problem of Hamiltonian simulation we mention that the corresponding partial order there has been solved completely [2]. The reason why the partial order established in result 3 only holds in the region of gates where \( \beta_1 + |\beta_2| \leq \pi/4 \) is again that we have to deal here with the rather involved periodic structure of \( su(4) \). It is exactly this restricted region where we can evade this difficulty by suppressing the otherwise essential optimization between \( \vec{\beta}_{(0,0,0)} \) and \( \vec{\beta}_{(-1,0,0)} \) (step 2 in the recipe given in section III C).

**IV. TRANSMISSION OF INFORMATION AND CLASSES OF GATES**

By now we analysed two-qubit gates in terms of the time expense they cause in the context of simulation. There the main objective is to perform a given gate on two qubits using a minimum time of interaction seen as a valuable resource. The notion of interaction cost thereby obtained gave a measure for how non-local (either relative to a specific interaction or absolute as in result 3) a gate is. In this section we change the perspective. We now want to prescribe the tasks a gate has to accomplish and ask how non-local it therefore has to be. In this setting we consider the gate and its inherent non-locality to be the valuable resource. The task we have in mind here is the transmission of information in form of classical as well as quantum bits.

This section is organized as follows: First we motivate why the capability of gates to transmit bits is a proper measure for their non-locality. After having given some basic definitions, we collect a number of known results for certain gates. Then we treat the problems of transmitting a qubit or a qubit in one direction as well as all possible combinations of them in both directions by using a two-qubit gate and determine the interaction content necessary to do so. The subsequent discussion of the results will allow us to distinguish various classes of gates differing in their capability for quantum-communication tasks which will give a characterization of the non-locality of a gate as well.

**A. Transmission capability and non-local content of gates**

Non-local gates result physically from an interaction taking place between the qubits by some means. Interaction between two physical systems conditions on the other hand the transmission of information between them since after having interacted (at least one of the subsystem's states will have changed depending on the states of both subsystems as they were before the interaction. Hence there must have been some kind of information exchange in the process of interaction. It is therefore natural to ask whether we can utilise a non-local gate to send (classical or quantum) information. The amount of information we can transmit using a gate will give us then a characterization of its degree of non-locality. A similar point of view was captured in [7], [8] where the amount of classical and quantum information necessary to implement a gate was adapted as a measure for its non-local content.

What do we mean by the transmission of classical or quantum information? Consider two parties Alice and Bob holding a qubit \( A \) and \( B \) respectively. Assume further that somehow they manage to perform a gate \( U \) on their qubits. Then we say that \( U \) allows for the transmission of a classical-bit from Alice to Bob (denoted by \( \text{cbit}_{A \rightarrow B} \)) if after the application of \( U \) Bob can distinguish with probability 1 whether Alice's qubit was in \( |0\rangle \) or \( |1\rangle \). We speak of the transmission of a quantum-bit from Alice to Bob (\( \text{qubit}_{A \rightarrow B} \)) if under the action of \( U \) Bob's qubit takes on the state of Alice's qubit.

Let us make some remarks here. (i) The essential difference between these two effects of a gate is, that in the case of \( \text{cbit}_{A \rightarrow B} \) we do not require superpositions of \( |0\rangle \) and \( |1\rangle \) to be transmitted faithfully whereas in the case \( \text{qubit}_{A \rightarrow B} \) we do. The possibility to send a qubit trivially includes the one to transmit a cbt resembling the fact that quantum information incorporates classical information. (ii) Without further specifying \( U \) we can state directly that in case \( \text{cbit}_{A \rightarrow B} \) Alice loses her state after sending it due to the no-cloning theorem. (iii) If Alice's qubit is maximally entangled to some ancilla qubit on her side then the transmission \( \text{qubit}_{A \rightarrow B} \) swaps the entanglement thus establishing a maximally entangled pair of qubits (\( \text{e-bit} \)) between Alice and Bob. That is why the authors of [7], [8] identified the capabilities of a gate to send a qubit and to create an e-bit. Here we want to distinguish between the actual creation
of entanglement without ancilla systems as treated in [3] and entanglement swapping by the transmission of a qubit. This differentiation is essential for example in the case of a CNOT gate which can be used to create an e-bit \( U_{CNOT}^{AB} |i\otimes i\rangle = |i\rangle \otimes |i\rangle \) but not to transmit a qubit as we shall show in the following.

