Friedmann-Robertson-Walker brane cosmological equations from the five-dimensional bulk (A)dS black hole

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ABSTRACT

In the first part of this work we review the equations of motion for the brane presented in Friedmann-Robertson-Walker (FRW) form, when bulk is five-dimensional (A)dS Black Hole. The spacelike (timelike) FRW brane equations are considered from the point of view of their representation in the form similar to two-dimensional CFT entropy, so-called Cardy-Verlinde (CV) formula. The following five-dimensional gravities are reviewed: Einstein, Einstein-Maxwell and Einstein with brane quantum corrections. The second part of the work is devoted to study FRW brane equations and their representation in CV form, brane induced matter and brane cosmology in Einstein-Gauss-Bonnet (GB) gravity. In particular, we focus on the inflationary brane cosmology. The energy conditions for brane matter are also analyzed. We show that for some values of GB coupling constant (bulk is AdS BH) the brane matter is not CFT. Its energy density and pressure are not always positive.

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1 Introduction

The recent astronomical data indicate that observable universe is currently accelerating [1]. This observation, in turn, indicates that the universe has a positive cosmological constant. As a result it is likely that universe evolves into the future (asymptotically) de Sitter phase.

The recent brane-world approach (as manifestation of holographical principle) to the description of the observable universe as a brane embedded in higher dimensional bulk space has brought many interesting ideas to the realization of (asymptotically) de Sitter brane universe. First of all, it became clear that brane matter may be induced by bulk space. Such brane matter which may play the role of dark matter may even violate the Dominant Energy Condition. Moreover, as a result of acceleration it looks that the cosmology with negative density energy and negative pressure does not
seem to contradict to the astronomical data (for recent initial steps to study such cosmology, see [2]). The brane-world approach may be also related with Cosmic Censorship via holographic principle (see recent discussion in [3]).

It is even more important that holographic principle is somehow encoded in the usual gravitational field equations. Indeed, as it has been shown by E. Verlinde [4] the usual four-dimensional Friedmann-Robertson-Walker (FRW) cosmological equations may be re-written in the form reminding about the entropy of two-dimensional conformal field theory (CFT). It again appears the connection between four-dimensional classical gravitational physics and quantum two-dimensional CFT. The corresponding two-dimensional CFT entropy has been extensively studied sometime ago in [5]. That is why, the two-dimensional CFT entropy representation of FRW equations is sometimes called generalized Cardy or Cardy-Verlinde formula.

Finally, holographic principle may suggest new interpretation of the evolution of the observable universe. Indeed, using dual description one can think about de Sitter phase as preferrable solution of cosmological equations. One possibility (which is not successfully worked out so far) could be provided by fixed points of the coupling constants for corresponding dual CFT where de Sitter space is realized.

The purpose of this work is two-fold. From one side we review the FRW brane cosmology where brane is embedded in AdS or dS Black Hole (BH), and induced brane matter in various higher dimensional gravitational theories from the point of view of the representation of corresponding field equations in Cardy-Verlinde form. From another side we study brane-world cosmology in Einstein-Gauss-Bonnet (Einstein-GB) gravity where brane description is extremely complicated. It is shown that for various values of GB coupling constant and AdS or dS bulk BHs the induced brane matter may have negative energy and pressure. It is demonstrated that de Sitter brane is not preferrable solution of Einstein-GB brane-world cosmology.

The paper is organized as follows. In the next section we give simple and pedagogical introduction to the Cardy-Verlinde (CV) formula in n+1-dimensional gravity with general state matter. In other words, FRW cosmological equations are rewritten (using several definitions of cosmological entropy) in the form similar to quantum two-dimensional CFT entropy. In the second appearance of CV formula the calculation of the universe entropy (supposing the valid first law of thermodynamics) gives it in generalized CV form. For radiation-dominated universe the cosmological entropy re-
duces to the standard CV form. In section three we give the introduction to
brane-world program on the example of five-dimensional Einstein gravity. Of
course, the study of brane cosmology when bulk space is AdS includes huge
number of works (see [6] and references therein). We consider both types of
bulks: AdS and dS BHs and discuss the brane equations of motion presented
in FRW form. Moreover, for each five-dimensional bulk the FRW equations
for space-like as well as time-like branes are reviewed. The induced brane
matter is introduced. The presentation of such FRW brane equations in CV
form is discussed as well as cosmological entropy bounds and the relation of
AdS BH entropy with cosmological entropy (second way appearance of CV
formula).

Section four is devoted to the review of FRW brane equations from five-
dimensional Reissner-Nordstrom-de Sitter (RNdS) BH and their relation with
CV formula. It is shown that while some contributions due to Maxwell field
appear in the intermediate identifications the corresponding FRW equations
may still be presented in the form similar to two-dimensional CFT entropy.
In section five we review the role of quantum brane matter to above FRW
equations. Using conformal anomaly induced effective action on the brane,
the quantum-corrected FRW brane equations are written. Bulk space is again
AdS BH. The values (signs) of induced brane matter pressure and energy are
considered in connection with Dominant and Weak Energy Conditions (DEC
and WEC). The quantum-corrected de Sitter brane solution is discussed.

Section six is devoted to the study of induced brane matter from bulk
AdS BH in Einstein-GB gravity. GB combination naturally appears in the
next-to-leading order term of the heterotic string effective action [7]. De-
spite the presence of higher derivative terms, the Einstein-GB field equations
(being much more complicated) include only second derivatives like in ordi-
nary Einstein gravity. That was the reason why there was much activity in
the study of brane-world aspects of Einstein-GB theory [8, 9, 10]. In par-
ticularly, we show that FRW brane equations being very complicated may
be still presented in CV form. The analysis of induced brane matter shows
that for some values of GB coupling constant and in different limits on scale
factor the dual brane matter is not always CFT. Moreover, there are cases
where matter pressure and energy are not positive. The careful analysis of
DEC and WEC is presented for various types of brane matter. These two
conditions are violated for some choices of GB coupling constant.

In section seven the FRW brane cosmology in Einstein-GB theory is dis-
cussed using the effective potential approach. The same cases of induced brane matter (for the same values of GB coupling) as in section 6 are considered. As is shown explicitly, there are various types of brane cosmology: de Sitter, hyperbolic and flat. Such brane universes may expand or contract, they maybe singular or non-singular. De Sitter (inflationary) brane cosmology does not seem to be the preferrable solution of FRW cosmological equations. Some summary and outlook are given in the last section.

2 Brief look to Cardy-Verlinde formula

In the seminal work [4] the very interesting approach to rewrite FRW cosmological equations in the form reminding about two-dimensional quantum field theory has been suggested. In fact, E. Verlinde [4] drew an interesting analogy between the FRW equations of a standard, closed, radiation-dominated universe and the two-dimensional entropy formula due to Cardy [5]. The physical origin of this analogy between classical gravity theory and two-dimensional QFT remains completely hidden.

Let us first review this analogy in Einstein gravity for the usual \((n + 1)\)-dimensional FRW Universe with a metric

\[
ds^2 = -d\tau^2 + a^2(\tau)g_{ij}dx^idx^j,
\]

where the \(n\)-dimensional spatial hypersurfaces with negative, zero or positive curvature are parametrized by \(K = -1, 0, 1\), respectively. For example, \(K = -1\) corresponds to hyperboloid (of one sheet), \(K = 0\) to flat surface, and \(K = 1\) to sphere, whose metric is given by

\[
g_{ij}dx^idx^j = \frac{dr^2}{1-Kr^2} + r^2d\Omega^2_{n-1}.
\]

Here \(d\Omega^2_{n-1}\) is the metric of \(n-1\)-dimensional sphere with unit radius. If \(R_{ij}\) is the Ricci tensor given by \(g_{ij}\), we have

\[
R_{ij} = (n - 1)Kg_{ij}.
\]

In our discussion of cosmology based on Einstein gravity, we parametrize the curvature of the spatial hypersurfaces in terms of \(K = -1, 0, 1\) since direct comparison with the standard cosmological equations is then straightforward.
In other sections, we often use the lower letter \( k \) defined by \( k \equiv (n-1)K \), instead of the capital letter \( K \). We limit our discussion mainly to that of the closed universe \( (K = 1) \), with a spatial volume defined by \( V = a^n \int d^n x \sqrt{g} \). The standard FRW equations, which follow from the Einstein equations may then be written as

\[
H^2 = \frac{16\pi G \rho - K}{n(n-1)a^2},
\]

\[
\dot{H} = -\frac{8\pi G}{(n-1)} (\rho + p) + \frac{K}{a^2},
\]

(2.4)

where \( \rho = \rho_m + \frac{\Lambda}{8\pi G} \), \( p = p_m - \frac{\Lambda}{8\pi G} \), \( \Lambda \) is a cosmological constant and \( \rho_m \) and \( p_m \) are the energy density and pressure of the matter contributions. The energy conservation equation is

\[
\dot{\rho} + n(\rho + p) \frac{\dot{a}}{a} = 0
\]

(2.5)

and for a perfect fluid matter source with equation of state \( p_m = \omega \rho_m \) (\( \omega = \) constant) Eq. (2.5) is solved as:

\[
\rho = \rho_0 a^{-n(1+\omega)} + \frac{\Lambda}{8\pi G}.
\]

(2.6)

When \( \omega = 0 \) the pressure vanishes \( p = 0 \), which corresponds to dust and \( \rho \) behaves as \( \rho \propto a^{-n} \), on the other hand, if \( \omega = \frac{1}{n} \), the trace of the energy momentum tensor vanishes: \( T_{\mu}^{\mu} = -\rho + np = 0 \) which corresponds to CFT or radiation:

\[
\omega = 0 \quad \text{dust} \quad p_m = 0, \quad \rho_m \propto a^{-n}
\]

\[
\omega = \frac{1}{n} \quad \text{radiation} \quad -\rho_m + np_m = 0, \quad p_m \propto \rho_m \propto a^{-(n+1)}.
\]

(2.7)

The definitions for the Hubble, Bekenstein [11] and Bekenstein-Hawking entropies are given as following [4]:

\[
S_H = (n-1)\frac{HV}{4G}, \quad S_{BH} = (n-1)\frac{V}{4Ga}, \quad S_B = \frac{2\pi a}{n} E,
\]

(2.8)

where the total energy, \( E \), is defined as \( E = \rho V \) and contains the contribution from the cosmological constant term. This differs from that of the standard case, where the definitions of the entropies \( S_{BH} \) and \( S_B \) may differ slightly in
their coefficients. This is specific to the presence of a cosmological constant [13, 14].

The Bekenstein entropy $S_B$ [11] gives the bound for the total entropy $S \leq S_{BH}$ for the system with limited gravity. The bound is useful for relatively low energy density or small volumes. Then the bound is not appropriate for strongly-gravitating universe, where $Ha \geq 1$. In this case $S_B \geq S_{BH}$. In the strongly-gravitating universe, the black hole production should be accounted for. $S_{BH}$ grows like an area rather than the volume and for the closed universe $S_{BH}$ reduces to the well-known expression of $A/4G$, where $A$ expresses the area. (This is, of course, typical for Einstein gravity as for higher derivative gravity the area law may not hold.) In [12], however, it has been argued that, when $Ha > 1$, the total entropy should be bounded by the Hubble entropy $S_H$, which is the entropy of the black hole with the radius of the Hubble size.

By employing the definitions (2.8), one can easily rewrite the FRW equations (2.4) as a cosmological Cardy-Verlinde (CV) formula:

$$S_H = \frac{2\pi}{n} a \sqrt{E_{BH} (2E - KE_{BH})},$$
$$KE_{BH} = n (E + pV - T_H S_H),$$

where the energy and Hawking temperature of the black hole are defined as

$$E_{BH} = n(n-1) \frac{V}{8\pi G a^2}, \quad T_H = -\frac{\dot{H}}{2\pi H}.$$  \hspace{1cm} (2.9)

and we have separated the energy into a matter part and a cosmological constant part, i.e., $E = E_m + E_{cosm}$, where $E_{cosm} = \frac{\Lambda}{8\pi G} V$. This is simply a way to rewrite the FRW equations in a form that resembles the equation defining the entropy of a two-dimensional CFT. However, the following remark is in order: the presence of cosmological constant may change some of the coefficients in Eq. (2.9) and this depends on precisely how the separation between the strongly and weakly interacting gravitational phases is made (compare with [13, 14]). In any case, the energy associated with the cosmological constant term is hidden in the expression for $E$, Eq. (2.9).

Later, we discuss the motion of the brane in Schwarzschild-(A)dS bulk space. The motion is again described by the FRW-like equation from which Cardy-Verlinde formula follows. The Hawking temperature $T_H$ (2.10) coincides with that of the black hole, when the radius $a$ of the universe is equal to the horizon radius, that is, the brane crosses the horizon.
Eq. (2.9) may also be rewritten in another form:

\[ S^2_H = S_{BH} (2S_B - KS_{BH}) . \]  

(2.11)

Since the definition of \( S_B \) normally contains only matter contributions, it is reasonable to define \( S_B \equiv S^m_B + S^\text{cosm}_B \), where the entropy associated with the cosmological constant is given by

\[ S^\text{cosm}_B = \frac{aV\Lambda}{4nG} . \]  

(2.12)

The appearance of such a new “cosmological constant” contribution to the entropy in the CV formula is quite remarkable.

