Phase-covariant quantum cloning of qudits

Heng Fan, Hiroshi Imai, Keiji Matsumoto, Xiang-Bin Wang
Quantum computation and quantum information project, ERATO,
Japan Science and Technology Corporation,
Daini Hongo White Bldg.201, Hongo 5-28-3, Bunkyo-ku, Tokyo 133-0033, Japan.

We study the phase-covariant quantum cloning machine for qudits, i.e. the input states in d-level quantum system have complex coefficients with arbitrary phase but constant module. A cloning unitary transformation is proposed. After optimizing the fidelity between input state and single qudit reduced density operator of output state, we obtain the optimal fidelity for phase-covariant quantum cloning of qudits and the corresponding cloning transformation.

I. INTRODUCTION

No-cloning theorem [1] is one of the most fundamental differences between classical and quantum information. It states an arbitrary quantum state cannot be cloned exactly. No-cloning theorem is also extended to other cases such as no broadcasting [2,3], no-imprinting [4]. A unified principle was proposed recently [5]. While an arbitrary quantum state cannot be cloned perfectly, we can clone it approximately [6] or probabilistically [7]. Thus some quantum cloning machines are proposed to study the cloning of quantum states.

A universal quantum cloning machine (UQCM) proposed by Bužek and Hillery [6] clones an arbitrary quantum state approximately. The quality of the copies is independent of the input state. This universal quantum cloning machine is studied and is generalized in several directions. Using fidelity as the measurement of quality of copies, Bužek and Hillery’s cloning machine is proved to be optimal [8]. Instead of a single input qubit and two copies, the UQCM with general N identical pure input qubits and M copies was studied in [9], the optimal fidelity is also obtained. By identifying the fidelity of copying N identical qubits to infinite copies with the fidelity of the corresponding quantum state estimation [10], the upper bound of fidelity of UQCM can be found [11]. Besides the cloning of qubits, the UQCM for d-level quantum states, qudits, is studied by completely positive map [12,13]. The unitary transformation for cloning of qudits was studied in [14] for 1 to 2 case, in [15] for general N to M case. The physical implementation of universal quantum cloning machine was proposed in [16,17]. Quantum networks to realize quantum cloning machine was studied in [18].

A UQCM copies arbitrary pure quantum states equally well. So, we can use UQCM in the case that the input state is completely unknown. However, sometimes, we already know partial information about the input state. If we know exactly the input quantum state, we can clone it perfectly. If we do not know it exactly, but have partial information about it, we can perhaps design a special quantum cloning machine for this kind of input state with a better quality than the UQCM. A phase-covariant quantum cloning is such a special quantum cloning machine. It is defined as a machine that optimally clone a special class of states, the states that have complex coefficients with arbitrary phase but constant module (see (6)). In 2-level quantum system, this special class of states is one kind of equatorial qubits. Here equatorial qubit means that one parameter of its Bloch vector is zero. We can change phase-covariant quantum cloning machine to the cloning machine for other equatorial qubits input via some unitary transformations. So, we generally do not distinguish phase-covariant quantum cloning machine with cloning machine for equatorial qubits. For 2-level quantum system, the 1 to 2 phase-covariant quantum cloning machine was studied in [19]. The 1 to M cloning machine for equatorial qubits was studied in [20] and the fidelity was proved to be optimal. The phase-covariant quantum cloning machine is of interest in particular in quantum key distribution. In the optimal eavesdropping of BB84 [21] quantum key distribution, instead of a UQCM, the eavesdropper should use the phase-covariant quantum cloning machine instead of the well studied UQCM [19]. If all 3 mutually unbiased states in 2-level system are used, i.e. the 6-state quantum key distribution scheme, a UQCM should be used [22]. Besides the 2-level quantum system, the phase-covariant quantum cloning machine in 3-level quantum system is also studied. By different methods, the optimal phase-covariant quantum cloning machine for 3-level quantum system is obtained by two groups [23,24].

