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CRITICAL EIGENVALUES FOR A NONLINEAR PROBLEM

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Abstract

In this work, we give the explicit solutions of the problem

$$(P_\lambda) \begin{cases} \Delta u + \lambda(1 + u) = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = 0 \text{ on } \partial B_1, \end{cases}$$

where $B_1 = \{x \in \mathbb{R}^3, \|x\| < 1\}$, and λ is a positive constant.

The supremum λ^* of values such that (P_λ) admits a solution is equal to π^2 ; we show that the problem (P_{π^2}) doesn't admit a solution (see Section 3).

We study the application

$$\lambda^* :]1, \infty[\longrightarrow \mathbb{R}_+,$$

where $\lambda^*(\alpha)$ is the supremum of positive values such that the problem

$$(P_\lambda^\alpha) \begin{cases} \Delta u + \lambda(1 + u)^\alpha = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = 0 \text{ on } \partial B_1, \end{cases}$$

admits a solution. We show that λ^* is a decreasing function, with

$$\lim_{\alpha \rightarrow \infty} \lambda^*(\alpha) = 0 \text{ and } \lim_{\alpha \rightarrow \infty} u_{\lambda^*(\alpha)}(0) = +\infty.$$

where $u_{\lambda^*(\alpha)}$ is the unique solution of the problem $(P_{\lambda^*(\alpha)}^\alpha)$ (see Section 4).

In the last section, we give a numerical approximation of $\lambda^*(2)$.

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1 Introduction

In this work, we deal with the following problem

$$(P_\lambda^\alpha) \begin{cases} \Delta u + \lambda(1+u)^\alpha = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = 0 \text{ on } \partial B_1, \end{cases}$$

where B_1 is the open unit ball of $\mathbb{R}^n$, $n \geq 3$, $\lambda > 0$ and $\alpha \geq 1$.

The problem (P_λ^α) is a particular form of the following general one

$$\begin{cases} Lu = -\sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) + c(x)u(x) = \lambda f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a regular bounded domain of $\mathbb{R}^n$, L is uniformly elliptic in Ω and $f(x, 0) > 0$, for $x \in \Omega$.

The previous problem arises in many mathematical models (combustion, thermal explosions, nonlinear heat generation and gravitational equilibrium of polytropic stars, see [2] for detailed references).

It is known (see [8],[2],[1] and [5]), that for every $\alpha > 1$, there exists a critical constant $\lambda^*(\alpha) > 0$, such that the problem (P_λ^α) admits two ordered solutions if $0 < \lambda < \lambda^*(\alpha)$, only one solution if $\lambda = \lambda^*(\alpha)$ and no solution if $\lambda > \lambda^*(\alpha)$.

Our aim is to get more details about the application

$$\lambda^* : [1, \infty[\rightarrow \mathbb{R}+.$$

We have already pointed out concrete situations where problem (P_λ^α) arises, let us add that if we know the exact value of $\lambda^*(\alpha)$, we can show (for instance) that the problem

$$\begin{cases} \Delta u + a(x)(1+u)^\alpha = 0 \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

doesn't admit a solution, if $a(x) > \lambda^*(\alpha)$ for every $x \in \overline{\Omega}$, and a is regular enough on Ω .

2 Some Previous Results

Let us consider the problem (P_λ^α) on Ω instead of B_1 , $\alpha > 1$. Let u_λ be a solution of this problem. If we put $v_\lambda = \lambda^{\frac{1}{\alpha-1}} u_\lambda + \lambda^{\frac{1}{\alpha-1}}$, then v_λ verifies

$$\begin{cases} \Delta v_\lambda + v_\lambda^\alpha = 0 \text{ in } \Omega, \\ v_\lambda > 0 \text{ in } \Omega, \\ v_\lambda = \lambda^{\frac{1}{\alpha-1}} = \epsilon \text{ on } \partial\Omega. \end{cases}$$

So one can study the previous Dirichlet's problem (with constant boundary data) to get existence results and critical boundary constants $\epsilon^*(\Omega, \alpha)$, such that, if $\epsilon > \epsilon^*(\Omega, \alpha)$, the previous problem doesn't admit a solution. One can study this problem with more general nonlinearities $f(x, u)$,

on adequate domains. This was the method in [5].