Thus far, we have discussed the appearance of the CV formula as a way to rewrite the FRW equations. However, the CV formula appears in a second formulation when one calculates the entropy \( S \), of the universe. Indeed, following Ref.[4], one can represent the total energy \( E = \rho V \) of the universe as the sum of the extensive energy, \( E_E \), and the subextensive (Casimir) energy \( E_C \):

\[ E(S, V) = E_E(S, V) + \frac{1}{2} E_C(S, V) . \]  

(2.13)

Note that unlike the case considered by Verlinde [4], the cosmological constant contribution appears in \( E_E \). Nevertheless, the constant rescaling of the energy is given by

\[
\begin{align*}
E_E(\lambda S, \lambda V) &= \lambda E_E(S, V) , \\
E_C(\lambda S, \lambda V) &= \lambda^{1 - \frac{2}{n}} E_C(S, V) .
\end{align*}
\]  

(2.14)

Now, if one assumes that the first law of thermodynamics is valid and that the expansion is adiabatic, one deduces that

\[ dS = 0 , \quad s = \frac{a^n}{T} (\rho + p) + s_0 , \]  

(2.15)

where the entropy \( S \equiv s \int d^n x \sqrt{g} \), \( s_0 \) is an integration constant and \( T \) is the temperature of the universe. It then follows that the Casimir energy is given by [15]

\[ E_C = n (E + pV - TS) = -nTs_0 \int d^n x \sqrt{g} \]  

(2.16)
and, consequently, that $E_C \sim a^{-n\omega}$ and $E_E - E_{\text{cosm}} \sim a^{-n\omega}$. This further implies that the products $E_C a^{n\omega}$ and $(E_E - E_{\text{cosm}}) a^{n\omega}$ are independent of the spatial volume of the universe, $V$. By employing the scaling relations (2.14) one then concludes that [15]:

$$E_E - E_{\text{cosm}} = \frac{\alpha}{4\pi a^{n\omega}} S^{\omega+1}, \quad E_C = \frac{\beta}{2\pi a^{n\omega}} S^{\omega+1-\frac{2}{n}},$$

where $\alpha$ and $\beta$ are some unknown constants. (They are known for CFT in four dimensions). Hence, the entropy is given by

$$S = \left[ \frac{2\pi a^{n\omega}}{\sqrt{\alpha \beta}} \sqrt{E_C (E_E - E_{\text{cosm}})} \right]^{\frac{n}{(\omega+1)n-1}}.$$  \hspace{1cm} (2.18)

Eq. (2.18) represents the generalization of the Cardy-Verlinde formula found by Youm [15] in the absence of a contribution from the cosmological constant. The negative term associated with such a cosmological entropy is quite remarkable. In the case of a radiation-dominated universe, Eq. (2.18) reduces to the standard CV formula with the familiar square root term[5]. This formulation will be used below when one writes the equation of brane motion as FRW equation.

3 FRW brane equations in the background of AdS and dS Schwarzschild black hole

In this section, we review briefly the relationship between the entropy of AdS and dS Schwarzschild space and those of the dual CFT which lives on the brane by using Friedmann-Robertson-Walker (FRW) equations and Cardy-Verlinde formula. The holographic principle between the radiation dominated FRW universe in $d$-dimensions and same dimensional CFT with a dual $d + 1$-dimensional AdS description was studied by E. Verlinde [4]. Especially, one can see the correspondence between black hole entropy and the entropy of the CFT which is derived by making the appropriate identifications for FRW equation with the generalized Cardy formula. The Cardy formula is originally the entropy formula of the CFT only for two dimensions [5], while the generalized Cardy formula expresses that of the CFT for any dimensions [4]. From the point of brane-world physics [16], the CFT/FRW
relation sheds further light on the study of the brane CFT in the background of AdS Schwarzschild black hole [17]. There was much activity on the studies of related questions[9, 10, 18, 19, 21, 22, 23, 24, 25] making use the connection with Cardy-Verlinde formula.

We will describe 4 kinds of FRW Eqs. following from AdS and dS Schwarzschild black hole, i.e., time(spacetime)-like FRW Eqs. from AdS(dS) Schwarzschild background. If all the vectors tangential to the brane are space-like, we call the brane space-like one. If there is any time-like tangential vector, we call the brane time-like one. We consider the FRW Eqs. for both cases.

One first considers a 4-dimensional time-like brane in 5-dimensional AdS Schwarzschild background. From the analogy with the AdS/CFT correspondence, one can regard that 4-dimensional CFT exists on the brane which is the boundary of the 5-dimensional AdS Schwarzschild background. The bulk action is given by the 5-dimensional Einstein action with cosmological term. The dynamics of the brane is described by the boundary action:

\[ \mathcal{L}_b = -\frac{1}{8\pi G_5} \int_{\partial\mathcal{M}} \sqrt{-g} \mathcal{K} + \frac{\kappa}{8\pi G_5} \int_{\partial\mathcal{M}} \sqrt{-g}, \quad \mathcal{K} = K^i_{ij} \quad (3.1) \]

Here \( G_5 \) is 5-dimensional bulk Newton constant, \( \partial\mathcal{M} \) denotes the surface of the brane, \( g \) is the determinant of the induced metric on \( \partial\mathcal{M} \), \( K_{ij} \) is the extrinsic curvature, \( \kappa \) is a parameter related to tension of the brane. From this Lagrangian, we can get the equation of motion of the brane as [17]:

\[ K_{ij} = \frac{\kappa}{2} g_{ij} \quad (3.2) \]

which implies that \( \partial\mathcal{M} \) is a brane of constant extrinsic curvature. The bulk action is given by 5-dimensional Einstein action with cosmological constant. The AdS Schwarzschild space is one of the exact solutions of bulk equations of motion and can be written in the following form,

\[ ds_5^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu = -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 d\Omega_3^2, \]

\[ e^{2\rho} = \frac{1}{a^2} \left( -\mu + a^2 + \frac{a^4}{l_{AdS}^2} \right). \quad (3.3) \]

\(^4\text{In this paper, lower case Latin indices span the world–volume, } (i, j) = (0, 1, 2, 3), \text{ lower case Greek indices span the bulk coordinates and a comma denotes partial differentiation.}\)
Here \( l_{\text{AdS}} \) is the curvature radius of AdS and \( \mu \) is the black hole mass. Following the method of the work [17], one rewrites AdS Schwarzschild metric (3.3) in the form of FRW metric by using a new time parameter \( \tau \). Note that the parameters \( t \) and \( a \) in (3.3) are the functions of \( \tau \), namely \( a = a(\tau), t = t(\tau) \). For the purpose of getting the 4-dimensional FRW metric, we impose the following condition,

\[
-e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial a}{\partial \tau} \right)^2 = -1.
\]  

(3.4)

Thus one obtains time-like FRW metric:

\[
ds_4^2 = g_{ij} dx^i dx^j = -d\tau^2 + a^2 d\Omega_3^2.
\]  

(3.5)

The extrinsic curvature, \( K_{ij} \), of the brane can be calculated and expressed in terms of the function \( a(\tau) \) and \( t(\tau) \). Thus one rewrites the equations of motion (3.2) as

\[
\frac{dt}{d\tau} = - \frac{\kappa a}{2} e^{-2\rho}.
\]  

(3.6)

Using (3.4) and (3.6), we can derive FRW equation for a radiation dominated universe where Hubble parameter \( H \) which is defined by \( H = \frac{1}{a} \frac{da}{d\tau} \) is given by

\[
H^2 = - \frac{1}{l_{\text{AdS}}^2} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4}.
\]  

(3.7)

From the point of view of brane-world physics [16], the tension of brane should be determined without ambiguity as \( \kappa = 2/l_{\text{AdS}} \), so we take it as above from now on. In fact, one can calculate \( \kappa \) requiring to cancel the leading divergence of bulk AdS Schwarzschild.

This equation can be rewritten by using 4-dimensional energy density \( \rho \) and volume \( V \) in the form of the standard FRW equation with the cosmological constant \( \Lambda \):

\[
H^2 = - \frac{1}{a^2} + \frac{8\pi G_4}{3} \rho + \frac{\Lambda}{3},
\]

\[
\rho = \frac{3\mu}{8\pi G_4 a^4}, \quad \Lambda = 0.
\]  

(3.8)
Here $G_4$ is the 4-dimensional gravitational coupling, which is defined by

$$G_4 = \frac{2G_5}{l_{\text{AdS}}} .$$

(3.9)

$\rho$ can be regarded as 4-dimensional energy density on the brane in AdS Schwarzschild background. We should note that when the bulk is pure AdS with $\mu = 0$ and therefore $\rho = 0$, the FRW equation (3.8) reduces to

$$H^2 = -\frac{1}{a^2} .$$

(3.10)

The equation (3.10) has no solution since l.h.s. is positive but r.h.s. is negative.

By differentiating Eq.(3.8) with respect to $\tau$, we obtain the second FRW equation:

$$\dot{H} = -4\pi G_4 (\rho + p) + \frac{1}{a^2} ,
\hspace{1cm} p = \frac{\mu}{8\pi G_4 a^4} .$$

(3.11)

Here $p$ is 4-dimensional pressure of the matter on the boundary.

From Eqs.(3.8) and (3.11), one finds that the energy-momentum tensor is traceless:

$$T^{\text{matter}}_{\mu \nu} = -\rho + 3p = 0 .$$

(3.12)

Therefore the matter on the brane can be regarded as the radiation. This result means the field theory on the brane should be CFT as in case of AdS Schwarzschild background [17].

Next, one considers space-like brane in 5-dimensional AdS Schwarzschild background. Similarly, we impose the following condition to obtain space-like brane metric instead of Eq.(3.4):

$$-e^{2\rho} \left( \frac{\partial t}{\partial \tau} \right)^2 + e^{-2\rho} \left( \frac{\partial a}{\partial \tau} \right)^2 = 1 .$$

(3.13)

Thus the following FRW-like metric is obtained:

$$ds_4^2 = g_{ij} dx^i dx^j = d\tau^2 + a^2 d\Omega_3^2 .$$

(3.14)
Note that this metric is also derived by Wick-rotation $\tau \rightarrow i\tau$ in Eq.(3.5). We again calculate the equations of motion and the extrinsic curvature of space-like brane instead of (3.2) and (3.6). These equations lead to FRW like equation as follows:

$$H^2 = \frac{1}{l_{\text{AdS}}^2} + \frac{1}{a^2} - \frac{\mu}{a^4} + \frac{\kappa^2}{4}. \tag{3.15}$$

One assumes this equation can be rewritten by using 4-dimensional energy density $\rho$ in the form analogous to the standard FRW equations:

$$H^2 = \frac{1}{a^2} - \frac{8\pi G_4}{3} - \frac{\Lambda}{3\rho}, \tag{3.16}$$
$$\rho = \frac{3\mu}{8\pi G_4 a^4}, \quad \Lambda = -\frac{l_{\text{AdS}}^2}{a^4}. \tag{3.17}$$

The reason why the sign of FRW equations is different from the standard FRW equations (3.8) results from the condition (3.13), namely $\tau \rightarrow i\tau$ in Eq.(3.5). From Eqs.(3.16) and (3.17), it follows the energy-momentum tensor is traceless again. It is interesting that the cosmological constant on the brane doesn’t appear for time-like FRW metric, while it appears for space-like FRW metric in AdS Schwarzschild background. We will see what happens in dS Schwarzschild case next.

When the bulk is pure AdS, $\mu = 0$ and therefore $\rho = 0$. The solution of (3.16) is given by

$$a = \frac{l_{\text{AdS}}}{\sqrt{2}} \sinh \left( \frac{\tau \sqrt{2}}{l_{\text{AdS}}} \right), \tag{3.18}$$

which is one of two sheet hyperboloid.

One assumes that there is some holographic relation between FRW universe which is reduction from dS Schwarzschild background and boundary CFT. The dS Schwarzschild space is also one of the exact solutions of bulk equations (with proper sign of bulk cosmological constant)

$$ds_5^2 = \hat{G}_{\mu\nu} dx^\mu dx^\nu$$
$$= -e^{2\rho} dt^2 + e^{-2\rho} da^2 + a^2 d\Omega^2_3,$$
$$e^{2\rho} = \frac{1}{a^2} \left( -\mu + a^2 - \frac{a^4}{l_{\text{dS}}^2} \right). \tag{3.19}$$

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Here \( l_{\text{dS}} \) is the curvature radius of dS and \( \mu \) is the black hole mass. This metric is very similar to the AdS Schwarzschild metric (3.3). The difference of Eq.(3.3) and Eq.(3.19) is in the sign in the third term of \( e^{2\rho} \), which corresponds to cosmological term in the bulk.

We first consider time-like brane in 5-dimensional dS Schwarzschild background. Similarly to AdS case, one imposes the same condition (3.4) in order to obtain time-like FRW metric (3.5). Then using equation of motion which has the same form as Eq.(3.6), we obtain FRW like equation as follows:

\[
H^2 = \frac{1}{l_{\text{dS}}^2} - \frac{1}{a^2} + \frac{\mu}{a^4} + \frac{\kappa^2}{4}.
\]  

(3.20)

From the standard FRW equations (3.8),(3.11), one defines \( E, \Lambda \) and \( p \) as

\[
\rho = \frac{3\mu}{8\pi G_4 a^4}, \quad \Lambda = \frac{6}{l_{\text{dS}}^2},
\]

\[
p = \frac{\mu}{8\pi G_4 a^4}.
\]  

(3.21)

The cosmological constant \( \Lambda \) has the opposite sign to AdS Schwarzschild case in Eq.(3.16), \( G_4 \) is the 4-dimensional gravitational coupling, which is defined by

\[
G_4 = \frac{2G_5}{l_{\text{dS}}}.
\]  

(3.22)

We now choose \( \kappa^2 = 4l_{\text{dS}}^2 \). When the bulk is pure de Sitter with \( \mu = 0 \) and therefore \( \rho = 0 \), the solution of (3.21) is given by

\[
a = \frac{l_{\text{dS}}}{\sqrt{2}} \cosh \left( \frac{\tau \sqrt{2}}{l_{\text{dS}}} \right),
\]  

(3.23)

which is so-called one-sheet hyperboloid.