In this paper, we shall study the phase-covariant quantum cloning machine in d-level quantum system. By analogy with the UQCM and the known situation for phase-covariant cloning machine when \(d = 2\) and \(d = 3\), we use the conjecture that each output state in the phase-covariant cloning machine is in scalar form, that is, the output reduced density operator state is the initial density operator state mixed with the fully mixed operator state as in a depolarizing channel. We also assume that the output states of the phase-covariant cloning machine are symmetric as in UQCM.
Using these assumptions, we find a simple unitary transformation for our d-level phase-covariant quantum cloning machine. Next, we optimize the fidelity over free parameters. As expected, the optimal fidelity for qudit is higher than the corresponding UQCM. For special case, $d = 2$ and $d = 3$, the optimal fidelity obtained in this paper agree with previous known results. For case $d$ is a prime number, we point out this optimal phase-covariant quantum cloning machine can be used in eavesdropping of quantum key distribution by using $d$ mutually unbiased states. However, it is not necessarily optimal for eavesdropping since optimal cloning is not known to be equivalent to optimal eavesdropping in general.

II. SOME KNOWN RESULTS ABOUT PHASE-COVARIANT QUANTUM CLONING MACHINE

We first introduce the notations and review some known results for qubits [19,20]. We consider the input state as

$$|\Psi\rangle^{(in)} = \frac{1}{\sqrt{2}}[|0\rangle + e^{i\phi}|1\rangle],$$

(1)

where $\phi \in [0, 2\pi)$. This state just has one arbitrary phase parameter $\phi$ instead of two free parameters for an arbitrary qubit. So, we already know partial information of this input state. One can check that the $y$ component of the Bloch vector of this state is zero. This case is equivalent to the case that the input state is $|\Psi\rangle = \cos \theta|0\rangle + \sin \theta|1\rangle$, in which the input state does not have arbitrary phase parameter. The optimal phase-covariant cloning transformation takes the form,

$$U|0\rangle^{(in)}|Q\rangle = \frac{1}{\sqrt{2}}[|00\rangle|0\rangle_\alpha + \frac{1}{2}(|01\rangle + |10\rangle)|1\rangle_\alpha],$$

$$U|1\rangle^{(in)}|Q\rangle = \frac{1}{\sqrt{2}}[|11\rangle|1\rangle_\alpha + \frac{1}{2}(|01\rangle + |10\rangle)|0\rangle_\alpha],$$

(2)

where $|Q\rangle$ is the blank state and initial state of the cloning machine. The first states in l.h.s. are input states. The single qubit reduced density matrix of output can be calculated as

$$\rho^{\text{out}}_{\text{red.}} = \frac{1}{\sqrt{2}}\rho^{(in)} + \left(\frac{1}{2} - \sqrt{\frac{1}{8}}\right)I,$$

(3)

where $I$ is the identity matrix, and the input density matrix is $\rho^{(in)} = |\Psi\rangle\langle\Psi|$ defined in (1). We use fidelity to define the quality of the copies. The general definition of fidelity takes the form $F(\rho_1, \rho_2) = [Tr\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}]^2$ [25]. The value of $F$ ranges from 0 to 1. A larger $F$ corresponds to a higher fidelity. $F = 1$ means two density matrices are equal. We only consider about the pure input states, and the fidelity can be simplified as $F = \langle \Psi | \rho^{\text{out}}_{\text{red.}} | \Psi \rangle^{(in)}$. The optimal fidelity of phase-covariant quantum cloning machine is obtained as

$$F_{\text{optimal}} = \frac{1}{2} + \sqrt{\frac{1}{8}}.$$  

(4)

As expected, this fidelity $F \approx 0.85$ is higher than the fidelity of UQCM $F \approx 0.83$.

In eavesdropping of well known BB84 quantum key distribution, because all four states $|0\rangle, |1\rangle, 1/\sqrt{2}(|0\rangle + |1\rangle), 1/\sqrt{2}(|0\rangle - |1\rangle)$ can be described by $|\Psi\rangle = \cos \theta|0\rangle + \sin \theta|1\rangle$. So, instead of the UQCM, we should at least use the cloning machine for equatorial qubits in eavesdropping. Actually in individual attack, we can not do better than the cloning machine for equatorial qubits [19,26]. The cloning machine presented in (2) can be used in analyzing the eavesdropping of other 2 mutually unbiased bases $1/\sqrt{2}(|0\rangle - |1\rangle), 1/\sqrt{2}(|0\rangle + |1\rangle), 1/\sqrt{2}(|0\rangle + i|1\rangle), 1/\sqrt{2}(|0\rangle - i|1\rangle)$.