Let us choose

$$\Omega = B_r = \{x \in \mathbb{R}^n, \|x\| < r\}, \quad r > 0,$$

and let $\lambda_r^*(\alpha)$ be the supremum of values λ for which the previous problem admits a solution, then we have

Lemma 1

$$\lambda_r^*(\alpha) = r^{-2} \lambda_1^*(\alpha), \quad \epsilon^*(B_r, \alpha) = r^{\frac{2}{1-\alpha}} \epsilon^*(B_1, \alpha)$$

and

$$\lambda_1^*(\alpha) = \epsilon^*(B_1, \alpha)^{\alpha-1}.$$

Proof

See [5] or [6].

The previous lemma gives a reason to deal firstly with the case $\Omega = B_1$.

Let us recall the Sobolev critical exponent case,

Proposition 1 *Consider the problem (P_λ^α) , if*

$$\alpha = \frac{n+2}{n-2}, \quad \text{then } \lambda^*(\alpha) = \frac{n(n-2)}{4}.$$

Proof

See Proposition 1 in [7].

Proposition 2 *Every C^2 -solution u of (P_λ^α) is radially symmetrical with respect to the origin and decreasing.*

Proof

See Theorem 1 in [3].

Proposition 3 *For every λ such that $0 < \lambda \leq \lambda^*(\alpha)$, let u_λ be the smaller solution of (P_λ^α) , then u_λ is analytic.*

Proof

We know from Proposition 2, that u_λ is spherically symmetrical. We also know (see [9], Theorem 3.2, p. 227) that

$$u_\lambda(r) = \lim_{i \rightarrow \infty} u_\lambda^{(i)}(r), \quad 0 \leq r \leq 1,$$

where $u_\lambda^{(i)}$ is the solution of the linear problem

$$(P_{\lambda,i}^\alpha) \left\{ \begin{array}{l} \Delta u_\lambda^{(i)} + \lambda \left(1 + u_\lambda^{(i-1)}\right)^\alpha = 0 \text{ in } B_1, \\ u_\lambda^{(i)} > 0 \text{ in } B_1, \\ u_\lambda^{(i)} = 0 \text{ on } \partial B_1, \end{array} \right.$$

with $u_\lambda^{(0)} \equiv 0$.

It is easy to verify that $u_\lambda^{(1)}$ is a radially symmetrical, decreasing analytic function of $r = \|x\|$, so are $u_\lambda^{(2)}$ and (by induction) $u_\lambda^{(i)}$ for every $i > 1$ (see Theorem 1' in [3], p.221). As the sequence $u_\lambda^{(i)} \leq u_\lambda^{(i+1)}$, one can use Dini's Lemma to conclude that u_λ is analytic.

3 Explicit Solutions for $\alpha = 1$, $n = 3$.

Let us consider the problem (P_λ) again with $n = 3$,

Proposition 4 *For every $0 < \lambda < \pi^2$, the problem (P_λ) admits the unique radially symmetrical solution*

$$u_\lambda(r) = \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda})} \frac{\sin r \sqrt{\lambda}}{r \sqrt{\lambda}} - 1, \quad 0 < r \leq 1, \quad \text{and} \quad u_\lambda(0) = \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda})} - 1.$$

$$\lim_{\lambda \rightarrow \pi^2} u_\lambda(r) = +\infty, \quad \text{for every } 0 \leq r < 1.$$

If $\lambda \geq \pi^2$, the problem (P_λ) doesn't admit a solution.