For the gates introduced in section III D it is well known and easy to see how they can be used to transmit bits. Regarding the definitions given there the following is effortlessly verified:

- \( U_{\text{CNOT}}^{AB} |i\rangle = |ii\rangle, i = 0, 1 \) and therefore the CNOT is sufficient to send a qubit from Alice to Bob. Since Alice's qubit does not change at all under the action of this gate it is impossible for her to send a qubit to Bob (see remark (ii) above). This is not true if Alice and Bob share entanglement as an additional resource. See the remark below.

- \( U_{\text{DCNOT}}^{AB} |\varphi 0\rangle = |0\varphi\rangle \) where \( |\varphi\rangle \) is an arbitrary qubit state transmitted by the action of \( U_{\text{DCNOT}}^{AB} \). Moreover we find \( U_{\text{DCNOT}}^{AB} |\varphi 1\rangle = |1\rangle \otimes \sigma_x |\varphi\rangle \) telling us that Bob may send at the same time a qubit to Alice under the condition that in case he sent \( |1\rangle \) he flips his qubit after the transmission in order to recover the correct state \( |\varphi\rangle \). Since he knows what he sent, as we can assume, this requires no additional communication.

- \( U_{\text{SWAP}}^{AB} |\varphi \psi\rangle = |\varphi \psi\rangle \) where \( |\varphi\rangle \) and \( |\psi\rangle \) are arbitrary states both being transmitted faithfully.

We can summarize this by the implications:

CNOT \( \rightarrow \text{cbit}_{A \rightarrow B} \)

DCNOT \( \rightarrow \text{qubit}_{A \rightarrow B} + \text{cbit}_{B \rightarrow A} \)

SWAP \( \rightarrow \text{qubit}_{A \rightarrow B} + \text{qubit}_{B \rightarrow A} \)

Obviously, due to the symmetry of the non-local content of two-qubit gates under exchange of parties, the same expressions hold if we make the substitutions A \( \rightarrow \) B. These relations hold strictly for the case where the communicating parties have no ancilla systems and no prior entanglement at hand, but have to be regarded as lower bounds on the capabilities of these gates to transfer information if we allow for additional resources of this kind. It is a central result in quantum information science that the capacities to transmit information can be increased if the parties possess shared entanglement (e-bits) [7],[8].

B. Transmission of information in the context of gate simulation

Assume now Alice and Bob want to send some given amount of information (possibly in both directions) by using some fixed interaction described by a Hamiltonian \( H \) and arbitrary local transformations of their qubits. They could do so by choosing appropriately one of the above gates providing the necessary transmission capability and then simulate it according to the results we derived so far. The interaction costs thereby incurred are given in section III D. But is this optimal? There might be gates which are suitable for the same task but have an interaction content different from the ones of CNOT, DCNOT or SWAP yielding smaller interaction costs. In the following we want to single out which gate is both sufficient for a certain transmission task and optimal in terms of interaction costs. We do this by deriving necessary and sufficient conditions on the interaction content of a gate to be capable of the transmission of a given amount of information. All we have to do then is to find the gate which fulfills the appropriate condition and causes the minimal interaction cost.

1. cbit_{A \rightarrow B}

Assume Alice encodes a classical bit into her qubit by preparing it in \( |0\rangle \) or \( |1\rangle \) and Bob holds some arbitrary state \( |\varphi\rangle \). Then the bit is by definition transmitted if after an application of a gate \( U \) Bob's qubit takes on a state \( |\psi\rangle \) or \( |\psi^\perp\rangle \) (some state orthogonal to \( |\psi\rangle \)) depending on whether Alice sent "0" or "1". At the same time Alice's qubit may change arbitrarily. The action of \( U \) we have to require is described by

\[
\begin{align*}
|0\varphi\rangle &\rightarrow |\chi\psi\rangle \\
|1\varphi\rangle &\rightarrow |\bar{\psi}\psi^\perp\rangle.
\end{align*}
\]