The optimal fidelity of phase-covariant quantum cloning machine for qutrits was obtained by D’Ariano et al [23] and Cerf et al [24];

$$F = \frac{5 + \sqrt{17}}{12}, \quad \text{for } d = 3.$$  

(5)
III. PHASE-COVARIANT CLONING OF QUDITS

We study the quantum cloning of d-level states in the form

$$|\Psi^{(in)}\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e^{i\phi_j} |j\rangle,$$

(6)

where the arbitrary phase parameters $\phi_j \in [0, 2\pi), j = 0, \cdots, d - 1$. A whole phase is not important, so we can assume $\phi_0 = 0$. The density operator of input state can be written as $\rho^{(in)} = \frac{1}{d} \sum_{jk} e^{i(\phi_j - \phi_k)} |j\rangle\langle k|$. For case 1 to M phase-covariant quantum cloning machine (with 1 input qudit and M output qudits), we will use the assumption that the most general cloning transformation takes the following form

$$U|j\rangle|Q\rangle = \sum_{\vec{k}} |\vec{k}\rangle|R_{j\vec{k}}\rangle,$$

(7)

where similar notations as in 2-level quantum system are used, and $\vec{k} \equiv \{k_0, \cdots, k_{d-1}\}$. The summation $\sum_{\vec{k}}^M$ means take summation over all possible values satisfy the restriction $\sum_{j=0}^{d-1} k_j = M$. The quantum state $|\vec{k}\rangle$ is a normalized symmetric state with $k_j$ states in $|j\rangle$. The ancilla states $|R_{j\vec{k}}\rangle$ are not necessarily orthogonal and normalized. The unitary relation means the restriction $\sum_{\vec{k}}^M \langle R_{j\vec{k}}|R_{j'\vec{k}}\rangle = \delta_{jj'}$. We remark here that as in UQCM, the output states are symmetrical so that every single qudit reduced density matrix of output are equal to each other. Except the assumption that the output states are symmetric as in UQCM [6,9,12], the relation (7) is the most general cloning transformation. Using this assumption, we find the optimal phase-covariant cloning machine from (7). Substituting the input state (6) into the general cloning machine (7), tracing out the ancilla states, we have the output state as follows,

$$\rho^{(out)} = \frac{1}{d} \sum_{jj'} \sum_{\vec{k}, \vec{k}'} e^{i(\phi_j - \phi_{j'})} |\vec{k}\rangle\langle \vec{k}'|\langle R_{j\vec{k}}|R_{j'\vec{k}'}\rangle.$$

(8)

Here we introduce the Bloch vector $\vec{\lambda}$ in d-dimension as presented in [27] and the reference therein. A density matrix in d-dimension can be expanded as

$$\rho = \frac{1}{2} \sum_{i=1}^{d^2-1} \lambda_i \tau_i + \frac{1}{d} I,$$

(9)

where $\tau_i$ are the generators of the group $SU(d)$ with

$$\text{Tr} \tau_i = 0, \quad \text{Tr}(\tau_i \tau_j) = 2\delta_{ij}.$$  

(10)

In UQCM, the property of Bloch vector invariance is often used. That is because we want the cloning machine to be universal, i.e., the quality of copies defined by fidelity between input state and output states does not depend on the input state, for detailed argument see [8,27]. The Bloch vector invariance means the output reduced density matrix can be written as a scalar form as in (3),

$$\rho^{(out)} = \frac{1}{2} s \sum_{i=1}^{d^2-1} \lambda_i \tau_i + \frac{1}{d} I = s\rho + \frac{1 - s}{d},$$

(11)

where $s$ is the shrinking factor. Furthermore, the optimal phase-covariant cloning quantum machine for 2-level system is known to have the Bloch vector invariance property. So, by analogy with the UQCM and the case $d = 2$, we assume this property for d-level phase cloning machine, i.e., we assume relation (11) hold. This is a very useful relation. We know the density matrix of input state (6) takes the form

$$\rho^{(in)} = \frac{1}{d} \sum_{jj'} e^{i(\phi_j - \phi_{j'})} |j\rangle\langle j'|.$$  