Proof

Let λ be a positive real such that the problem (P_λ) admits a solution u_λ . The unicity of such a solution was proved in [8]. From Proposition 3, we infer that u_λ is radially symmetrical and analytic. So we can rewrite (P_λ) as follows

$$(P'_\lambda) \begin{cases} u_\lambda''(r) + \frac{2}{r} u_\lambda'(r) + \lambda(1 + u_\lambda(r)) = 0 & \forall 0 < r < 1, \\ u_\lambda(0) = c_0, \quad u_\lambda'(0) = 0, \quad u_\lambda(1) = 0. \end{cases}$$

So we obtain that

$$r(u_\lambda''(r) + \lambda(1 + u_\lambda(r))) = -2u_\lambda'(r), \quad 0 \leq r < 1.$$

If

$$u_\lambda(r) = \sum_{j=0}^{\infty} c_j r^j \quad \text{on } [0, 1],$$

we get that

$$(j+1)(j)c_{j+1} + \lambda c_{j-1} = -2(j+1)c_{j+1}, \quad \forall j \geq 2 \quad \text{and} \quad 2c_2 + \lambda(1 + c_0) = -4c_2.$$

From these equalities, we obtain

$$c_j = -\frac{\lambda}{(j+1)j} c_{j-2}, \quad \forall j \geq 3 \quad \text{and} \quad c_2 = -\frac{\lambda(1 + c_0)}{6}.$$

As $c_1 = u_\lambda'(0) = 0$, we infer that $c_j = 0$ for every odd integer j and if j is even

$$c_j = \frac{(-\lambda)^{\frac{j-2}{2}} c_2}{(j+1)(j) \dots (4)} = \frac{(-\lambda)^{\frac{j}{2}} (1 + c_0)}{(j+1)!}.$$

Finally, we get

$$u_\lambda(r) = \sum_{j=0}^{\infty} \frac{(-\lambda)^j (1 + c_0)}{(2j+1)!} r^{2j} - 1.$$

When $j = 0$, $\frac{(-\lambda)^j(1+c_0)}{(2j+1)!} = 1 + c_0$, that's why we have the term -1, on the right, in the expression of u_λ .

We can rewrite u_λ as follows

$$u_\lambda(r) = \frac{(1+c_0)}{r\sqrt{\lambda}} \sum_{j=0}^{\infty} \frac{(-1)^j (r\sqrt{\lambda})^{2j+1}}{(2j+1)!} - 1.$$

We can identify the well-known Taylor's series of the function $\sin(r\sqrt{\lambda})$ and get

$$u_\lambda(r) = \frac{(1+c_0)}{r\sqrt{\lambda}} \sin(r\sqrt{\lambda}) - 1 ; \quad u_\lambda(1) = 0 = \frac{(1+c_0)}{\sqrt{\lambda}} \sin(\sqrt{\lambda}) - 1.$$

We infer that

$$c_0 = \frac{\sqrt{\lambda} - \sin(\sqrt{\lambda})}{\sin(\sqrt{\lambda})} = \frac{\sqrt{\lambda}}{\sin(\sqrt{\lambda})} - 1.$$

Substituting in the expression of u_λ , we obtain

$$u_\lambda(r) = \frac{1}{r \sin(\sqrt{\lambda})} \sin(r\sqrt{\lambda}) - 1 = \frac{\sqrt{\lambda}}{\sin \sqrt{\lambda}} \frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}} - 1.$$

As $u_\lambda(r) \geq 0$ on $[0, 1]$, one can verify that necessarily $\lambda < \pi^2$.

Let us consider the function

$$\begin{cases} g : [0, \pi] \rightarrow \mathbb{R} \\ g(x) = \frac{\sin x}{x}, \text{ if } x \neq 0, \quad g(0) = 1. \end{cases}$$

On $]0, \pi]$, the function g is C^∞ and verifies $g'(x) = \frac{x \cos x - \sin x}{x^2}$. As