More precisely we can state: A necessary condition for a gate \( U \) to be capable of transmitting a bit is that there exist states \( |\varphi\rangle, |\chi\rangle, |ar{\psi}\rangle, |\psi\rangle \) and \( |\psi^\perp\rangle \) such that relations (11) hold. Assume now that this is indeed the case. What can we say about the interaction content of \( U \)? Since independent local transformations before and after the application of \( U \) do not affect its interaction content, we can look for unitaries fulfilling \( Z |\varphi\rangle = |\bar{\psi}\rangle, Y |\chi\rangle = |\bar{\psi}\rangle, X |\psi\rangle = |0\rangle \) and \( X |\psi^\perp\rangle = |1\rangle \) and define \( \mathcal{U}' = (X_A \otimes Y_B)U(X_A \otimes Z_B) \) having a simpler action given by

\[
\begin{align*}
|00\rangle &\rightarrow |00\rangle \\
|10\rangle &\rightarrow |11\rangle.
\end{align*}
\]

where \( |0\rangle = Y |\bar{\psi}\rangle \). \( \mathcal{U}' \) and \( U \) are locally equivalent and therefore have the same interaction content. To derive conditions on this interaction content we apply \( \mathcal{U}' \) to the state \( \phi := c_1 |\bar{\psi}\rangle + c_2 |0\rangle \) - transforming under the terms of (12) - and take the partial trace with respect to system \( A \):

\[
\text{tr}_A \{ \mathcal{U}' \rho \mathcal{U}'^\dagger \} = \frac{1}{2} \text{tr} \{ |00\rangle\langle 00| + |11\rangle\langle 11| \} = \frac{1}{2} I_B.
\]
When we on the other hand assume a decomposition $\mathcal{U}' = (\widetilde{U} \otimes \widetilde{V})\mathcal{U}_3(\widetilde{U} \otimes \widetilde{V})$ we find

$$tr_A \left\{ \mathcal{U}'_g\mathcal{U}'^\dagger \right\} = \frac{1}{2} tr_A \left\{ \left[ \widetilde{V}_B\mathcal{U}'_{gB}\mathcal{V}_B^\dagger \right] 1_A \otimes [\mathcal{U}'_{gB}\mathcal{V}_B^\dagger] \right\},$$

(14)

Equating the right hand sides of (13) and (14) and multiplying from the left by $\widetilde{V}_B^\dagger$ and from the right by $\widetilde{V}_B$ yields

$$1_B = tr_A \left\{ \mathcal{U}'_{gB} 1_A \otimes |\omega\rangle_B (\omega|\mathcal{U}'_{gB}^\dagger) \right\}$$

where we have abbreviated $V|0\rangle = |\omega\rangle$. Expressing without loss of generality $|\omega\rangle = \cos(\omega)|0\rangle + e^{i\phi} \sin(\omega)|1\rangle$ one can work out the trace explicitly and finds

$$1_B = \begin{pmatrix} 1 - a & b \\ b^* & 1 + a \end{pmatrix}$$

(15)

$$a = \cos(2\omega) \cos(2\beta_1) \cos(2\beta_2)$$

$$b = \sin(2\omega) \cos(2\beta_1) \cos(2\beta_2) + i \sin(\theta) \sin(2\beta_1).$$

Let us stop here and consider what equation (15) tells us. The left hand side was an immediate consequence of the necessary conditions on $\mathcal{U}$ to properly transmit a qubit while the right hand side results from the general ansatz $\mathcal{U}' = (\widetilde{U} \otimes \widetilde{V})\mathcal{U}_3(\widetilde{U} \otimes \widetilde{V})$ where the unitary $\widetilde{V}$ contains the parameters $\omega$, $\theta$ and $\beta = (\beta_1, \beta_2, \beta_3)$ characterizes the interaction content $\mathcal{U}_3$. Equation (15) thus puts certain conditions on the parameters in the decomposition of $\mathcal{U}'$. Obviously we have to require $a = b = 0$. This in turn is fulfilled in various cases, for example whenever two of the coefficients $\beta_k = \pi/4$, the third being arbitrary. However it is also easy to see that there are solutions, where only one of the coefficients $\beta_k = \pi/4$. In this case we have to choose either $\omega = 0$ appropriately. This puts conditions on the state $|\varphi\rangle$ in (11) denoting the input state Bob has to choose in order to properly receive the qubit Alice aims to send him. Three solutions of this kind are for example given by $\{\beta_1 = \pi/4, \omega = 0\}$, $\{\beta_2 = \pi/4, \omega = 0\}$ and $\{\beta_3 = \pi/4, \omega = \pi/4\}$ where in each case the remaining parameters can be chosen arbitrarily. All in all we have shown that it is a necessary condition for the transmission of a qubit to have at least one of the coefficients $\beta_k$ equal to $\pi/4$ and without loss of generality we can always require this to be $\beta_1$.