On the other hand, by using the same method for AdS Schwarzschild background with space-like FRW metric, from FRW-like equation,

\[
H^2 = -\frac{1}{l_{\text{dS}}^2} + \frac{1}{a^2} - \frac{\mu}{a^4} + \frac{\kappa^2}{4},
\]  

(3.24)

13
one can derive the $E_4, \Lambda$ and $p$ with $\kappa^2 = 4l_{dS}^2$, as

\[
\begin{align*}
\rho &= \frac{3\mu}{8\pi G_4 a^4}, \quad \Lambda = 0, \\
p &= \frac{\mu}{8\pi G_4 a^4}.
\end{align*}
\]  

(3.25)

Therefore we find the energy-momentum tensor is traceless (dual QFT is CFT) for the dS Schwarzschild background too. When the bulk is pure de Sitter, the solution of (3.24) is given by

\[
a = \tau,
\]  

(3.26)

which is the cone.

We point out that[19] the cosmological constant on the brane appears in AdS Schwarzschild background with space-like brane, while it appears in dS Schwarzschild background with time-like brane. The energy-momentum tensor is traceless for all cases.

Moreover, for all cases Hubble parameter $H$ takes the same form as $\pm 1/l_{\text{AdS}}$ or $\pm 1/l_{\text{dS}}$ when brane crosses the horizon. Here the plus sign corresponds to the expanding brane universe and the minus one to the contracting universe. Let us choose the expanding case below. The four-dimensional Hubble entropy is defined as[4]

\[
S = \frac{HV}{2G_4}
\]  

(3.27)

It takes the following forms

\[
S_4 = \frac{V}{2l_{\text{AdS}} G_4}, \quad \frac{V}{2l_{\text{dS}} G_4} = \frac{V}{4G_5}.
\]  

(3.28)

when brane crosses the horizon. Here Eqs.(3.9), (3.22) are used. The entropy (3.28) is nothing but the Bekenstein-Hawking entropy of 5-dimensional AdS (dS) black hole similarly to ref.[17, 18].

Coming back to the discussion of previous section where it was shown that the $d$-dimensional FRW equation can be regarded as an analogue of the Cardy formula of 2-dimensional CFT [4] one gets

\[
S_4 = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{c}{24} \right)}.
\]  

(3.29)
Let us use the Cardy formula for AdS(dS) Schwarzschild background with time-like and space-like branes.

For time-like brane of AdS Schwarzschild background, identifying
\[
\frac{2\pi}{3} V \rho a \Rightarrow 2\pi L_0, \\
V \Rightarrow \frac{c}{24}, \\
HV \Rightarrow S_4, \\
\] (3.30)

one can rewrite FRW equation (3.8) in the form of Cardy formula Eq.(3.29). Note that the identification of Eq.(3.30) is identical with original one \[4\] exactly.

For space-like brane of AdS Schwarzschild background, identifying
\[
\frac{2\pi}{3} \left( V \rho a + \frac{\Lambda V a}{8\pi G_4} \right) \Rightarrow 2\pi L_0, \\
V \Rightarrow \frac{c}{24}, \\
-HV \Rightarrow S_4, \\
\] (3.31)

one rewrites FRW equation (3.16). Here $H$ changes as $H \rightarrow -iH$ since $H$ is defined by $H = \frac{1}{a} \frac{da}{d\tau}$ and $\frac{d}{d\tau}$ change as $-i\frac{d}{d\tau}$ by the Wick-rotation, $\tau \rightarrow i\tau$.

For time-like brane of dS Schwarzschild background, we assume the identification as
\[
\frac{2\pi}{3} \left( V \rho a + \frac{\Lambda V a}{8\pi G_4} \right) \Rightarrow 2\pi L_0, \\
V \Rightarrow \frac{c}{24}, \\
HV \Rightarrow S_4, \\
\] (3.32)

In above two cases, space-like brane of AdS and time-like brane of dS Schwarzschild background, the effect of the cosmological constant appears in Cardy formula. We included contribution of the cosmological constant in $L_0$ because it shifts
the vacuum energy. This means the cosmological entropy bound [4] should be changed. The Bekenstein bound in 4-dimensions is

$$S \leq S_B, \quad S_B \equiv \frac{2\pi}{3}\rho V a. \quad (3.33)$$

Using Eq.(3.32), the Bekenstein entropy bound should be changed as follows:

$$S \leq S_B, \quad S_B \equiv \frac{2\pi}{3}V a \left(\rho + \frac{\Lambda}{8\pi G_4}\right). \quad (3.34)$$

Thus, the effect of the cosmological constant appears in the change of the Bekenstein entropy bound.

For space-like brane of dS Schwarzschild background, the identification looks as follows:

$$\frac{2\pi}{3}V \rho a \Rightarrow 2\pi L_0,$$

$$\frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24},$$

$$-\frac{iHV}{2G_4} \Rightarrow S_4, \quad (3.35)$$

which represents the FRW-like equation (3.24) in the form of Cardy formula Eq.(3.29) again. Thus, we presented the review of brane motion as FRW equation (or Cardy formula) in AdS (dS) Schwarzschild black hole background when the theory is Einstein gravity with cosmological constant.

In order to discuss the second way of getting the Cardy formula by using Casimir energy like in Eq.(2.18) one should use the holographic principle. For simplicity, we examine only the 5-dimensional AdS Schwarzschild black hole case here.

The horizon radius, $a_H$, is deduced by solving the equation $e^{2\rho_0(a_H)} = 0$ in (3.3), i.e.,

$$a_H^2 = -\frac{l_{\text{AdS}}^2}{2} + \frac{1}{2}\sqrt{l_{\text{AdS}}^4 + 4\mu l_{\text{AdS}}^2}. \quad (3.36)$$

The Hawking temperature, $T_H$, is then given by

$$T_H = \left(e^{2\rho}\right)'_{a=a_H} = \frac{1}{2\pi a_H} + \frac{a_H}{\pi l_{\text{AdS}}^2}, \quad (3.37)$$
where a prime denotes differentiation with respect to $r$. One can also rewrite the mass parameter, $\mu$, using $a_H$ or $T_H$ from Eq. (3.36) as follows:

$$\mu = \frac{a_H^4}{l_{AdS}^2} + a_H^2 = a_H^2 \left( \frac{a_H^2}{l_{AdS}^2} + 1 \right)$$

$$= \frac{1}{4} \left( \pi l_{AdS}^2 T_H \pm \sqrt{(\pi l_{AdS}^2 T_H)^2 - 2l_{AdS}^2} \right)^2$$

$$\times \left( \frac{1}{4l_{AdS}^2} \left( \pi l_{AdS}^2 T_H \pm \sqrt{(\pi l_{AdS}^2 T_H)^2 - 2l_{AdS}^2} \right)^2 + 1 \right). \quad (3.38)$$

The entropy $S$ and the thermodynamical energy $E$ of the black hole are given in [17, 18]

$$S = \frac{V_3 \pi a_H^3}{2} \frac{8}{16\pi G_5}$$

$$= \frac{V_3 \pi}{32\pi G_5} \left( \pi l^2 T_H \pm \sqrt{(\pi l^2 T_H)^2 - k l^2} \right)^3. \quad (3.39)$$

$$E = \frac{3V_3 \mu}{16\pi G_5}. \quad (3.40)$$

On the other hand, the 4-dimensional energy can be derived from the FRW equations of the brane universe in the SAdS background. It is given by Eq.(3.8)

$$E_4 = \frac{3V_3 l_{AdS} \mu}{16\pi G_5 a}. \quad (3.41)$$

Then the relation between 4-dimensional energy $E_4$ on the brane and 5-dimensional energy in Eq.(3.40) is as follows [17]

$$E_4 = \frac{l_{AdS}}{a} E. \quad (3.42)$$

It is assumed that the total entropy $S$ of the dual CFT on the brane is given by Eq. (3.39). If this entropy is constant during the cosmological evolution, the entropy density $s$ is given by

$$s = \frac{S}{a^3 r_3} = \frac{r_H^3}{2a^3 G_4 l_{AdS}} \quad (3.43)$$
If one further assumes that the temperature $T$ on the brane differs from the Hawking temperature $T_H$ by the factor $l_{\text{AdS}}/a$ like energy relation, it follows that

$$T = \frac{l_{\text{AdS}}}{a} T_H = \frac{r_H}{\pi a l_{\text{AdS}}} + \frac{l_{\text{AdS}}}{2\pi a r_H}$$  \hspace{1cm} (3.44)$$

and, when $a = r_H$, this implies that

$$T = \frac{1}{\pi l_{\text{AdS}}} + \frac{l_{\text{AdS}}}{2\pi r_H^2}.$$  \hspace{1cm} (3.45)$$

If the energy and entropy are purely extensive, the quantity $E_4 + pV - TS$ vanishes. In general, this condition does not hold and one can define the Casimir energy $E_C$ as in case for Einstein gravity (section two):

$$E_C = 3(E_4 + pV - TS).$$  \hspace{1cm} (3.46)$$

Then, by using Eqs. (3.39), (3.41), and (3.44), and the relation $3p = E_4/V$, we find that

$$E_C = \frac{3l_{\text{AdS}} r_H^2 V_3}{8\pi G_5 a}.$$  \hspace{1cm} (3.47)$$

Finally, by combining Eqs. (3.39), (3.41), and (3.47) one gets

$$S = \frac{4\pi a}{3\sqrt{2}} \sqrt{|E_C \left(E_4 - \frac{1}{2} E_C\right)|}.$$  \hspace{1cm} (3.48)$$

Thus, we have demonstrated how the FRW equation which is written in CV form (second way) can be related to the thermodynamics of the bulk black hole. Similarly, one can consider bulk dS black hole thermodynamics.

4 FRW brane equations from 5-dimensional Einstein-Maxwell gravity

In this section, we discuss the Cardy-Verlinde formula from the five-dimensional Einstein-Maxwell gravity. The example of such sort is necessary in order to understand the role of non-gravitational fields in rewriting of FRW equations in the Cardy form.
Let us consider 4-dimensional brane in 5-dimensional Reissner-Nordstrom-de Sitter (RNdS) background following ref.[21]. The bulk solution is given as:

\[ ds^2 = -h(a)dt^2 + \frac{1}{h(a)}da^2 + a^2d\Omega_3^2, \quad (4.1) \]

\[ h(a) = e^{2\rho} = 1 - \frac{a^2}{l_{dS}^2} - \frac{\omega_4 M}{a^2} + \frac{3\omega_4^2 Q^2}{16a^4}, \quad (4.2) \]

\[ \omega_4 = \frac{16\pi G_5}{3V_3}, \quad \phi_{dS}(a) = \frac{3\omega_4 Q}{8a^2}. \quad (4.3) \]

Here \( l_{dS} \) is the curvature radius of dS background, \( d\Omega_3^2 \) is a unit 3-dimensional sphere with volume \( V_3 \), \( G_5 \) is the 5-dimensional Newton constant, \( M \) and \( Q \) are the conserved quantities of black hole mass and charge respectively. \( \phi_{dS}(a) \) is a measure of the electrostatic potential at \( a \).

To derive FRW equations, one adopts the same method of Section 3. The dynamics of the brane is assumed to be described by the boundary action (3.1) even in RNdS bulk. Then one can get the equation of motion of the brane from this Lagrangian as in (3.2). Following the method of Section 3, we rewrite RNdS metric (4.1) in the form of FRW metric by using a new time parameter \( \tau \). On the brane the coordinates \( t \) and \( a \) in (4.1) are the functions of \( \tau \), namely \( a = a(\tau) \), \( t = t(\tau) \) as in (3.5). By imposing the condition (3.4), we can rewrite the equations of motion (3.2) as (3.6), again. From Eqs. (3.4) and (3.6), one can derive FRW equation for a radiation dominated universe (Hubble parameter \( H \) is defined by \( H^2 = \frac{1}{a^2} - \frac{1}{l_{dS}^2} - \frac{\omega_4 M}{a^2} - \frac{3\omega_4^2 Q^2}{16a^4} + \kappa^2 \)).

\[ H^2 = \frac{1}{l_{dS}^2} - \frac{1}{a^2} + \frac{\omega_4 M}{a^2} - \frac{3\omega_4^2 Q^2}{16a^4} + \kappa^2 \quad (4.4) \]

The tension of brane is chosen as \( \kappa = 2/l_{dS} \). The above equation can be written in the form of the standard FRW equation with \( Q \) and the cosmological constant \( \Lambda \):

\[ H^2 = -\frac{1}{a^2} + \frac{8\pi G_4}{3} \left( \rho - \frac{1}{2} \phi \rho Q \right) + \frac{\Lambda}{3}, \]

\(^5\)In our conventions, black hole mass \( \mu \) represents the \( \omega_4 M \) in Eq.(4.2). Hereafter we adopt \( M \) as black hole mass instead of \( \mu \) for later convenience.
\[ \rho = \frac{E_4}{V} = \frac{3\omega_4 M}{8\pi G_4 a^4}, \quad V = a^3 V_3, \quad \Lambda = \frac{6}{l_{\text{dS}}^2}, \]
\[ \rho_Q = \frac{Q}{V}, \quad \phi = \frac{l_{\text{dS}}}{a} \phi_{\text{dS}}, \quad G_4 = \frac{2G_5}{l_{\text{dS}}}. \quad (4.5) \]

Here \( G_4 \) is the 4-dimensional gravitational coupling again and \( \rho \) and \( \rho_Q \) can be regarded as 4-dimensional energy density and charge density on the brane in RNdS background respectively.