$(x \cos x - \sin x)' = -x \sin x \leq 0$, on $]0, \pi]$, it is straightfoward to verify that g is a decreasing function on $[0, \pi]$, with $g(\pi) = 0$. As

$$u_\lambda(r) = \frac{\sqrt{\lambda}}{\sin \sqrt{\lambda}} \frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}} - 1 = \frac{\frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}}}{\frac{\sin(\sqrt{\lambda})}{\sqrt{\lambda}}} - 1 = \frac{g(r\sqrt{\lambda})}{g(\sqrt{\lambda})} - 1 \text{ and } r\sqrt{\lambda} \leq \sqrt{\lambda},$$

we infer that

$$\forall 0 < \lambda < \pi^2, \quad u_\lambda(r) = \frac{g(r\sqrt{\lambda})}{g(\sqrt{\lambda})} - 1 \geq 0, \quad \forall 0 \leq r \leq 1.$$

It is immediate to verify that

$$\lim_{\lambda \rightarrow \pi; \lambda < \pi} u_\lambda(r) = +\infty, \quad \forall 0 \leq r < 1.$$

We conclude that $\lambda^*(1) = \pi^2$ and there is no solution for the problem $(P_{\lambda^*(1)})$.

Remark 1 The problem $(P_{\lambda^*(1)})$ doesn't admit a solution, whereas it is known that the problem $(P_{\lambda^*(\alpha)})$ admits a unique solution, if $\alpha > 1$.

4 Some Properties of the Application λ^*

Let $u_{\lambda^*(\alpha)}$ be the unique solution of the problem $(P_{\lambda^*(\alpha)}^\alpha)$, when $\alpha > 1$.

Proposition 5 *The application*

$$\lambda^* :]1, \infty[\longrightarrow \mathfrak{R}_+,$$

is decreasing and verifies

$$\lim_{\alpha \rightarrow \infty} \lambda^*(\alpha) = 0 \text{ and } \lim_{\alpha \rightarrow \infty} u_{\lambda^*(\alpha)}(0) = +\infty.$$

Proof

Let $1 < \alpha_1 < \alpha_2$ and $u_{\lambda^*(\alpha_2)}$ be the unique solution of

$$\begin{cases} \Delta u + \lambda^*(\alpha_2)(1 + u)^{\alpha_2} = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = 0 \text{ on } \partial B_1. \end{cases}$$

As $\alpha_2 > \alpha_1$, using the Maximum Principle, we get that

$$0 = \Delta u_{\lambda^*(\alpha_2)} + \lambda^*(\alpha_2)(1 + u_{\lambda^*(\alpha_2)})^{\alpha_2} > \Delta u_{\lambda^*(\alpha_2)} + \lambda^*(\alpha_2)(1 + u_{\lambda^*(\alpha_2)})^{\alpha_1} \text{ in } B_1.$$

So we infer that $u_{\lambda^*(\alpha_2)}$ is a supersolution of the problem

$$\begin{cases} \Delta v + \lambda^*(\alpha_2)(1 + v)^{\alpha_1} = 0 \text{ in } B_1, \\ v > 0 \text{ in } B_1, \\ v = 0 \text{ on } \partial B_1. \end{cases}$$

We can choose the function $\phi \equiv 0$ as a subsolution for the previous problem and obtain (by Nagumo's Theorem) a solution of this problem. Using the maximality of $\lambda^*(\alpha_1)$ and the uniqueness of u_{α_1} , we infer that

$$\lambda^*(\alpha_2) < \lambda^*(\alpha_1).$$

As we have $\lambda^*(\alpha_2) < \lambda^*(\alpha_1)$, we know that there exists a “small” solution w such that

$$\begin{cases} \Delta w + \lambda^*(\alpha_2)(1 + w)^{\alpha_1} = 0 \text{ in } B_1, \\ w > 0 \text{ in } B_1, \\ w = 0 \text{ on } \partial B_1. \end{cases}$$

We have already mentioned that for every λ , the “small” solution, when it exists, verifies

$$u_\lambda(r) = \lim_{j \rightarrow \infty} u_\lambda^{(j)}(r),$$

where $u_\lambda^{(0)} \equiv 0$ and $u_\lambda^{(j)}$ is the solution of the linear problem $(P_{\lambda,j}^\alpha)$. Let us show that $u_{\lambda^*(\alpha_2)} \geq w$, on B_1 .