To be systematic we should now continue and show, that any gate characterized by a vector $\vec{\beta} = (\pi/4, \beta_2, \beta_3)$ is also sufficient for this task. But at this point we will not do so for two reasons. Firstly we already know that an interaction content $\vec{\beta} = (\pi/4, 0, 0)$ is sufficient to transmit a qubit because this basically fixes a CNOT or any gate locally equivalent to a CNOT. Secondly we find $(\pi/4, 0, 0) \prec (\pi/4, \beta_2, \beta_3)$ for all $0 \leq |\beta_2| \leq \beta_3 \leq \pi/4$ and therefore $\mathcal{G}_H(CNOT) \leq \mathcal{G}_H(\mathcal{U}_{\pi/4, \beta_2, \beta_3})$ for all $H$. Thus looking for gates other than ones out of the CNOT-class has no advantage in terms of interaction costs. Let us state this as

**Result 4** The cheapest (time optimal) way to transmit a qubit using some given interaction is to simulate a CNOT gate. The interaction cost is $C_B(\text{cbit} \rightarrow \text{b}) = \frac{1}{16}$.

The following results will show that the transmission capability scales up with the coefficients $\beta_k$ becoming bigger. Just by continuity it follows then right away that any gate having an interaction content $\vec{\beta} = (\pi/4, \beta_2, \beta_3)$ is also sufficient to transmit at least a qubit.

2. $\text{cbit} \rightarrow \text{b}$ and $\text{cbit} + \text{a}$

Again let Alice encode a logical bit into her qubit as $|0\rangle$ or $|1\rangle$. Further assume Bob wants to send “0” and therefore prepares $|0\rangle$. To properly transmit their two messages they have to find a gate, which transforms the states like

$$|00\rangle \rightarrow |\varphi\rangle$$

$$|10\rangle \rightarrow |\varphi\rangle^\perp.$$ (16)

To detect the messages being sent to him, Bob has to measure the observable $\sigma_z = \frac{1}{2}(|\chi\rangle \langle \chi| - |\chi^\perp\rangle \langle \chi^\perp|)$. Conversely Alice has to measure $\sigma_x$ or $\sigma_{\phi}$ (defined similarly) depending on whether her message was “0” or “1”. Consider now the same situation but let Bob’s message be “1”. The same reasoning as before yields

$$|11\rangle \rightarrow |\varphi\rangle_{\chi}$$

(17)

Now Bob has to measure $\sigma_x$. The transformation behavior characterized so far lacks one of essential condition: it is not unitary. Unitary transformations map an orthonormal basis into another one and this is so long not fulfilled, since $\langle \varphi\rangle |\psi\rangle_{\chi} = 0$. Imposing that the vectors on the right hand side of (16) and (17) build again a basis one finds four possible cases: (i) $\langle \varphi\rangle |\psi\rangle_{\chi} = 0$ and $|\psi\rangle_{\chi} = 0$, (ii) $|\chi\rangle |\psi\rangle_{\chi} = 0$ and $|\psi\rangle_{\chi} = 0$, (iii) $|\varphi\rangle |\psi\rangle_{\chi} = 0$ and $|\chi\rangle |\psi\rangle_{\chi} = 0$ and (iv) $|\varphi\rangle |\psi\rangle_{\chi} = 0$ and $|\chi\rangle |\psi\rangle_{\chi} = 0$. The last two cases are more restrictive than (i) and (ii) since there the states of both qubits have to meet certain conditions. We are however interested to stay as less restrictive as possible so that we are going to focus on (i) in which case we have to require $|\psi\rangle = e^{-i\alpha} |\varphi\rangle$ and $|\psi\rangle = e^{-i\beta} |\varphi\rangle$. Let us summarize what we have found so far:

$$|00\rangle \rightarrow |\varphi\rangle$$

$$|10\rangle \rightarrow e^{-i\alpha} |\varphi\rangle^\perp$$

$$|01\rangle \rightarrow |\varphi\rangle_{\chi}$$

$$|11\rangle \rightarrow e^{-i\beta} |\varphi\rangle_{\chi}^\perp.$$ Including the phases into $|\chi\rangle$ and $|\chi^\perp\rangle$ and again adjusting the axes by local transformations to cleanse the notation (as we did for the cbit$_{A \rightarrow B}$-problem) we can
write equivalently
$$
\begin{align*}
|00\rangle & \rightarrow |00\rangle \\
|10\rangle & \rightarrow |01\rangle \\
|01\rangle & \rightarrow |1\rangle \omega \\
|11\rangle & \rightarrow |\omega^1\rangle \\
\end{align*}
$$

(18)

We can see that Bob has to measure a different observable depending on what he sent. For the case (ii) above we would find similar transformations but then being Alice the one who has to adapt her observable. Therefore case (i) gets identical with (ii), if we let Alice and Bob exchange their names which in turn cannot have any relevance for the interaction content of the gate they use. Or more mathematically: (i) can be transformed into (ii) by conjugating the gate with the SWAP and this does not alter the interaction content.

We can now parametrize $|\omega\rangle = \cos(\omega)|0\rangle + e^{-i\phi}\sin(\omega)|1\rangle$ and $|\omega^1\rangle = e^{-i\theta}(-\sin(\omega)|0\rangle + e^{i\phi}\cos(\omega)|1\rangle)$

and determine the interaction content of the gate

$$
U(\eta, \theta, \omega) = e^{-i\pi/4}e^{(i\eta+\phi)/4} \times
\begin{pmatrix}
e^{-i(\theta+\phi)}\cos(\omega) & 0 & e^{-i\phi}\sin(\omega) \\
-e^{-i(\theta+\phi)}\sin(\omega) & 0 & \cos(\omega) \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

(19)

written in the computational basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ in this order. The global phase assures $U(\eta, \theta, \omega)$ being a special unitary operator. Following appendix A one finds for the vector $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ characterizing the interaction content $U_\beta$ of $U(\eta, \theta, \omega)$

$$
\begin{align*}
\beta_1 &= \pi/4 \\
\beta_2 &= \pi/4 \\
\beta_3 &= \pi/4 + \phi
\end{align*}
$$

where $\phi$ is a solution to $\tan^2(2\phi) = \sec^2\left(\frac{\pi}{4}\right)\sec^2(\omega) - 1$. $\phi$ therefore parametrizes a family of gates, of which each element has the desired capability to transmit $\text{cbit}_{A+B}$.

Note especially that the DCNOT $\vec{\beta} = (\pi/4, \pi/4, 0)$ and the SWAP $\vec{\beta} = (\pi/4, \pi/4, \pi/4)$ belong to this family as we should expect according to the discussion in section III D. These gates are attained for the choice $\theta = \pi/4$ and $\phi = 0$ respectively. In terms of $\{\eta, \theta, \omega\}$ this corresponds i.e. to set $\{\eta = \pi, \theta = 0, \omega = \pi/2\}$ and $\{\eta = 0, \theta = 0, \omega = 0\}$ for the DCNOT and the SWAP respectively yielding the expected result when inserted in (19).

If we want to transmit the cbits using some given interaction we can freely choose the parameter $\phi$ out of $[0, \pi/2]$ in order to keep down the interaction costs. Let us present the optimal choice in

\begin{itemize}
\item \textbf{Result 5} The cheapest (time optimal) way to transmit cbits in both directions using some given interaction is to simulate a gate holding an interaction content $\vec{\beta} = \frac{\pi}{4}(1, 1, 2\omega)$. The corresponding interaction cost is $C_H(\text{cbit}_{A+B}) = \frac{\pi}{2} \frac{1}{1+\omega^2}$.
\end{itemize}