By differentiating Eq.(4.5) with respect to \( \tau \), we obtain the second FRW equation:
\[ \dot{H} = -4\pi G_4 (\rho + p - \phi \rho_Q) + \frac{1}{a^2}, \]
\[ p = \frac{\omega_4 M}{8\pi G_4 a^4}. \quad (4.6) \]

Here \( p \) is 4-dimensional pressure of the matter on the boundary.

Next, we recall the Cardy formula of 2-dimensional CFT:
\[ S_4 = 2\pi \sqrt{\frac{c}{6}} \left( L_0 - \frac{c}{24} \right). \quad (4.7) \]

One can get now the Cardy formula for RNdS background with time-like branes.\(^6\) For time-like brane on RNdS background, the following identifications may be done
\[ \frac{2\pi}{3} \left( E_4 a - \frac{1}{2} \phi Q a + \frac{\Lambda V a}{8\pi G_4} \right) \Rightarrow 2\pi L_0, \]
\[ \frac{V}{8\pi G_4 a} \Rightarrow \frac{c}{24}, \]
\[ \frac{HV}{2G_4} \Rightarrow S_4, \quad (4.8) \]

It is clear that with above identifications FRW equations take the form of Cardy formula. Note that the effect of \( Q \) and the cosmological constant appears in Cardy formula (in the shift of energy for Hamiltonian). We included the contribution of the cosmological constant in \( L_0 \) because it shifts

\(^6\)For space-like brane, this may be easily repeated like for AdS(dS) Schwarzschild background.
the vacuum energy. This means the cosmological entropy bound[4] should be changed. The Bekenstein bound in 4-dimensions is

\[ S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} E_4 a. \]  

(4.9)

Using Eq.(4.8), the Bekenstein entropy bound should be changed as follows:

\[ S \leq S_B, \quad S_B \equiv \frac{2\pi}{3} a \left( E_4 - \frac{1}{2} \phi Q + \frac{\Lambda V}{8\pi G_4} \right). \]  

(4.10)

Thus, the matter fields (vectors in the example under consideration) modify some quantities which appear in Cardy formula via the contribution of the electric potential in the operator of Virasoro algebra of zero level. Moreover, the Bekenstein entropy bound is also modified. In the same way, the other fields (fermions, tensor fields, etc) will influence to Cardy representation of FRW equations. Of course, the explicit equations may be quite complicated. Note also that similarly to the discussion of the previous section one can relate thermodynamic entropy of RNdS BH with dual CFT entropy and to get the CV formula from such relation. AdS/CFT correspondence is again used in such calculation.

5 Brane New World from five-dimensional AdS-Schwarzschild Black Hole

The interesting question now is: what is the role of quantum brane effects to FRW brane cosmology and to representation of FRW equations in the form of two-dimensional entropy equation? In this section based on [25], we review the appearance of quantum matter effects in the brane equations of motion.

We assume the brane connects two bulk spaces and we may also identify the two bulk spaces as in [16] by imposing $Z_2$ symmetry. One starts with the Minkowski signature action $S$ which is the sum of the Einstein-Hilbert action $S_{EH}$ with the cosmological term, the Gibbons-Hawking surface term $S_{GH}$, the surface counter term $S_1$ and the trace anomaly induced action $\mathcal{W}$:

\[ S = S_{EH} + S_{GH} + 2S_1 + \mathcal{W}. \]  

(5.1)
\[ S_{\text{EH}} = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g(5)} \left( R(5) + \frac{12}{l^2} \right), \quad (5.2) \]

\[ S_{\text{GH}} = \frac{1}{8\pi G_5} \int d^4x \sqrt{-g(4)} \nabla_\mu n^\mu, \quad (5.3) \]

\[ S_1 = -\frac{6}{16\pi G_5 l_{\text{AdS}}} \int d^4x \sqrt{g(4)}, \quad (5.4) \]

\[ W = b \int d^4x \sqrt{-\tilde{g}} \tilde{F} A + b' \int d^4x \sqrt{\tilde{g}} \left\{ A \left[ 2 \tilde{\Box}^2 + \tilde{R}_\mu \tilde{\nabla}_\mu \tilde{\nabla}_\nu \right] \right. \]
\[ - \left[ 4 \tilde{R} \tilde{\Box} + \frac{2}{3} (\tilde{\nabla}^\mu \tilde{R}) \tilde{\nabla}_\mu \right] A + \left( \tilde{G} - \frac{2}{3} \tilde{\Box} \tilde{R} \right) A \}
\[ - \frac{1}{12} \left\{ b'' + \frac{2}{3} (b + b') \right\} \int d^4x \sqrt{\tilde{g}} \left[ \tilde{R} - 6 \tilde{\Box} A - 6 (\tilde{\nabla}_\mu A) (\tilde{\nabla}^\mu A) \right]^{2}. \quad (5.5) \]

Here the quantities in the 5 dimensional bulk spacetime are specified by the suffixes \((5)\) and those in the boundary 4 dimensional spacetime are specified by \((4)\) (for details, see [26]). In (5.3), \(n^\mu\) is the unit vector normal to the boundary. The Gibbons-Hawking term \(S_{\text{GH}}\) is necessary in order to make the variational method well-defined when there is boundary in the spacetime. In (5.4), the coefficient of \(S_1\) is determined from AdS/CFT [27]. The factor 2 in front of \(S_1\) is coming from that we have two bulk regions which are connected with each other by the brane. In (5.5), one chooses the 4 dimensional boundary metric as \(g(4)_{\mu\nu} = e^{2A} \tilde{g}_{\mu\nu}\), where \(\tilde{g}_{\mu\nu}\) is a reference metric. \(G \left( \tilde{G} \right)\) and \(F \left( \tilde{F} \right)\) are the Gauss-Bonnet invariant and the square of the Weyl tensor. \(W\) can be obtained by integrating the conformal anomaly with respect to the scale factor \(A\) of the metric tensor since the conformal anomaly should be given by the variation of the quantum effective action with respect to \(A\). Note that quantum effects of brane CFT are taken into account via Eq.(5.5).

In the effective action (5.5) induced by brane quantum conformal matter, in general, with \(N\) scalar, \(N_{1/2}\) spinor, \(N_1\) vector fields, \(N_2\) (= 0 or 1) gravitons and \(N_{\text{HD}}\) higher derivative conformal scalars, \(b\), \(b'\) and \(b''\) are [26]

\[ b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{\text{HD}}}{120(4\pi)^2} \]
\[ b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{\text{HD}}}{360(4\pi)^2} , \quad b'' = 0. \quad (5.6) \]
For typical examples motivated by AdS/CFT correspondence one has: 

a) $\mathcal{N} = 4$ $SU(N)$ SYM theory: $b = -b' = \frac{N^2 - 1}{4(4\pi)^2}$; 
b) $\mathcal{N} = 2$ $Sp(N)$ theory: $b = \frac{12N^2 + 18N - 2}{24(4\pi)^2}$ and $b' = -\frac{12N^2 + 12N - 1}{24(4\pi)^2}$. Note that $b'$ is negative in the above cases. It is important to note that brane quantum gravity may be taken into account via the contribution to correspondent parameters $b$, $b'$.

Then on the brane, we have the following equation which generalizes the classical brane equation of the motion:

$$0 = \frac{48l^4_{\text{AdS}}}{16\pi G_5} \left( A_{,z} - \frac{1}{l_{\text{AdS}}} \right) e^{4A} + b' \left( 4\partial^4 A + 16\partial^2_\tau A \right) - 4(b + b') \left( \partial^4 A - 2\partial^2_\tau A - 6(\partial_\tau A)^2 \partial^2_\tau A \right). \quad (5.7)$$

This equation is derived from the condition that the variation of the action on the brane, or the boundary of the bulk spacetime, vanishes under the variation over $A$. The first term proportional to $A_{,z}$ expresses the bulk gravity force acting on the brane and the term proportional to $\frac{1}{l_{\text{AdS}}}$ comes from the brane tension. The terms containing $b$ or $b'$ express the contribution from the conformal anomaly induced effective action (quantum effects). In (5.7), one uses the form of the metric as

$$ds^2 = dz^2 + e^{2A(z,\tau)} \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \quad \tilde{g}_{\mu\nu} dx^\mu dx^\nu \equiv l^2 \left( -d\tilde{t}^2 + d\Omega^2_3 \right). \quad (5.8)$$

Here $d\Omega^2_3$ corresponds to the metric of 3 dimensional unit sphere.

As a bulk space, one considers 5d AdS-Schwarzschild black hole spacetime (3.3). By putting $a = r$, $h = e^{2\rho}$, and $\mu = \frac{16\pi G_5 M}{3V_3}$ ($V_3$ is the volume of the unit 3 sphere) and by choosing new coordinates $(z, \tau)$ as

$$\frac{e^{2A}}{h(a)} A_{,z}^2 - h(a)t^2_z = 1, \quad \frac{e^{2A}}{h(a)} A_{,\tau} - h(a)t_z t_\tau = 0$$

$$\frac{e^{2A}}{h(a)} A_{,\tau}^2 - h(a)t^2_\tau = -e^{2A}. \quad (5.9)$$

the metric takes the warped form (5.8). Here $a = l_{\text{AdS}} e^{A}$. Further choosing a coordinate $\tilde{t}$ by $d\tilde{t} = le^A d\tau$, the metric on the brane takes FRW form:

$$e^{2A} \tilde{g}_{\mu\nu} dx^\mu dx^\nu = -d\tilde{t}^2 + l^2_{\text{AdS}} e^{2A} d\Omega_3^2. \quad (5.10)$$

By solving Eqs.(5.9), we have

$$H^2 = A_{,z}^2 - he^{-2A} = A_{,z}^2 - \frac{1}{l^2_{\text{AdS}}} - \frac{1}{a^2} + \frac{16\pi G_5 M}{3V_3 a^4}. \quad (5.11)$$

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Here the Hubble constant $H$ is introduced: $H = \frac{1}{a} \frac{da}{dt} = \frac{dA}{dt}$. On the other hand, from (5.7) one gets

$$A_z = \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( (H_{,z} + 4H_i^2 + 7HH_{,i} + 18H^2H_i + 6H^4) ight) ight.$$

$$+ \frac{4}{a^2} \left( H_i + H^2 \right) \left. \right) + 4(b + b') \left( (H_{,ii} + 4H_i^2 + 7HH_{,i} + 12H^2H_i) - \frac{2}{a^2} \left( H_i + H^2 \right) \right) \right\}. \tag{5.12}$$

Then combining (5.11) and (5.12), we find

$$H^2 = -\frac{1}{l_{\text{AdS}}^2} - \frac{1}{a^2} + \frac{16\pi G_5 M}{3V_3 a^4} + \left[ \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( (H_{,ii} + 4H_i^2$$

$$+ 7HH_{,i} + 18H^2H_i + 6H^4) \right) \right. \right. \right.$$

$$\left. \left. + \frac{4}{a^2} \left( H_i + H^2 \right) \right) + 4(b + b') \left( (H_{,ii} + 4H_i^2 + 7HH_{,i} + 12H^2H_i) - \frac{2}{a^2} \left( H_i + H^2 \right) \right) \right\} \right]\right]^2. \tag{5.13}$$