$$\begin{cases} \Delta \left(u_{\lambda^*(\alpha_2)}^{(1)} - w \right) = -\lambda^*(\alpha_2) \left((1 + w)^{\alpha_2} - 1 \right) \leq 0 \text{ in } B_1, \\ u_{\lambda^*(\alpha_2)}^{(1)} - w = 0 \text{ on } \partial B_1. \end{cases}$$

The Maximum Principle implies that

$$u_{\lambda^*(\alpha_2)}^{(1)} \geq w, \text{ on } B_1.$$

One can reiterate the previous argument to show that

$$u_{\lambda^*(\alpha_2)}^{(j)} \geq w, \text{ on } B_1, \forall j \geq 1,$$

so we deduce that

$$(1) \quad u_{\lambda^*(\alpha_2)} \geq w, \text{ on } B_1, \forall \alpha_2 \geq \alpha_1.$$

As the function λ^* is a positive decreasing function on $]1, \infty[$, there are only two possible cases

$$\begin{cases} (i) & \lim_{\alpha \rightarrow \infty} \lambda^*(\alpha) = 0, \\ \text{or} \\ (ii) & \lim_{\alpha \rightarrow \infty} \lambda^*(\alpha) = l > 0. \end{cases}$$

Let us suppose that (ii) is true, so we infer that

$$l < \lambda^*(\alpha), \forall \alpha > 1.$$

Let us choose a fixed $\alpha_1 > 1$, if w is the “small” solution of

$$\begin{cases} \Delta h + l(1+h)^{\alpha_1} = 0 \text{ in } B_1, \\ h > 0 \text{ in } B_1, \\ h = 0 \text{ on } \partial B_1. \end{cases}$$

Using the inequality (1), we infer that

$$u_{\lambda^*(\alpha)} \geq w, \forall \alpha \geq \alpha_1,$$

where $u_{\lambda^*(\alpha)}$ is the unique solution of

$$\begin{cases} \Delta u + \lambda^*(\alpha)(1+u)^\alpha = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = 0 \text{ on } \partial B_1. \end{cases}$$

Let G be the Green’s function of the first kind for $-\Delta$, on B_1 (see [4], p. 19), we have the following identity

$$u_{\lambda^*(\alpha)}(x) = \lambda^*(\alpha) \int_{B_1} G(x, y) \left(1 + u_{\lambda^*(\alpha)}(y)\right)^\alpha dy, \forall x \in \overline{B_1}.$$

Let ϵ and δ be two positive fixed constants, such that $0 < \epsilon < 1 - \delta < 1$, consider the set

$$B(\epsilon, \delta) = \{x \in B_1, \epsilon < \|x\| < 1 - \delta\}.$$

The set $B(\epsilon, \delta)$ is not empty. Let us choose $x_0 \in B(\epsilon, \delta)$, so we infer that

$$\frac{1}{\lambda^*(\alpha)} = \frac{\int_{B_1} G(x_0, y) \left(1 + u_{\lambda^*(\alpha)}(y)\right)^\alpha dy}{u_{\lambda^*(\alpha)}(x_0)} \geq \frac{\int_{B(\epsilon, \delta)} G(x_0, y) \left(1 + u_{\lambda^*(\alpha)}(y)\right)^\alpha dy}{u_{\lambda^*(\alpha)}(x_0)}.$$

As we have (see Lemma 1)

$$u_{\lambda^*(\alpha)}(x_0) = u_{\lambda^*(\alpha)}(\|x_0\|) \leq \epsilon^*(B_{\|x_0\|}, \alpha) \leq \|x_0\|^{\frac{2}{1-\alpha}} \epsilon^*(B_1, \alpha).$$

We get that

$$\frac{1}{\lambda^*(\alpha)} \geq \frac{\int_{B(\epsilon, \delta)} G(x_0, y) (1 + u_{\lambda^*(\alpha)}(y))^\alpha dy}{\|x_0\|^{\frac{2}{1-\alpha}} \epsilon^*(B_1, \alpha)}.$$