\begin{proof}
Define $b := 1/2 - 2/\pi \cdot \phi$ and parametrize $\vec{\beta}(\phi) = \frac{\pi}{4}(1, 1, 2b)$. We have to find $b \in [-1/2, 1/2]$ and $t_\beta \geq 0$ such that either $\vec{\beta}(b) \prec_3 \vec{\alpha}_t$ or $\vec{\beta}_{(-1,0,0)}(b) \prec_3 \vec{\alpha}_t$ holds and $t_\beta$ is minimal. First note that $\vec{\beta}_{(-1,0,0)}(b) = \vec{\beta}(-b)$. The optimization with respect to $b$ therefore includes the one with respect to $\vec{\beta}$ and $\vec{\beta}_{(-1,0,0)}$. The minimal time such that $\vec{\beta}(b) \prec_3 \vec{\alpha}_t$ is fulfilled is given by $t_{\text{min}}(b) = \max\left\{\frac{\pi}{4}, \frac{1}{2\omega}, \frac{1}{1+\omega}, \frac{1}{1+\omega^2}\right\}$. Optimization with respect to $b$ yields the interaction cost $C_H(\text{cbit}_{A+B}) = \min_b t_{\text{min}}(b) = \min_{b \in [-1/2, 1/2]} \max\left\{\frac{1}{4}, \frac{1}{2\alpha}, \frac{1}{1+\omega}, \frac{1}{1+\omega^2}\right\}$.

This is an exercise in linear optimization which has to be solved under the condition $\pi/4 \geq \alpha_1 \geq \alpha_2 \geq |\alpha_3|$. An elementary calculation yields $C_H(\text{cbit}_{A+B}) = \frac{\pi}{2} \frac{1}{1+\omega^2}$ for $b = \frac{\alpha_3}{\alpha_1 + \alpha_2}$.

3. $\text{qubit}_{A+B}$ and $(\text{qubit}_{A+B}$ and $\text{cbit}_{B+A})$

To reliably transmit a qubit we have to require
$$
\begin{align*}
|00\rangle & \rightarrow |\phi\chi\rangle \\
|10\rangle & \rightarrow |\phi^+\chi^+\rangle
\end{align*}
$$

The remaining vectors $|01\rangle$ and $|11\rangle$ may transform arbitrarily but have to stay orthogonal to both among themselves and with respect to $|\phi\chi\rangle$ and $|\phi^+\chi^+\rangle$. The least restrictive choice yields similar to the following section
$$
\begin{align*}
|01\rangle & \rightarrow |\phi^+\omega\rangle \\
|11\rangle & \rightarrow |\phi^{-}\omega^+\rangle
\end{align*}
$$

Without loss of generality we can identify $|\phi\rangle = |0\rangle$, $|\phi^+\rangle = |1\rangle$, $|\chi\rangle \equiv |0\rangle$ and $|\chi^+\rangle \equiv |1\rangle$ ending up with the same gate (18) as for $\text{cbit}_{A+B}$. The optimal interaction content and cost to send a $\text{qubit}_{A+B}$ is therefore the same as in result 5.

Regarding the transformations given in (18) it is obvious that this gate is also capable to send at the same time a $\text{cbit}_{B+A}$. To do so Bob encodes his bit into $|0\rangle$ or $|1\rangle$. Applying $H$ sends the bit to Alice. The qubit Bob gets from Alice comes in faithfully if Bob sent “0”. In the other case he has to recover the qubit by a local transformation obeying $V(\omega) = |0\rangle$ and $V(\omega^+\rangle = |1\rangle$. An interaction content $\vec{\beta} = (\pi/4, \pi/4, \pi/4 + \phi)$ is therefore sufficient for the transmission $\text{qubit}_{A+B}$ and $\text{cbit}_{B+A}$. This is also necessary since any interaction content showing less than $\pi/4$ in the first two entries is not sufficient
to send a qubit\(^{A\rightarrow B}\). Again we can refer to the values given in result 5 for the optimal interaction content and cost.

4. qubit\(^{A\rightarrow B}\)

This problem is trivial since the exchange of the two quantum states completely fixes the transformation of the basis states and therefore also the gate. The SWAP is the only gate providing the required action. Interaction content and cost are given in section III D.