This expresses the quantum correction to the corresponding brane equation in [17]. In fact, if we put $b = b' = 0$, Eq.(5.13) reduces to the classical FRW equation

$$H^2 = -\frac{1}{a^2} + \frac{16\pi G_5 M}{3V_3 a^4}. \tag{5.14}$$

Further by differentiating Eq.(5.13) with respect to $\tilde{t}$, one arrives to second FRW equation. One can rewrite FRW equations in more familiar form

$$H^2 = -\frac{1}{a^2} + \frac{8\pi G_4 \rho}{3} \tag{5.15}$$

$$\rho = \frac{l_{\text{AdS}}}{a} \left[ \frac{M}{V_3 a^3} + \frac{3a}{16\pi G_5} \left[ \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( (H_{,ii} + 4H_i^2 + 7HH_{,i} + 18H^2H_i + 6H^4) \right) \right. \right.$$

$$\left. \left. + \frac{4}{a^2} \left( H_i + H^2 \right) \right) + 4(b + b') \left( (H_{,ii} + 4H_i^2 + 7HH_{,i} + 12H^2H_i) - \frac{2}{a^2} \left( H_i + H^2 \right) \right) \right\} \right]^2 - \frac{1}{l_{\text{AdS}}^2} \right], \tag{5.16}$$

$$H_i = \frac{1}{a^2} - 4\pi G_4 (\rho + p) \tag{5.17}$$

$$\rho + p = \frac{l_{\text{AdS}}}{a} \left[ \frac{4M}{3V_3 a^3} - \frac{1}{24l_{\text{AdS}}^2} \left[ \frac{1}{l_{\text{AdS}}} + \frac{\pi G_5}{3} \left\{ -4b' \left( (H_{,ii} + 4H_i^2 + 7HH_{,i} + 18H^2H_i + 6H^4) \right) \right. \right.$$

$$\left. \left. + \frac{4}{a^2} \left( H_i + H^2 \right) \right) + 4(b + b') \left( (H_{,ii} + 4H_i^2 + 7HH_{,i} + 12H^2H_i) - \frac{2}{a^2} \left( H_i + H^2 \right) \right) \right\} \right]^2 - \frac{1}{l_{\text{AdS}}^2} \right].$$
$$+ 18 H^2 H_{,\dot{t}} + 6 H^4 + \frac{4}{a^2} \left( H_{,\dot{t}} + H^2 \right)$$

$$+ 4 (b + b') \left( \left( H_{,\dddot{t}} + 4 H_{,\dot{t}}^2 + 7 H H_{,\dddot{t}} + 12 H^2 H_{,\dot{t}} \right) - \frac{2}{a^2} \left( H_{,\dot{t}} + H^2 \right) \right)$$

$$\times \left\{ -4 b' \left( \left( H_{,\dddot{t}} + 15 H_{,\dot{t}} H_{,\dddot{t}} + 7 H H_{,\dddot{t}} + 18 H^2 H_{,\dddot{t}} + 36 H H_{,\dot{t}}^2 \right) + 24 H^3 H_{,\dot{t}} + \frac{4}{a^2} \left( H_{,\dddot{t}} - 2 H^3 \right) \right) + 4 (b + b') \left( \left( H_{,\dddot{t}} + 15 H_{,\dot{t}} H_{,\dddot{t}} + 24 H H_{,\dot{t}}^2 \right) - \frac{2}{a^2} \left( H_{,\dddot{t}} - 2 H^2 \right) \right) \right\} .$$

(5.18)

Here 4d Newton constant $G_4$ is given by (3.9) and quantum corrections from CFT are included into the definition of energy (pressure). These quantum corrected FRW equations are written from quantum-induced brane-world perspective. As the correction terms include higher derivatives, these terms become relevant when the universe changes its size very rapidly as in the very early universe. It is also very important to note that doing the same identification as in the previous section one easily rewrites above FRW equations in the Cardy formula form. This is caused by the fact that quantum effects are included into the definition of energy and pressure. In other words, formally these equations look like classical FRW equations.

It is not so clear if the energy density $\rho$ and the pressure $p$ satisfy the energy conditions because quantum effects generally may violate the energy conditions. For the solution of (5.15), however, $\rho$ is always positive since (5.15) can be rewritten

$$\rho = \frac{3}{8\pi G_4} \left( H^2 + \frac{1}{a^2} \right) > 0. \quad (5.19)$$

We also have from (5.17)

$$\rho + p = \frac{1}{4\pi G_4} \left( \frac{1}{a^2} - H_{,\dot{t}} \right) . \quad (5.20)$$

Therefore the weak energy condition should be satisfied if $\frac{1}{a^2} - H_{,\dot{t}} > 0$ in the solution. In order to clarify the situation, we consider the specific case of $b + b' = 0$ as in $\mathcal{N} = 4$ theory and we assume that $b'$ is small. Then from (5.15) and (5.17) and by differentiating (5.17) with respect $\dot{t}$, one gets

$$H^2 = -\frac{1}{a^2} + \frac{8\pi G_4 M \Lambda_{\text{AdS}}}{3V_3 a^4} \mathcal{O} (b') , \quad H_{,\dot{t}} = \frac{1}{a^2} - \frac{16\pi G_4 M \Lambda_{\text{AdS}}}{3V_3 a^4} \mathcal{O} (b') ,$$

25
\[
H_{\ddot{t}t} = -\frac{2}{a^2} H + \frac{64\pi G_4 M l_{AdS}}{3V_3 a^4} H + \mathcal{O}(b') , \quad \text{etc.} \tag{5.21}
\]

Then by using (5.16) and (5.18), we find
\[
\rho = \frac{M l_{AdS}}{V_3 a^4} - \frac{b'}{2} \left( \frac{8\pi G_4 M l_{AdS}}{V_3 a^6} - \frac{128\pi^2 G_4^2 M^2 l_{AdS}^2}{3V_3^2 a^8} \right) + \mathcal{O}(b') \tag{5.22}
\]
\[
p = \frac{M l_{AdS}}{3V_3 a^4} - \frac{b'}{2} \left( \frac{8\pi G_4 M l_{AdS}}{V_3 a^6} - \frac{64\pi^2 G_4^2 M^2 l_{AdS}^2}{9V_3^2 a^8} \right) + \mathcal{O}(b') \tag{5.23}
\]

The correction part of \( \rho \) is not always positive but \( \rho \) itself should be positive, which is clear from (5.19). One also gets
\[
\rho + p = \frac{4M l_{AdS}}{3V_3 a^4} - \frac{b'}{2} \left( \frac{16\pi G_4 M l_{AdS}}{V_3 a^6} - \frac{1024\pi^2 G_4^2 M^2 l_{AdS}^2}{9V_3^2 a^8} \right) + \mathcal{O}(b') \tag{5.24}
\]

Then the correction part seems to be not always positive and the weak energy condition might be broken. As the above discussion is based on the perturbation theory, we will discuss the weak energy condition later using the de Sitter type brane universe solution.

Let us consider the solution of quantum-corrected FRW equation (5.13). Assume the de Sitter type solution
\[
a = A \cosh B\tilde{t} \tag{5.24}
\]

Substituting (5.24) into (5.13), one finds the following equations should be satisfied:
\[
0 = -\frac{1}{B^2} - \frac{1}{l_{AdS}^2} + \left( \frac{1}{l_{AdS}} - 8\pi G_5 b' B^4 \right)^2 \tag{5.25}
\]
\[
0 = B^2 - \frac{1}{A^2} + 2 \left( \frac{1}{l_{AdS}} - 8\pi G_5 b' B^4 \right) \frac{\pi G_5}{3} (24b' + 8b) \left( B^4 - \frac{B^2}{A^2} \right) \tag{5.26}
\]
\[
0 = \frac{16\pi G_5 M}{3V_3} + \left( \frac{\pi G_5}{3} \right)^2 (24b' + 8b)^2 \left( B^4 - \frac{B^2}{A^2} \right)^2. \tag{5.27}
\]

Eq.(5.25) tells that there is no de Sitter type solution if there is no quantum correction, or if \( b' = 0 \). Eq.(5.27) tells that if the black hole mass \( M \) is
non-vanishing and positive, there is no any solution of the de Sitter-like brane. When $M = 0$, Eqs.(5.26) and (5.27) are trivially satisfied if $A^2 = \frac{1}{B^2}$. Actually this case corresponds to well-known anomaly-driven inflation [28] (for recent discussion, see [29]). Eq.(5.25) has unique non-trivial solution for $B^2$, which corresponds to the de Sitter brane universe in [27, 26]. This brane-world is called Brane New World.

When $M < 0$, there is no horizon and the curvature singularity becomes naked. We will, however, formally consider the case since there is no de Sitter-like brane solution in the classical case ($b' = 0$) even if $M$ is negative. If $M \neq 0$ or $A^2 \neq \frac{1}{B^2}$, Eq.(5.26) has the following form:

\[
0 = 1 + 2 \left( \frac{1}{l_{AdS}^2} - \frac{8\pi G_5 b'}{l_{AdS}^4 B^4} \right) \frac{\pi G_5}{3l_{AdS}^4} (24b' + 8b) B^2 .
\]  

(5.28)

Eq.(5.28) is not always compatible with Eq.(5.25) and gives a non-trivial constraint on $G_5$, $l_{AdS}$, $b$ and $b'$. If the constraint is satisfied, $B^2$ can be uniquely determined by (5.25) or (5.28). Then (5.27) can be solved with respect to $A^2$.

Now we consider the above constraint and solution for $B^2$. By combining (5.25) and (5.28), one obtains

\[
0 = B^6 + \frac{1}{l_{AdS}^2} B^4 - \frac{1}{\eta} , \quad \eta \equiv 4 (24b' + 8b) \left( \frac{\pi G_5}{3} \right)^2
\]  

(5.29)

\[
0 = \left( \frac{1}{l_{AdS}^3} + B^2 \right) - \left\{ \frac{1}{l_{AdS}} \left( \frac{1}{l_{AdS}^2} + B^2 \right) - \zeta \right\} ,
\]

\[
\zeta \equiv \frac{6b'}{24b' + 8b} \left( \frac{3}{\pi G_5} \right) .
\]  

(5.30)

In most of cases, $\eta$ is negative and $\zeta$ is positive. The explicit solution of (5.29) is given by

\[
B^2 = - \frac{1}{3l_{AdS}^2} + \left( \frac{1}{27l_{AdS}^6} - \frac{1}{2\eta} + \sqrt{\frac{1}{4\eta^2} - \frac{1}{27l_{AdS}^6\eta}} \right)^\frac{1}{3}
\]

\[
+ \left( \frac{1}{27l_{AdS}^6} - \frac{1}{2\eta} - \sqrt{\frac{1}{4\eta^2} - \frac{1}{27l_{AdS}^6\eta}} \right)^\frac{1}{3} .
\]  

(5.31)
On the other hand, if \( \frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^2} > 0 \), the solution of (5.30) is given by

\[
B^2 = -\frac{2}{3l_{AdS}^2} + \left( -\frac{1}{27l_{AdS}^6} + \frac{\zeta}{3l_{AdS}^3} - \frac{\zeta^2}{2} + \frac{\left( \frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^4} \right)^{\frac{1}{2}}}{27l_{AdS}^6} \right) \left( -\frac{1}{27l_{AdS}^6} + \frac{\zeta}{3l_{AdS}^3} - \frac{\zeta^2}{2} - \frac{\left( \frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^4} \right)^{\frac{1}{2}}}{27l_{AdS}^6} \right) + \frac{\zeta^3}{27l_{AdS}^4}.
\]

(5.32)

or if \( \frac{\zeta^4}{4} - \frac{\zeta^3}{27l_{AdS}^2} < 0 \), the solutions are

\[
B^2 + \frac{2}{3l_{AdS}^2} = \xi + \xi^*, \quad \xi \omega + \xi^* \omega^2, \quad \xi \omega^2 + \xi^* \omega.
\]

(5.33)

Here

\[
\xi = \left( -\frac{1}{27l_{AdS}^6} + \frac{\zeta}{3l_{AdS}^3} - \frac{\zeta^2}{2} + i\sqrt{\frac{\zeta^3}{27l_{AdS}^4} - \frac{\zeta^4}{4}} \right)^{\frac{1}{2}}, \quad \omega = e^{\frac{2\pi}{3}}.
\]

(5.34)

Then if the solution (5.31) coincides with any of the solutions (5.32) or (5.33), there occurs quantum-induced de Sitter-like brane realized in d5 AdS BH. In a sense, we got the extension of scenario of refs.[27, 26] for quantum-induced brane-worlds within AdS/CFT set-up when bulk is given by d5 AdS BH.

For the de Sitter type solution (5.24), Eq.(5.20) has the following form:

\[
\rho + p = \frac{1}{4\pi G_4} \left( \frac{1}{A^2} - B^2 \right) \frac{1}{\cosh^2 Bi}.
\]

(5.35)

Then the weak energy condition can be satisfied if

\[
\frac{1}{A^2} \geq B^2.
\]

(5.36)

For the exact de Sitter solution corresponding to \( M = 0 \), we have \( \frac{1}{A^2} = B^2 \) and Eq.(5.36) is satisfied. For more general solution in (5.31) or (5.33), \( B \) and \( A \) non-trivially depend on the parameters \( G_5, M, b \) and \( b' \) and it is not so clear if Eq.(5.36) is always satisfied.

The more detailed analysis shows that quantum corrections induce de Sitter brane not only in the case of zero black hole mass but also in the case
of negative black hole mass. Of course, the specific details of such brane-world inflation depend on the fields content on the brane. As a final remark one can note that above picture may be considered also in the case when bulk space is de Sitter black hole (see first work in ref.[24]). The presentation of brane equations in Cardy form is again possible.

6 Brane matter induced by five-dimensional Einstein-Gauss-Bonnet gravity

In the present section we study more complicated theory, i.e. Einstein-GB gravity. As the bulk space, Anti-de Sitter black hole is considered. The question is: how looks the brane matter (which may be considered as dark matter) induced in such brane-world theory? As it is shown explicitly such brane matter is quite complicated what is caused by higher derivatives terms.

The action of the \((d+1)\)-dimensional Einstein–GB bulk action is given by

\[
S = \int d^{d+1}x \sqrt{g} \left\{ \frac{1}{\kappa_y^2} R - \Lambda_{\text{bulk}} + c \left( R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} \right) \right\}, \tag{6.1}
\]

where \(c\) is an arbitrary coupling constant, \(\kappa_y^2 = 16\pi G_5\) parametrizes the \((d+1)\)-dimensional Planck mass, the Riemann tensor, \(R_{\mu\nu\xi\sigma}\), and its contractions are constructed from the metric, \(g_{\mu\nu}\), and its derivatives, \(g = \det g_{\mu\nu}\) and \(\Lambda_{\text{bulk}}\) represents the bulk cosmological constant.