We deduce that

$$\frac{\epsilon^*(B_1, \alpha)}{\lambda^*(\alpha)} \geq \inf_{z \in B(\epsilon, \delta)} \left(1 + u_{\lambda^*(\alpha)}(z)\right)^\alpha \frac{\int_{B(\epsilon, \delta)} G(x_0, y) dy}{\|x_0\|^{\frac{2}{1-\alpha}}}.$$

We use Lemma 1 again, to verify that

$$(2) \quad \frac{1}{(\lambda^*(\alpha))^{1-\frac{1}{\alpha-1}}} \geq \inf_{z \in B(\epsilon, \delta)} \left(1 + u_{\lambda^*(\alpha)}(z)\right)^\alpha \frac{\int_{B(\epsilon, \delta)} G(x_0, y) dy}{\|x_0\|^{\frac{2}{1-\alpha}}}.$$

But for every $\alpha \geq \alpha_1$, we have

$$\inf_{z \in B(\epsilon, \delta)} \left(1 + u_{\lambda^*(\alpha)}(z)\right)^\alpha \geq \inf_{z \in B(\epsilon, \delta)} (1 + w(z))^\alpha.$$

As w is radially symmetrical and decreasing, we have

$$\inf_{z \in B(\epsilon, \delta)} (1 + w(z))^\alpha = (1 + w(1 - \delta))^\alpha.$$

The Maximum Principle guarantees that, $w(1 - \delta) > 0$. So we get that

$$\lim_{\alpha \rightarrow \infty} (1 + w(1 - \delta))^\alpha = +\infty,$$

as

$$\lim_{\alpha \rightarrow \infty} \frac{1}{(\lambda^*(\alpha))^{1-\frac{1}{\alpha-1}}} = \frac{1}{l} > 0$$

and

$$\lim_{\alpha \rightarrow \infty} \|x_0\|^{\frac{2}{1-\alpha}} = 1,$$

we get a contradiction from (2). This completes the proof of the first part of Proposition 4.

As we have

$$u_{\lambda^*(\alpha)}(x) = \lambda^*(\alpha) \int_{B_1} G(x, y) (1 + u_{\lambda^*(\alpha)}(y))^\alpha dy, \quad \forall x \in \overline{B_1}.$$

Let us choose $x = O$, using the fact that $u_{\lambda^*(\alpha)}$ is radially symmetrical and has its maximum at the origin, we infer that

$$\frac{1}{\lambda^*(\alpha)} \leq \frac{(1 + u_{\lambda^*(\alpha)}(0))^\alpha}{u_{\lambda^*(\alpha)}(0)} \int_{B_1} G(O, y) dy.$$

As $\lambda^*(\alpha) \rightarrow 0$, when $\alpha \rightarrow \infty$, we infer that necessarily,

$$u_{\lambda^*(\alpha)}(0) \rightarrow \infty, \text{ as } \alpha \rightarrow \infty.$$

Proposition 6 *Let us suppose that there exists $b_0 > 0$ and $r_0 \in]0, 1[$, such that*

$$u_{\lambda^*(\alpha)}(r_0) \leq b_0, \quad \forall \alpha > 1.$$

Then there exists a positive constant c , such that

$$\lim_{\alpha \rightarrow \infty} u_{\lambda^*(\alpha)}(r) = u_0(r) \equiv \frac{c}{n-2}(r^{2-n} - 1), \quad \forall r \in [r_0, 1].$$

Proof

As $u_{\lambda^*(\alpha)}$ is a decreasing function, we infer that

$$u_{\lambda^*(\alpha)}(r) \leq b_0, \quad \forall \alpha > 1, \quad \forall r \in [r_0, 1].$$

So we can use the L^p -estimates and Morrey Theorem (see [4]), to show that

$$u_{\lambda^*(\alpha)} \rightarrow u_0 \text{ in } C^{1,\delta}(\overline{B(r_0, 1)}), \quad \forall 0 < \delta < 1.$$

As, $\lambda^*(\alpha) \rightarrow 0$, when $\alpha \rightarrow \infty$, we deduce that

$$\Delta u_0(x) = 0 \text{ in } B(r_0, 1).$$

As the function u_0 is radially symmetrical, we get that

$$r^{n-1} \left(r^{n-1} u_0'(r) \right)' = 0, \quad \forall r \in [r_0, 1].$$

So we infer that, there exists a positive constant c , such that

$$u_0(r) = \frac{c}{n-2}(r^{2-n} - 1), \quad \forall r \in [r_0, 1].$$

This ends the Proof.