C. Classes of gates

Let us summarize the results of the foregoing sections in the following table:

<table>
<thead>
<tr>
<th>interaction content</th>
<th>transmission capability</th>
<th>interaction cost (\mathcal{C}_P(U))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Controlled-(U) (x\ 0\ 0)</td>
<td>(\times)</td>
<td>(\times)</td>
</tr>
<tr>
<td>CNOT (\pi/4\ 0\ 0)</td>
<td>(\checkmark)</td>
<td>(\times)</td>
</tr>
<tr>
<td>I (\pi/4\ \ z)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>DCNOT (\pi/4\ \pi/4\ \ 0)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>II (\pi/4\ \pi/4\ \ z)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>SWAP (\pi/4\ \pi/4\ \pi/4)</td>
<td>(\checkmark)</td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

What we can see, is that the capability of a gate to transmit information increases when the coefficients \(\beta_k\) characterizing its interaction content approach their maximal values \(\pi/4\). Especially when one of them takes on this maximum value, the corresponding gate acquires a new feature. The special gates CNOT, DCNOT, and SWAP (and all their local equivalents) mark these thresholds and that is why we announced them being “landmarks” in the set of two-qubit gates. This allows us to distinguish four classes of gates differing in their transmission capability: (i) gates with \(\pi/4 > \beta_1 > \beta_2 > |\beta_3|\) (no transmission capability), (ii) CNOT and type I (iii) DCNOT and type II and (iv) SWAP. This classification endows the coefficients \(\beta_k\) with physical significance and therefore complements earlier work, where a gate’s interaction content \(U_3\) was associated with its capability to create entanglement [5].

V. CONCLUSIONS

In this work we addressed the problem of simulating two-qubit gates using some given interaction and local unitary transformations in the fast control limit. For this to be possible we presented a necessary and sufficient condition linking the gate, the Hamiltonian characterizing the interaction and the total time of simulation. Optimization with respect to time gave a measure \(\mathcal{C}_H(U)\) - termed interaction cost - for how costly such a simulation in terms of time of interaction is and thereby recovered a result already attained in [4]. The interaction cost has been computed for various gates and was shown to induce a partial order in a region of the set of two-qubit gates thus establishing a meaningful notion of and measure for the non-locality of a gate.

To give an application as well as a supplementation of these results we then turned to the problem of transmitting information between two parties using two-qubit gates. Necessary and sufficient conditions on gates were established to be capable of transferring classical and quantum bits in all combinations and directions. This allowed us to compute explicitly the interaction costs for these tasks. Beyond it the transmission capability of a gate provided a classification of two-qubit gates.

All results derived here concern two-qubit systems. All the underlying problems can naturally be extended to higher dimensional systems and therefore it would be desirable to generalize the results. The main obstacle to do so is that in higher dimensions there is no decomposition like in (1) for a general unitary operator.

VI. ACKNOWLEDGMENTS

K. H. would like to thank Barbara Kraus for kind and generous help and the referee of this paper for a hint to simplify the proof of result 1. We thank C.H. Bennett, A. Harrow, D.-W. Leung and J. A. Smolin for communications about their results on the use of bipartite Hamiltonians to communicate information [13]. This work was supported by the European Community project EQUIP (contract IST-1999-11053) and by the National Science Foundation of USA, grant No. EIA-0086038.
APPENDIX A: INTERACTION CONTENT OF NON-LOCAL GATES

In lemma 1 we presented a decomposition for two-qubit gates of the form \( \mathcal{U} = \tilde{U}_A \otimes \tilde{V}_B e^{-iH} \tilde{U}_A \otimes \tilde{V}_B \) where \( H = \sum_{k=1}^{\frac{3}{2}} \alpha_k \sigma_3 \). Here we demonstrate a method based on [5] to determine the \( \alpha_k \) for a general given \( \mathcal{U} \).