(12)
The output reduced density matrix can be obtained from the output state (8) by taking trace over all but one qudits. If we let the output reduced density matrix and the input state (12) has the relation of scalar form as in (3), it implies the following restrictions for the cloning transformation (7),

\begin{align}
\langle R_j | R_j \rangle &\propto \langle R_j | R_j \rangle \delta_{k,k'}, \\
\langle R_j' | R_j \rangle &\propto \langle R_j' | R_j \rangle \delta_{k_0,k'_0} \cdots \delta_{k_{d-1},k'_{d-1}} \delta_{k_j,k_j' + 1} \delta_{k_{k_j'} + 1},
\end{align}

where in \cdots, we do not have \(\delta_{k_j,k_j'}\) and \(\delta_{k_j,k_j'}\), the same notations will be used later. The output single qudit reduced density matrix can be written as the following form

\[
\rho_{\text{red.}}^{(\text{out})} = \sum_{l=0}^{d-1} |l\rangle \langle l| + \frac{1}{d} \sum_{j=0}^{M} \sum_{k} \frac{k_l}{M} \langle R_j | R_j \rangle + \sum_{j \neq j'} e^{i(\phi_j - \phi_{j'})} |j\rangle \langle j'| \left[ \sum_{k,k'} \frac{\sqrt{k_j k_j'}}{M} \langle R_j' | R_j \rangle \delta_{k_0,k'_0} \cdots \delta_{k_{d-1},k'_{d-1}} \delta_{k_j,k_j' + 1} \delta_{k_{k_j'} + 1} \right].
\]

The corresponding fidelity is written as

\[
F = \frac{1}{d} + \frac{1}{d^2} \sum_{j} \sum_{l} \sqrt{\frac{\langle R_j | R_j \rangle \delta_{k_0,k'_0} \cdots \delta_{k_{d-1},k'_{d-1}} \delta_{k_j,k_j' + 1} \delta_{k_{k_j'} + 1}}{\langle R_j | R_j \rangle \delta_{k_0,k'_0} \cdots \delta_{k_{d-1},k'_{d-1}} \delta_{k_j,k_j' + 1} \delta_{k_{k_j'} + 1}}}.
\]

Next, we shall pay our attention to 1 to 2 phase-covariant quantum cloning machine. Considering the restriction that the reduced density matrix of output should be written as a scalar form, and also considering the symmetric property of the input state (6), we have the following phase-covariant quantum cloning transformation

\[
U |j\rangle |Q\rangle = \alpha |jj\rangle |R_j\rangle + \frac{\beta}{\sqrt{2(d-1)}} \sum_{l \neq j} (|jl\rangle + |lj\rangle) |R_l\rangle,
\]

where \(\alpha, \beta\) are real numbers, and \(\alpha^2 + \beta^2 = 1\). Actually letting \(\alpha, \beta\) to be complex numbers does not improve the fidelity. \(|R_j\rangle\) are orthonormal ancilla states. This is a simplified cloning transformation. Here we show this cloning transformation can be derived from (7) under the restriction (13,14) for \(M = 2\). The ancilla states \(|R_j\rangle\) are orthogonal is due to the relation (13). In the most general cloning transformation (7), the ancilla states should be denoted as \(|R_j\rangle\). In case 1 to 2 cloning, for fixed \(j, j'\) if we choose \(k_j = 2\), then \(k_{j'} = 0\). According to relation (14), the ancilla state \(|R_j'\rangle\) can be identified with \(|R_j\rangle\) when \(k'_{j'} = 1\) with some normalization. So, we actually just need one ancilla state \(|R_j\rangle\) to represent \(|R_j\rangle\) and \(|R_{j'}\rangle\) if we have relations \(k_j = 2, k_{j'} = 0; k'_{j} = 1, k'_{j'} = 1\). Without other states in (17), the cloning transformation (17) can achieve the optimal fidelity due to relation (16). In short, we can find the optimal cloning transformation from (17).