By extremising the variations of the action (6.1) with respect to the metric tensor we obtain the field equations

\[
0 = \frac{1}{2} g^{\mu\nu} \left\{ c \left( R^2 - 4 R_{\rho\sigma} R^{\rho\sigma} + R_{\mu\lambda\xi\sigma} R^{\mu\lambda\xi\sigma} \right) + \frac{1}{\kappa_y^2} R - \Lambda_{\text{bulk}} \right\} \tag{6.2}
+ c \left( -2 R R^{\mu\nu} + 4 R_{\mu} R^{\mu\rho} + 4 R^{\mu\rho\sigma\tau} R_{\rho\sigma\tau} - 2 R_{\mu\rho\sigma\tau} R^{\mu\rho\sigma\tau} \right) - \frac{1}{\kappa_y^2} R^{\mu\nu} .
\]

\[7\]In this section, upper case Latin indices run from \((A, B) = (1, 2, 3)\) over the spatial sections of the world–volume of the brane and \(y\) is the coordinate associated with the fifth dimension.
One considers the case where the bulk spacetime corresponds to a static, hyper-spherically symmetric geometry with a line element given by

\[ ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 \sum_{A,B=1}^{d-1} \tilde{g}_{AB} dx^A dx^B, \]

(6.3)

where \{\nu(r), \lambda(r)\} are functions of the radial coordinate, \( r \), and the metric \( \tilde{g}_{ij} \) is the metric of the \((d-1)\)-dimensional Einstein manifold with a Ricci tensor defined by \( \tilde{R}_{ij} = \kappa g_{ij} \). The constant \( k \) has values \( k = \{d-2, 0, -(d-2)\} \) for a \((d-1)\)-dimensional unit sphere, a flat Euclidean space and a \((d-1)\)-dimensional unit hyperboloid, respectively.

Restricting to the five-dimensional case \((d = 4)\), Eq. (6.2) admits the black hole solution [20, 9]:

\[
e^{2\nu} = e^{-2\lambda} = \frac{1}{2c} \left\{ c k + \frac{r^2}{\kappa^2} \right\} \pm \sqrt{\frac{r^4}{4\kappa^4} \left( \frac{4c^2\kappa^2}{\lambda^2} - 1 \right)^2 - \frac{2c\mu}{\kappa^2} \left( \frac{4c^2\kappa^2}{\lambda^2} - 1 \right)}, \]

(6.4)

where the constant \( \mu \) is related to the gravitational mass of the black hole and

\[
\frac{1}{l^2} \equiv \frac{1}{4c^2\kappa^2} \left( 1 \pm \sqrt{1 + \frac{2c\Lambda_{\text{bulk}}\kappa^4}{3}} \right). \]

(6.5)

The constant, \( l^2 \), is determined by the Gauss–Bonnet coupling parameter, \( c \), and the bulk cosmological constant, \( \Lambda_{\text{bulk}} \). It corresponds to the length parameter of the asymptotically AdS space when \( r \) is large. If \( c\Lambda_{\text{bulk}} > 0 \), \( l^2 \) can be formally negative and the spacetime then becomes asymptotically de Sitter. In principle, the above solution (6.4) for positive \( l^2 \) generalizes the well-known Schwarzschild-AdS black hole solution to Einstein–GB gravity.

We now proceed to consider the motion of a domain wall (three-brane) along a timelike geodesic of the five-dimensional, static background defined by Eqs.(6.3) and (6.4). The equation of motion of the brane is interpreted by an observer confined to the brane as an effective Friedmann equation describing the expansion or contraction of the universe. From this Friedmann
equation, we can deduce the energy and entropy of the matter in the brane universe. Specifically, we consider a brane action of the form:

$$S_{br} = -\eta \int d^4x \sqrt{-h}.$$ (6.6)

where $\eta$ is a positive constant representing the tension associated with the brane and $h$ is the determinant of the boundary metric, $h_{ij}$, induced by the bulk metric, $g_{\mu\nu}$.

We employ the method developed in Ref.[18] to derive the Friedmann equation. (Note that it is more complicated than Einstein gravity case considered in section 3.) The metric (6.3) is rewritten by introducing new coordinates $(y, \tau)$ and a scalar function $A = A(y, \tau)$ that satisfies the set of constraint equations:

$$l^2 e^{2A+2\lambda} A_y^2 - e^{-2\lambda} t_y^2 = 1, $$

$$l^2 e^{2A+2\lambda} A_y A_{\tau} - e^{-2\lambda} t_y t_{\tau} = 0, $$

$$l^2 e^{2A+2\lambda} A_{\tau}^2 - e^{-2\lambda} t_{\tau}^2 = -l^2 e^{2A},$$ (6.7)

where a comma denotes partial differentiation. When $\lambda = -\nu$, as in Eq. (6.4), the metric (6.3) may then be written in the form

$$ds^2 = dy^2 + e^{2A(y, \tau)} \sum_{i,j=1}^4 \tilde{g}_{ij} dx^i dx^j,$$ (6.8)

where $r = l \exp(A)$. Since we are interested in the cosmological implications, we assume that the metric, $\tilde{g}_{ij}$, respects the same symmetries as the metric of the Friedmann–Robertson–Walker (FRW) models, i.e., we assume that

$$\tilde{g}_{ij} dx^i dx^j \equiv l^2 \left(-d\tau^2 + d\Omega^2_{k,3}\right),$$ (6.9)

where $d\Omega^2_{k,3}$ is the metric of unit three–sphere for $k > 0$, three–dimensional Euclidean space for $k = 0$ and the unit three–hyperboloid for $k < 0$. Thus, choosing a timelike coordinate, $\tilde{t}$, such that $dt \equiv l e^{A} d\tau$, implies that the induced metric on the brane takes the FRW form:

$$ds^2_{brane} = -d\tilde{t}^2 + l^2 e^{2A} d\Omega^2_{k,3}.$$ (6.10)
It follows by solving Eqs. (6.7) that

\[ H^2 = A^2_y - e^{-2\lambda}e^{-2A/l^2}, \]  

(6.11)

where the Hubble parameter on the brane is defined by \( H \equiv dA/d\tilde{t} \). Thus, for a vacuum brane that has no matter confined to it, the cosmic expansion (contraction) is determined once the functional forms of \( \{A, \lambda\} \) have been determined. Hereafter we use the scale factor of the metric on the brane as \( a \equiv e^{A(\tilde{t})} \), then the Hubble parameter is rewritten in the standard form

\[ H = \frac{1}{a} \frac{da}{d\tilde{t}}. \]

Then the Friedmann equation describing the motion of the 3-dimensional brane in the AdS BH bulk space-time is given by the following equation (see work by Lidsey et al from ref.[10]) where the function \( f(a) \) is introduced for later convenience:

\[ H^2 = \frac{G^2}{H^2} - \frac{X(a)}{a^2} \equiv f(a), \]  

(6.12)

where

\[ G = 4\eta \pm \frac{12X^{1/2}}{Y^{3/2}} \left\{ 16\epsilon^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3} \right\} \]

\[ \eta \equiv \frac{6}{\kappa_g^2 l} (12\epsilon - 1). \]  

(6.13)

and

\[ H = -\frac{48}{\kappa_g^2} \pm \frac{24}{Y^{3/2}} \left( 5(2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-2} + \frac{(4\epsilon - 1)^2}{4\kappa_g^4} a^6 \right. \]

\[ \left. - \frac{9(4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4 a^2} \right), \]  

(6.14)

\[ X \equiv \frac{k}{2} + \frac{a^2}{4\epsilon l^2} \pm \frac{Y^{1/2} \kappa_g^2}{2\epsilon l^2}, \quad Y \equiv -2\epsilon \tilde{\mu}(4\epsilon - 1) + \frac{(4\epsilon - 1)^2}{4\kappa_g^4} a^4. \]  

(6.15)

Here we have defined the rescaled parameters:

\[ \epsilon \equiv \frac{c\kappa_g^2}{l^2}, \quad \tilde{\mu} \equiv \frac{l^2 \mu}{\kappa_g^4}. \]  

(6.16)
The standard FRW equations for 4-dimensions can be written as

\[
H^2 = \frac{8\pi G}{3} \rho - \frac{k}{2a^2},
\]

\[
\dot{H} = -4\pi G (\rho + p) + \frac{k}{2a^2},
\]

(6.17)

here, \(\dot{}\) is the derivative with respect to cosmological time \(\tilde{t}\). Then \(\rho\) and \(p\) are

\[
\rho = \frac{3}{8\pi G} \left(f(a) + \frac{k}{2a^2}\right)
\]

(6.18)

\[
p = -\frac{1}{8\pi G} \left(\frac{k}{2a^2} + af'(a) + 3f(a)\right),
\]

(6.19)

where \(\prime\) denotes the derivative with respect to \(a\). Similarly, FRW equations from the bulk dS BH maybe constructed (see work by Lidsey et al in ref.[10].

In [4], it was shown that the standard FRW equation in \(d\) dimensions can be regarded as a \(d\)-dimensional analogue of the Cardy formula for a two-dimensional CFT [5]:

\[
\tilde{S} = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{k}{d-2}\frac{c}{24}\right)},
\]

(6.20)

where \(c\) is the analogue of the two-dimensional central charge and \(L_0\) is the analogue of the two-dimensional Hamiltonian. In the present case \((d = 4)\), we make the following identifications (similarly to the case of Einstein-Maxwell gravity in section 4):

\[
\frac{2\pi \rho V a}{3} \rightarrow 2\pi L_0,
\]

\[
\frac{2V}{16\pi Ga} \rightarrow \frac{c}{24},
\]

\[
\frac{8\pi HV}{16\pi G} \rightarrow \tilde{S},
\]

\[
V = a^3 V_3,
\]

(6.21)

where \(V_3\) is the volume of the three-dimensional sphere with unit radius. Then one finds the first FRW-like equation (6.17) has the same form as Eq. (6.20).

---

8In our conventions, \(k\) takes \(-2, 0, 2\).
In order to get explicit form of \( p \), one calculates \( f'(a) \) as

\[
f'(a) = \frac{2G^2}{H^2} \left( \frac{G'}{G} - \frac{\mathcal{H}'}{\mathcal{H}} \right) - \frac{X'}{a^2} + \frac{2X}{a^3} . \tag{6.22}
\]

From Eqs.(6.15), we get

\[
X' = \frac{a}{2\epsilon l^2} \pm \frac{Y^{-1/2} \kappa_g^2 (4\epsilon - 1)^2}{4\epsilon l^2} a^3 , \quad Y' = \frac{(4\epsilon - 1)^2}{\kappa_g^4} a^3 . \tag{6.23}
\]

Then, the last two terms of Eq.(6.22) are written as

\[
- \frac{X'}{a^2} + \frac{2X}{a^3} = \pm \frac{\kappa_g^2}{4\epsilon l^2} \frac{(4\epsilon - 1)^2}{\kappa_g^4} a^{-1/2} + \frac{k}{a^3} \pm \frac{\kappa_g^2}{a^3\epsilon l^2} Y^{1/2} . \tag{6.24}
\]

To calculate the first two terms of Eq.(6.22), the derivatives of \( G \) and \( \mathcal{H} \) are needed

\[
G' = \pm \frac{12 X^{1/2}}{Y^{3/2}} \left\{ 16 \epsilon^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3} \right\} \left( \frac{1}{2} \frac{X'}{X} - \frac{3}{2} \frac{Y'}{Y} - \frac{3}{a} \right)
\]

\[
\mathcal{H}' = \pm \frac{24}{Y^{3/2}} \left\{ - \frac{3}{2} \frac{Y'}{Y} \left( 5 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-2} + 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^6 \right. 
- \frac{9 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4} a^2 
+ \left. \left( -10 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-3} 
+ 36 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^{-5} - \frac{18 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4} a \right) \right\} \tag{6.25}
\]

Substituting above equations into Eq.(6.22), one obtains

\[
f'(a) = \frac{2G^2}{H^2} \times \left( \pm \frac{12 X^{1/2}}{G} \frac{1}{Y^{3/2}} \left\{ 16 \epsilon^2 \tilde{\mu}^2 (4\epsilon - 1)^2 a^{-3} \right\} \left( \frac{1}{2} \frac{X'}{X} - \frac{3}{2} \frac{Y'}{Y} - \frac{3}{a} \right) 
\pm \frac{1}{\mathcal{H} Y^{3/2}} \left\{ - \frac{3}{2} \frac{Y'}{Y} \left( 5 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-2} + 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^6 
- \frac{9 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4} a^2 
+ \left( -10 (2\epsilon \tilde{\mu} (4\epsilon - 1))^2 a^{-3} 
+ 36 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^{-5} - \frac{18 (4\epsilon - 1)^3 \epsilon \tilde{\mu}}{2\kappa_g^4} a \right) \right\} \right) \right)
\]
\[ +36 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^5 - \frac{18 (4\epsilon - 1)^3 \epsilon \bar{\mu}}{2\kappa_g^4 a} \right) \right) \\
+ \frac{\kappa_g^2}{4\epsilon l^2} (4\epsilon - 1)^2 a Y^{-1/2} + \frac{k}{a^3} \pm \frac{\kappa_g^2}{\epsilon l^2 a^3} Y^{1/2} \right]. \quad (6.26) \]

This equation has very complicated form, so we consider mainly the limit \( a \to \infty \) or \( a \to 0 \).