In section II A we gave an alternative representation of \( H \) in terms of its eigenvalues \( \lambda_k \). The method actually admits to compute the \( \lambda_k \) and relies on the following two observations: (i) Hamiltonians of the special form considered here are diagonal in the magic basis as we have already shown in section II A. (ii) Local unitaries are real in the magic basis [9]. Especially they become real orthogonal matrices since of course they stay to be unitary. This fact resembles the homomorphism \( su(2, \mathbb{C}) \times su(2, \mathbb{C}) \approx SO(4, \mathbb{R}) \) [14] becoming manifest in the magic basis. Using these two facts the decomposition takes on the form \( \mathcal{U} = \tilde{O} \tilde{D} \tilde{O} \) when written in the magic basis where \( \tilde{D} = \text{diag}(e^{-i\lambda_1}, e^{-i\lambda_2}, e^{-i\lambda_3}, e^{-i\lambda_4}) \) and \( \tilde{O} \) and \( \tilde{O} \) are real orthogonal matrices corresponding to \( \tilde{U}_A \otimes \tilde{V}_B \) and \( \tilde{U}_A \otimes \tilde{V}_B \). Therefore \( \mathcal{U}^T \mathcal{U} \approx \tilde{D} \tilde{D}^T \tilde{O} \tilde{O} \) hence, if we compute the eigenvalues of \( \mathcal{U}^T \mathcal{U} \) we will find them to be \( \{ e^{-2i\lambda_1}, e^{-2i\lambda_2}, e^{-2i\lambda_3}, e^{-2i\lambda_4} \} \). Taking the arguments of these phases and dividing by two will give us the \( \lambda_k \)'s and via (4) the \( \alpha_k \)s.

As an example let us determine the \( \alpha_k \)s for the CNOT gate. In the computational basis [in the order \( |11\rangle , |10\rangle , |01\rangle , |00\rangle \) and the magic basis [the order given by the enumeration in (2)] we find respectively

\[
\mathcal{U}_{CNOT}^{AB} = e^{-i\pi/4} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{CB}
\]

\[
= \frac{e^{-i\pi/4}}{2} \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ -1 & -i & 1 & -i \\ -i & i & -1 & i \end{pmatrix}_{MB}
\]

\[
= \frac{e^{-i\pi/4}}{2} \begin{pmatrix} 1 & -i & 1 & -i \\ 1 & i & 1 & i \\ -1 & -i & 1 & -i \\ -i & i & -1 & i \end{pmatrix}_{MB}
\]

The overall phase included assures that \( \det(\mathcal{U}_{CNOT}) = 1 \) and therefore \( \mathcal{U}_{CNOT} \in su(4) \). The eigenvalues of \( \mathcal{U}_{CNOT}^{AB} \) turn out to be \( \{ 1, i, -i, -1 \} \). Taking the square root and then ordering the arguments in decreasing order we find \( \lambda = \pi/4 (1, 1, 1, -1) \). Solving equations (4) we get \( \alpha = \pi/4(1, 0, 0) \).

However in some cases simple algebraic considerations provide a more elegant way to find the interaction content. We shall demonstrate this on the basis of the class of controlled-U gates. These gates are of the form \( \mathcal{U}_{ctrl-U}^{AB} = P_B P_{1} \otimes \mathbb{1} \) where \( P_1 = |\tilde{U}_A \rangle \langle \tilde{U}_A | \otimes \mathbb{1} \) as we mentioned in section III D. If we now take the transpose \( \mathcal{U}_{ctrl-U}^{T} \) in the magic basis and take into account that \( P_B^T = P_B \) and \( \mathbb{1} \otimes U^T = \mathbb{1} \otimes U \) we find

\[
\mathcal{U}_{ctrl-U}^{T} \mathcal{U}_{ctrl-U} = (P_B^T P_B^{1} \otimes U^T)(P_B P_{1}^{1} \otimes U) = P_B P_{1}^{1} \otimes U^{T} + P_B P_{1} \otimes U = |0\rangle \langle 0| \otimes U^{T} + |1\rangle \langle 1| \otimes U.
\]

This operator is block diagonal in the computational basis and therefore has the same eigenvalues as \( U \) with multiplicity 2, i.e., has a spectrum \( \{ e^{\pm i\phi}, e^{\mp i\phi}, e^{\pm i\phi}, e^{\mp i\phi} \} \) where \( e^{\pm i\phi} \) are the eigenvalues of \( U \). Solving equations (4) for \( \lambda_1 = \lambda_3 = \beta, \lambda_2 = \lambda_4 = -\beta \) we find \( \alpha = (\beta, 0, 0) \) and \( \beta \leq \pi/4 \). For the CNOT we have especially \( U = \sigma_x \) and thus \( \beta = \pi/4 \) as it shall be.

[12] P. Wojcik, M.Rötteler, D. Janzing and T.Beth, Quant...

[15] A matrix is called doubly stochastic if its entries are all nonnegative and each row and column adds up to one.

[16] Throughout the paper we put $\hbar = 1$ and consider time to be dimensionless.