Substituting the input state (6) into the cloning transformation and tracing out the ancilla states, the output state takes the form

\[
\rho^{(\text{out})} = \frac{\alpha^2}{d} \sum_{j} |jj\rangle \langle jj| + \frac{\alpha \beta}{d \sqrt{2(d-1)}} \sum_{j \neq k} e^{i(\phi_j - \phi_k)} \|jj\rangle \langle jj| + \langle lj\rangle + \langle jl\rangle + \langle lj\rangle |jj\rangle |jj\rangle
\]

\[
+ \frac{\beta^2}{2d(d-1)} \sum_{j' \neq j} \sum_{l \neq j} e^{i(\phi_{j'} - \phi_j)} (|jl\rangle + |lj\rangle) (|lj\rangle + \langle lj\rangle) |jj\rangle |jj\rangle.
\]

Taking trace over one qudit, we obtain the single qudit reduced density matrix of output

\[
\rho_{\text{red.}}^{(\text{out})} = \frac{1}{d} \sum_{j} |j\rangle \langle j| + \left( \frac{\alpha \beta}{d} \sqrt{\frac{2}{d-1}} + \frac{\beta^2(d-2)}{2d(d-1)} \right) \sum_{j \neq k} e^{i(\phi_j - \phi_k)} |j\rangle \langle k|.
\]

The fidelity can be calculated as

\[
F = \frac{1}{d} + \alpha \beta \sqrt{\frac{2}{d-1}} + \frac{\beta^2 d - 2}{2d}.
\]
These relations are the special case of the general 1 to M case that we obtained in (15,16), but with $M = 2$. Now, we need to optimize the fidelity under the restriction $\alpha^2 + \beta^2 = 1$. We can find the optimal fidelity of 1 to 2 phase-covariant quantum cloning machine can be written as

$$F_{\text{optimal}} = \frac{1}{d} + \frac{1}{4d}(d - 2 + \sqrt{d^2 + 4d - 4}).$$

(22)

In case $d = 2, 3$, this results agree with previous known results (4,5), respectively. As expected, this optimal fidelity of phase-covariant quantum cloning machine is higher than the corresponding optimal fidelity of UQCM $F_{\text{optimal}} > F_{\text{universal}} = (d + 3)/(2(d + 1))$. The optimal fidelity can be achieved when $\alpha, \beta$ take the following values,

$$\alpha = \left(\frac{1}{2} - \frac{d - 2}{2\sqrt{d^2 + 4d - 4}}\right)^{\frac{1}{2}},$$

$$\beta = \left(\frac{1}{2} + \frac{d - 2}{2\sqrt{d^2 + 4d - 4}}\right)^{\frac{1}{2}}.$$

(23)

In case $d = 2$, the cloning transformation (17) recovers the previous result (2). Thus we find the optimal phase-covariant quantum cloning machine for qudits (17, 23) and the corresponding optimal fidelity (22).

**IV. DISCUSSIONS AND SUMMARY**

Quantum measurements by mutually unbiased bases provide the optimal way of determining a quantum state. And the mutually unbiased bases have close relations with quantum cryptography. In d-dimension, when d is prime, there are $d + 1$ mutually unbiased bases. Except the standard basis {\ket{0}, \ket{1}, \ldots, \ket{d - 1}}}, the other $d$ mutually unbiased bases take the form [28]

$$\ket{\psi^l_t} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} (\omega^l)^{d-j} (\omega^{-k})^s \ket{j}, \quad t = 0, \ldots, d - 1,$$

(24)

where $s_j = j + \cdots + (d - 1)$. And $l = 0, \ldots, d - 1$ represent $d$ mutually unbiased bases. The phase-covariant quantum cloning machine of qudits can clone all of these states equally well. So, we see if one uses $d$ mutually unbiased bases (24) to perform quantum key distribution, the eavesdropper could use phase-covariant quantum cloning machine to attack instead of the UQCM. If all $d + 1$ mutually unbiased bases are used, we should use UQCM. However, it is not known whether using phase-covariant cloning machine in eavesdropping is optimal or not when $d > 3$ bases are used even though the cloning machine itself is optimal. We see the difference between $d$ and $d + 1$ mutually unbiased bases decreases when $d$ becomes larger. Correspondingly the gap between the fidelities of phase-covariant cloning machine and UQCM decreases when $d$ becomes larger. When $d$ is large enough, this gap becomes negligible.

In summary, we present in this paper the optimal phase-covariant quantum cloning machine for qudits (17, 23). The corresponding optimal fidelity (22) was found. In $d = 2$ case, the results recover the previous result [19,15]. In $d = 3$, the optimal fidelity agree with the result obtained by D’Ariano et al [23] and Cerf et al [24].

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