One first considers the \( \rho \) and \( p \) until the order of \( a^{-4} \), in the limit of \( a \to \infty \). Then

\[
X = \frac{k + a^2}{2} \pm \frac{\kappa_g^2}{2\epsilon l^2} Y^{1/2}, \\
= \frac{k}{2} + \frac{a^2}{4\epsilon l^2} \pm \frac{4\epsilon - 1}{4\epsilon l^2} a^2 \left( 1 - \epsilon \bar{\mu} \frac{4\kappa_g^4}{(4\epsilon - 1)a^4} \right), \\
Y = \frac{(4\epsilon - 1)^2}{4\kappa_g^4} a^4 \left( 1 - \epsilon \bar{\mu} \frac{8\kappa_g^4}{(4\epsilon - 1)a^4} \right), \\
G \to 4\eta, \\
H \to -\frac{48}{\kappa_g^2} \pm 24 \frac{8\kappa_g^6}{(4\epsilon - 1)^3} a^{-6} \left( 1 - \epsilon \bar{\mu} \frac{8\kappa_g^4}{(4\epsilon - 1)a^4} \right)^{-3/2} \\
\times \left( 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 a^6 - \frac{9(4\epsilon - 1)^3 \epsilon \bar{\mu} a^2}{2\kappa_g^4} \right), \\
\approx -\frac{48}{\kappa_g^2} \pm 24 \frac{8\kappa_g^6}{(4\epsilon - 1)^3} \left( 1 + \epsilon \bar{\mu} \frac{12\kappa_g^4 a^{-4}}{(4\epsilon - 1)} \right) \\
\times \left( 6 \left( \frac{(4\epsilon - 1)^2}{4\kappa_g^4} \right)^2 - \frac{9(4\epsilon - 1)^3 \epsilon \bar{\mu} a^{-4}}{2\kappa_g^4} \right) \\
= -\frac{24}{\kappa_g^2} \left( 2 \pm 3(4\epsilon - 1) \right). \quad (6.27) \]

Thus,

\[
f(a) = \frac{1}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} \\
- \frac{k}{2a^2} - \frac{1}{4\epsilon l^2} + \frac{4\epsilon - 1}{4\epsilon l^2} \left( 1 - \epsilon \bar{\mu} \frac{4\kappa_g^4}{(4\epsilon - 1)a^4} \right) \quad (6.28) \]
\[
\rho = \frac{3}{8\pi G} \left( \frac{1}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2} \right.
\nonumber
- \frac{1}{4\epsilon l^2} \mp \frac{4\epsilon - 1}{4\epsilon l^2} \left( 1 - \epsilon \bar{\mu} \frac{4\kappa_5^4}{(4\epsilon - 1) a^4} \right) \right)
\]
\tag{6.29}
\]
\[
p = -\frac{1}{8\pi G} \left( \frac{3}{l^2} (12\epsilon - 1)^2 (2 \pm 3(4\epsilon - 1))^{-2}
\nonumber
- \frac{3}{4\epsilon l^2} + \frac{3(4\epsilon - 1)}{4\epsilon l^2} + \frac{1}{l^2} \bar{\mu} \kappa_5^4 a^{-4} \right).
\tag{6.30}
\]

If we choose the upper sign, that is, + of \(\pm\) and - of \(\mp\) in Eqs.(6.28), (6.29), (6.30), then
\[
f(a) = -\frac{k}{2a^2} + \frac{\bar{\mu} \kappa_5^4}{l^2 a^4},
\tag{6.31}
\]
\[
\rho = \frac{3}{8\pi G} \frac{\bar{\mu} \kappa_5^4}{l^2 a^4},
\tag{6.32}
\]
\[
p = \frac{1}{8\pi G} \frac{\bar{\mu} \kappa_5^4}{l^2 a^4}.
\tag{6.33}
\]

This shows that energy-momentum tensor is traceless, \(T^\mu_\mu = \rho - 3p = 0\). This means the theory on the brane is CFT.

However, if we take the lower sign, that is - of \(\pm\) and + of \(\mp\) in Eqs.(6.28), (6.29), (6.30), there is a constant term and \(a^{-4}\) term which comes from conformal matter. Since the original Friedmann equation which includes cosmological constant \(\Lambda\) has the following form:
\[
H^2 = \frac{8\pi G}{3} \rho_m - \frac{k}{2a^2} + \frac{\Lambda}{3},
\]
\[
\dot{H} = -4\pi G (\rho + p) + \frac{k}{2a^2},
\tag{6.34}
\]
one can divide \(\rho\) and \(p\) into the sum of the contributions from matter fields \(\rho_m\) and \(p_m\) and those from the cosmological constant:
\[
\rho = \rho_m + \rho_0, \quad p = p_m - \rho_0, \quad \rho_0 = \frac{\Lambda}{8\pi G}.
\tag{6.35}
\]

Then the constant term in (6.29) corresponds to the effective cosmological constant on the brane:
\[
\rho_0 = \frac{\Lambda}{8\pi G} = \frac{3}{8\pi G} \left\{ \frac{1}{l^2} (12\epsilon - 1)^2 (5 - 12\epsilon)^{-2} - \frac{1 - 2\epsilon}{2\epsilon l^2} \right\}.
\tag{6.36}
\]
The matter parts $\rho_m$ and $p_m$ in $\rho$ and $p$ are given by

$$\rho_m = -\frac{3}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4},$$  \hspace{1cm} (6.37)$$

$$p_m = -\frac{1}{8\pi G} \frac{\tilde{\mu} \kappa_g^4}{l^2 a^4},$$  \hspace{1cm} (6.38)$$

and the matter energy-momentum tensor $T^m_{\mu\nu}$ is traceless, $T^m_{\mu\nu} = \rho_m - 3p_m = 0$. Thus, having the effective cosmological term in FRW equations, the brane matter is again the conformal one.

Next, one takes $a \to 0$ limit in case $-2\epsilon \tilde{\mu} (4\epsilon - 1) > 0$. Then

$$X \to \frac{k}{2} \pm \frac{\kappa_g^2}{2\epsilon l^2} \sqrt{-2\epsilon \tilde{\mu} (4\epsilon - 1)}, \quad Y \to -2\epsilon \tilde{\mu} (4\epsilon - 1)$$

$$G \to \pm 48 (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \left\{ \frac{k}{2} \pm \frac{\kappa_g^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \right\}^{1/2} a^{-3},$$

$$\mathcal{H} \to \mp 120 (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} a^{-2}. \hspace{1cm} (6.39)$$

These equations give $\rho$ and $p$ as

$$\rho = \frac{3}{8\pi G} \left\{ \frac{2k}{25} + \frac{21}{25} \frac{\kappa_g^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \right\} a^{-2}$$

$$p = -\frac{1}{8\pi G} \left\{ \frac{2k}{25} + \frac{21}{25} \frac{\kappa_g^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \right\} a^{-2}. \hspace{1cm} (6.40)$$

The energy-momentum tensor is not traceless. The case that $\rho$ and $p$ is proportional to $a^{-2}$ is known as curvature dominant case. The original Friedmann equation (6.34) can be rewritten in the following form:

$$H^2 = \frac{8\pi G}{3} \left( \frac{\rho_m}{3} - \frac{k}{8\pi G 2a^2} \right) + \frac{\Lambda}{3},$$

$$= \frac{8\pi G}{3} \tilde{\rho} - \frac{\tilde{k}}{2a^2} + \frac{\Lambda}{3} \hspace{1cm} (6.41)$$

Here $\tilde{\rho}$ and $\tilde{k}$ are effective energy density and effective $k$ respectively defined by

$$\tilde{\rho} = \rho_m - \frac{3}{8\pi G 2a^2} \frac{k}{k}, \quad \tilde{k} = 0. \hspace{1cm} (6.42)$$

\footnote{When curvature $k$ becomes large, $k$ can be divided as $k = k + \tilde{k}$. The $\tilde{k}$ takes the original value, namely 0, ±2.}
Such effective energy density is proportional to $a^{-2}$ when $k$ is dominant, that is $a \to \infty$, because the density $\rho_m$ must decrease with increasing $a$ at least as fast as $a^{-3}$. It is interesting that the original behavior of $\tilde{\rho}$ is proportional to $a^{-2}$ in the limit $a \to \infty$, while our $\rho$ (6.40) behaves like $a^{-2}$ in the limit of $a \to 0$.

Note that if $-2\epsilon\tilde{\mu}(4\epsilon - 1)$ is less than zero, $X, \mathcal{H}, \mathcal{G}$ become imaginary when $a = 0$. Then we consider $Y = 0$ case instead of $a \to 0$ limit. When $Y = 0$, $a$ and $X$ are

$$a = \left( \frac{8\epsilon\tilde{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} \kappa_g,$$

$$X = \frac{k}{2} + \frac{a^2}{4\epsilon l^2}$$

$$= \frac{k}{2} + \frac{1}{4\epsilon l^2} \left( \frac{2\epsilon\tilde{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} (2\kappa_g^2)$$

which leads to

$$G^2 = 4 \frac{2k}{81} \left\{ \frac{2\epsilon\tilde{\mu}}{4\epsilon - 1} \right\}^{\frac{1}{2}} (2\kappa_g^2)^{-1} + \frac{1}{4\epsilon l^2} \right\},$$

which leads to

$$\rho = \frac{3}{8\pi G} \left\{ \frac{2k}{81} \left( \frac{2\epsilon\tilde{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} (2\kappa_g^2)^{-1} - \frac{77}{81} \frac{1}{4\epsilon l^2} \right\},$$

$$p = -\frac{1}{8\pi G} \left\{ \frac{2k}{81} \left( \frac{2\epsilon\tilde{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} (2\kappa_g^2)^{-1} - \frac{77}{81} \frac{3}{4\epsilon l^2} \right\}.$$ (6.45)

One can divide $\rho$ and $p$ in (6.45) into the sum of the contributions from matter fields and the cosmological constant as in (6.35). Then one arrives at

$$\rho_0 = -\frac{3}{8\pi G} \frac{77}{81} \frac{1}{4\epsilon l^2},$$

$$\rho_m = \frac{3}{8\pi G} \frac{2k}{81} \left( \frac{2\epsilon\tilde{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} (2\kappa_g^2)^{-1}$$

$$p_m = -\frac{1}{8\pi G} \frac{2k}{81} \left( \frac{2\epsilon\tilde{\mu}}{4\epsilon - 1} \right)^{\frac{1}{2}} (2\kappa_g^2)^{-1}.$$ (6.46)
For simplicity, we consider $\epsilon = 1/4$ case where $\mathcal{G}$, $\mathcal{H}$ become

$$
\mathcal{G} = \frac{48}{\kappa_g^2 l} , \quad \mathcal{H} = -\frac{48}{\kappa_g^2} . 
$$

(6.47)

$f(a), f'(a)$ take simple forms

$$
f(a) = -\frac{k}{2a^2}, \quad f'(a) = \frac{k}{a^3} . 
$$

(6.48)

Note that $k$ should be negative , i.e. $k = -2$ since $f(a) = H^2$ is always positive . Thus $\rho$ and $p$ are

$$
p = 0, \quad \rho = 0 .
$$

(6.49)

Therefore the energy-momentum tensor is zero. Next, we consider $\epsilon = 1/4 - \delta^2$ case. Here $\delta^2 > 0$ and $|\delta| \ll 1$. In this case, $X,Y$ are

$$
X = \frac{k}{2} + \frac{a^2}{l^2} \pm \frac{2\sqrt{2\bar{\mu}\kappa_g^2 \delta}}{l^2} + \mathcal{O}(\delta^2), \quad Y \sim 2\bar{\mu} \delta^2 + \mathcal{O}(\delta^4)
$$

(6.50)

Using Eqs.(6.50), one obtains $\mathcal{G}, \mathcal{H}$ as

$$
\mathcal{G} = \frac{48}{\kappa_g^2 l} \pm 48 \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} (2\bar{\mu})^{1/2} a^{-3} \delta ,
$$

$$
\mathcal{H} = -\frac{48}{\kappa_g^2} \pm 120(2\bar{\mu})^{1/2} a^{-2} \delta .
$$

(6.51)

until the order of $\delta$. Then $f(a)$ is

$$
f(a) = -\frac{k}{2a^2} \pm \frac{1}{l^2} \left\{ 2\kappa_g^2 l \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} a \right\} \frac{7\kappa_g^2}{a^2} (2\bar{\mu})^{1/2} \delta ,
$$

(6.52)

This leads to the following $\rho$

$$
\rho = \pm \frac{3}{8\pi G l^2} \left\{ \frac{2\kappa_g^2 l}{a^4} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} a - 7 \right\} (2\bar{\mu})^{1/2} \delta .
$$

(6.53)

$f'(a)$ is

$$
f'(a) = \frac{k}{a^3} \pm \frac{\kappa_g^2}{l^2} \left\{ -\frac{6\kappa_g^2 l}{a^4} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} \right\} + \frac{2}{a^2 l^2} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} + \frac{14}{a^3} (2\bar{\mu})^{1/2} \delta .
$$
which leads to the following $p$

$$p = \pm \frac{1}{8\pi G l^2} \left\{ \frac{2}{a l} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{7}{a^2} \right\} (2\bar{\mu})^{1/2} \delta . \quad (6.54)$$

In the limit of $a \to 0$, $\rho$ which is much larger than $p$ is proportional to $a^{-3}$. This means there is “dust” on the brane. In the limit of $a \to \infty$, $\rho$ and $p$ are proportional to $a^{-2}$ like in Eq.(6.40), which agrees with the behavior of the original effective energy density $\tilde{\rho}$ as it was mentioned.