We conjecture that

$$\lim_{\alpha \rightarrow \infty} u_{\lambda^*(\alpha)}(r) = \frac{c}{n-2}(r^{2-n} - 1), \quad \forall r \in]0, 1].$$

Let us give an immediate corollary of Proposition 5. Let a be a positive regular function on $\overline{B_1}$, then,

Corollary 1 *The problem*

$$\begin{cases} \Delta u + a(x)(1 + u(x))^\alpha = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

doesn't admit a solution, for α "great" enough.

5 Some Numerical Computations

Let us choose $n = 3$ and consider the “small” solution u_λ (when it exists) of the following problem

$$\begin{cases} \Delta u + \lambda(1 + u)^2 = 0 \text{ in } B_1, \\ u > 0 \text{ in } B_1, \\ u = 0 \text{ on } \partial B_1, \end{cases}$$

As we have already mentioned in Section 1, we can put $v_\lambda(x) = \lambda u_\lambda(x) + \lambda$, so we obtain

$$\begin{cases} \Delta v_\lambda + v_\lambda^2 = 0 \text{ in } B_1, \\ v_\lambda > 0 \text{ in } B_1, \\ v_\lambda = \lambda \text{ on } \partial B_1. \end{cases}$$

As v_λ is radially symmetrical, we rewrite the first equation as follows,

$$r(v_\lambda''(r) + v_\lambda^2(r)) = -2v_\lambda'(r), \quad \forall 0 \leq r < 1.$$

If

$$v_\lambda(r) = \sum_{j=0}^{\infty} c_j r^j,$$

we obtain

$$c_j = -\frac{1}{(j+1)j} \sum_{k=0}^{j-2} c_k c_{j-2-k}, \quad \forall j \geq 2.$$

As $v_\lambda'(0) = 0$, we infer that

$$c_{2j+1} = 0, \quad \forall j \geq 0, \text{ and} \\ v_\lambda(r) = c_0 - \sum_{j=1}^{\infty} \left(\frac{1}{(2j+1)(2j)} \sum_{k=0}^{j-1} c_{2k} c_{2j-2-2k} \right) r^{2j}.$$

As we have

$$\lambda = v_\lambda(1) = c_0 - \sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k)} \sum_{j=0}^{k-1} c_{2j} c_{2k-2-2j},$$

let us compute the values of the application (putting $c_0 = c$)

$$f : c \longrightarrow c - \sum_{k=1}^{\infty} \frac{1}{(2k+1)(2k)} \sum_{j=0}^{k-1} c_{2j} c_{2k-2-2j}.$$

We have written a short Maple Procedure (see the Appendix) which computes and plots this curve. The following curve is the output.

As it appears from the curve, one can compute $\lambda^*(2) \approx 2.343255111$, with $c = c_0 \approx 6.4407$. When $c \approx 15$, the computations of $f(c_0)$ “cease” to converge. Is it a loss of analyticity or an unstability effect?

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A mes chères enfants Kénizé et Maïna.

APPENDIX

```
interface(quiet=true);

g:=proc(a::nonneg)

local M,i,j,k;

M[0]:=a;
for i from 1 to 201 do
M[i]:=-(1/(2*i*(2*i+1)))*sum(M[j]*M[i-1-j],j=0..i-1)
od; evalf(sum(M[k],k=0..201));
end:

plotsetup(ps,plotoutput='test.ps');

with(plots);

plot(g,0..14,labels=['c','f(c)']);
```

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