The trace of the energy-momentum tensor is

$$T^\mu_\mu = -\rho + 3p \quad (6.55)$$

$$= \mp \frac{3}{8\pi G l^2} \left\{ \frac{2l}{a^3} \left( \frac{k}{2} - \frac{a^2}{l^2} \right)^{1/2} - \frac{7}{a^2} \right\} (2\bar{\mu})^{1/2} \delta$$

$$= \mp \frac{3}{8\pi G l^2} \left\{ \frac{2}{a l} \left( \frac{k}{2} - \frac{a^2}{l^2} \right)^{-1/2} - \frac{7}{a^2} \right\} (2\bar{\mu})^{1/2} \delta$$

$$= \mp \frac{3}{8\pi G l^2} \left\{ \frac{2l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} + \frac{2}{a l} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{14}{a^2} \right\} (2\bar{\mu})^{1/2} \delta ,$$

which is not zero. In the limit $a \to \infty$, the energy-momentum tensor is traceless. This indicates that dual CFT description is valid only in such a limit.

Another simple case is $\epsilon = 1/12$, which gives $\eta = 0$. Then

$$X = \frac{k}{2} + \frac{3a^2}{l^2} \pm \frac{2a^2}{l^2} \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right)^{1/2} \quad Y = \frac{a^4}{9\kappa_g^4} \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right) . \quad (6.56)$$

$$G = \pm 16\bar{\mu}^2 a^{-9} \kappa_g^6 \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right)^{-3/2} \left( \frac{k}{2} + \frac{3a^2}{l^2} \pm \frac{2a^2}{l^2} \right)^{1/2} \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right)^{1/2}$$

$$H = -\frac{48}{\kappa_g^6} \mp a^{-6} \kappa_g^6 \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right)^{-3/2} \left( 40\bar{\mu} a^{-2} + 48 \frac{a^6}{\kappa_g^4} + 72 \bar{\mu} \kappa_g^4 a^2 \right) . \quad (6.57)$$

Above equations lead to the following $f(a)$

$$f(a) = 256\bar{\mu}^4 a^{-18} \kappa_g^{12} \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right)^{-3} \left( \frac{k}{2} + \frac{3a^2}{l^2} \pm \frac{2a^2}{l^2} \right)^{1/2} \left( \frac{\bar{\mu} \kappa_g^4}{a^4} + 1 \right)^{1/2}$$
\[
\times \left\{ \frac{48}{\kappa_g^2} \mp a^{-6}\kappa_g^6 \left( \frac{\mu \kappa_g^4}{a^4} + 1 \right)^{-3/2} \left( 40 \tilde{\mu} a^{-2} + 48 \frac{a^6}{\kappa_g^6} + 72 \frac{\tilde{\mu}}{\kappa_g^4} a^2 \right) \right\}^{-2}
\]
\[
- \frac{k}{2a^2} - \frac{3}{l^2} \mp \frac{2}{l^2} \left( \frac{\kappa_g^4}{a^4} + 1 \right)^{1/2}.
\]

(6.58)

The structure of $\rho, p$ is very complicated, so we consider them in the limit $a \to \infty$ or $a \to 0$. Taking $a \to \infty$, one gets

\[
f(a) \to - \frac{k}{2a^2} - \frac{3}{l^2} \mp \frac{2}{l^2} \left( \frac{\kappa_g^4}{a^4} + 1 \right)^{1/2},
\]
\[
\rho \to \frac{3}{8\pi G} \left( - \frac{3}{l^2} \mp \frac{2}{l^2} \mp \frac{\mu \kappa_g^4}{l^2 a^4} \right),
\]
\[
p \to \frac{1}{8\pi G} \left( - \frac{9}{l^2} \mp \frac{6}{l^2} \pm \frac{\mu \kappa_g^4}{l^2 a^4} \right).
\]

(6.59)

Similarly to Eqs.(6.29) and (6.30), there are constant and $a^{-4}$ terms which correspond to effective cosmological constant on the brane and the effect of conformal matter, respectively. Then $\rho$ and $p$ in (6.59),(6.60) can be divided into the sum of the contributions from matter and from the cosmological constant as in (6.35) again. Then they look as

\[
\rho_0 = \frac{1}{8\pi G} \left( - \frac{9}{l^2} \mp \frac{6}{l^2} \right),
\]
\[
\rho_m = \mp \frac{1}{8\pi G l^2 a^4},
\]
\[
p_m = \mp \frac{1}{8\pi G l^2 a^4}.
\]

(6.61)

and the matter energy-momentum tensor $T_{\mu \nu}^{m}$ is traceless, $T_{\mu \nu}^{m} = \rho_m - 3p_m = 0$. On the other hand, when $a$ is small, we find

\[
\rho \to \mp \frac{6}{8\pi G} \frac{21 \sqrt{\mu \kappa_g^2}}{25 l^2 a^2}, \quad p \to \pm \frac{2}{8\pi G} \frac{21 \sqrt{\mu \kappa_g^2}}{25 l^2 a^2}.
\]

(6.62)

This corresponds to the curvature dominant case as in (6.40).

It is quite interesting now to check the Weak Energy Condition (WEC) and the Dominant Energy Condition (DEC) for above four-dimensional cases.
The WEC is defined as the condition where

\[ \rho_m + p_m \geq 0, \quad \rho_m \geq 0, \]  

(6.63)

and the DEC is given by

\[ \rho_m + p_m \geq 0, \quad \rho_m + 3p_m \geq 0, \]  

(6.64)

In the general case it follows from Eqs. (6.18), (6.19):

**WEC**

\[ \rho + p = \rho_m + p_m \geq 0 \iff k \geq a^3 f'(a), \]
\[ \rho_m \geq 0 \iff \frac{k}{2a^2} \geq -f(a) - \rho_0, \]  

(6.65)

**DEC**

\[ \rho + p = \rho_m + p_m \geq 0 \iff k \geq a^3 f'(a), \]
\[ \rho_m + 3p_m \geq 0 \iff -2f(a) \geq af'(a) + 2\rho_0. \]  

(6.66)

Note that \( f(a) \) is always positive, \( f(a) \geq 0 \). We will check some limits of \( a \) and specific \( \epsilon \) cases mentioned above.

1. In the limit \( a \to \infty \), including the order of \( a^{-4} \), one finds

\[ \rho_m + p_m = \pm \frac{4 \mu k_\mu^4}{8\pi G \ell^2 a^4}, \quad \rho_m = \pm \frac{3 \mu k_\mu^4}{8\pi G \ell^2 a^4} \]
\[ \rho_m + 3p_m = \pm \frac{6 \mu k_\mu^4}{8\pi G \ell^2 a^4}. \]  

(6.67)

Then for the upper signs both of the DEC and WEC are satisfied but for the lower signs, both of the conditions are not satisfied.

2. The limit \( a \to 0 \): If \( \frac{2k}{25} + \frac{21}{25} \frac{\kappa_g^2}{2 \ell^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \geq 0 \),

\[ \rho + p = \frac{2}{8\pi G} \left\{ \frac{2k}{25} + \frac{21}{25} \frac{\kappa_g^2}{2 \ell^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \right\} a^{-2} \geq 0 \]
\[ \rho = \frac{3}{8\pi G} \left\{ \frac{2k}{25} + \frac{21}{25} \frac{\kappa_g^2}{2 \ell^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \right\} a^{-2} \geq 0 \]
\[ \rho + 3p = 0. \]  

(6.68)

Then both of the DEC and WEC are satisfied. If \( \frac{2k}{25} + \frac{21}{25} \frac{\kappa_g^2}{2 \ell^2} (-2\epsilon \tilde{\mu} (4\epsilon - 1))^{1/2} \leq 0 \), both WEC and DEC do not hold.
3. $Y = 0$ case. If $k \geq 0$,

$$\rho_m + p_m = \frac{2}{8\pi G} \left\{ \frac{2k}{81} \left( \frac{2\epsilon \tilde{\mu}}{(4\epsilon - 1)} \right)^{\frac{1}{2}} \left( 2\kappa_g^2 \right)^{-1} \right\} \geq 0, \quad (6.69)$$

$$\rho_m = \frac{3}{8\pi G} \frac{2k}{81} \left( \frac{2\epsilon \tilde{\mu}}{(4\epsilon - 1)} \right)^{\frac{1}{2}} \left( 2\kappa_g^2 \right)^{-1} \geq 0$$

$$\rho_m + 3p_m = 0. \quad (6.70)$$

Then both of the DEC and WEC are satisfied. If $k \leq 0$ both WEC and DEC do not hold.

4. When $\epsilon = 1/4$, we find

$$\rho + p = 0, \quad \rho = 0, \quad \rho + 3p = 0. \quad (6.71)$$

Then both of the DEC and WEC are trivially satisfied.

5. When $\epsilon = 1/4 - \delta^2$

$$\rho + p = \pm \frac{1}{8\pi G \ l^2} \left\{ \frac{6l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} - \frac{2}{al} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} - \frac{14}{a^2} \right\} (2\tilde{\mu})^{1/2} \delta$$

$$\rho + 3p = \pm \frac{1}{8\pi G \ l^2} \left\{ \frac{6l}{a^3} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{1/2} - \frac{6}{al} \left( \frac{k}{2} + \frac{a^2}{l^2} \right)^{-1/2} \right\} (2\tilde{\mu})^{1/2} \delta \quad (6.72)$$

If the upper signs are chosen, that is, $+\$ of $\pm$ in above equations and in the limit of $a \rightarrow \infty$, then

$$\rho + p \leq 0, \quad \rho + 3p \leq 0. \quad (6.73)$$

Then both of the DEC and WEC are not satisfied. In the limit of $a \rightarrow 0$,

$$\rho + p \geq 0, \quad \rho \geq 0 \quad \rho + 3p \geq 0. \quad (6.74)$$
Then both of the DEC and WEC are satisfied. On the other hand if
we choose the lower sings, that is, − of ± in Eqs.(6.72), in the limit of
\( a \to \infty \), both of WEC and DEC hold, while in the limit of \( a \to 0 \), both
of WEC and DEC do not hold.

6. When \( \epsilon = 1/12 \) and \( a \) is large

\[
\rho_m + p_m = \pm \frac{1}{8\pi G l^2} \frac{4\tilde{\mu} \kappa_4^4}{a^4} \quad (6.75)
\]

\[
\rho_m + 3p_m = \pm \frac{1}{8\pi G l^2} \frac{6\tilde{\mu} \kappa_4^4}{a^4} \quad (6.76)
\]

If we choose + of \( \mp \) in above equations, both of the DEC and WEC
are satisfied. If we choose − of \( \mp \) in Eq.(6.75), however, both of the
DEC and WEC are not satisfied. On the other hand when \( a \) is small
one gets

\[
\rho_m + p_m = \mp \frac{4}{8\pi G l^2} \frac{21\sqrt{\mu \kappa_2^2}}{25 a^2} \quad (6.77)
\]

\[
\rho_m + 3p_m = 0 \quad . \quad (6.78)
\]

Then for + of \( \mp \) in above equations both of the DEC and WEC are
satisfied again but for − both of the DEC and the WEC are not satis-
fied.

In the above analysis one finds that the matter on the brane shows the
singular behavior when \( \epsilon = \frac{1}{4} \) or \( \epsilon = \frac{1}{12} \). In [9] it has been shown that the
black hole entropy is given by

\[
S = \frac{V_3}{\kappa_4^2} \left( \frac{1 - 12\epsilon}{1 - 4\epsilon} \right) \left( 4\pi r_H^3 + 24\epsilon k \pi r_H \right) + S_0 \quad . \quad (6.79)
\]

Here \( r_H \) is the horizon radius and \( V_3 \) is the volume of the Einstein manifold
with unit radius. \( S_0 \) is a constant of the integration and if we assume \( S = 0 \)
when \( r_H = 0 \), we have \( S_0 = 0 \). Then the entropy vanishes when \( \epsilon = \frac{1}{12} \)
and diverges when \( \epsilon = \frac{1}{4} \). Note that above entropy should be identified with
cosmological entropy of dual QFT (second way appearance of CV formula).

If we take \( \epsilon \) between 1/12 and 1/4, that is, \( 1/12 < \epsilon < 1/4 \) then the
entropy (6.79) takes negative value. (The appearance of negative entropy
black holes in HD gravity has been discussed in [9]. As it was shown in last work of ref.[10] black hole with negative entropy is very instable and quickly decays.) Let us check what happens in the region around $1/4$ and $1/12$.

How looks the behavior around $\epsilon = 1/4$. For this purpose, we extend $\epsilon$ to the complex value. Since Eqs.(6.13), (6.14) and (6.15) contain the half-integer power of $Y$, the expressions have branch points in the complex $\epsilon$-plane when $Y = 0$, that is, at

$$\epsilon = \epsilon_1 \equiv \frac{1}{4}, \quad \epsilon = \epsilon_2 \equiv \frac{1}{4} \left( 1 - \frac{2\bar{\mu}\kappa^2}{a^4} \right).$$

and there is a cut connecting two branch points as in Fig.1. In the limit $a \to \infty$, the two branch points coincide with each other, $\epsilon_2 \to \epsilon_1$. Then if we consider the limit $a \to \infty$ first, the cut does not appear as in (6.29) and (6.30) or (6.37) and (6.38). When $\epsilon$ is real, Eqs.(6.53) and (6.54) tell that $\rho$ and $p$ becomes complex on the cut, that is, when $\epsilon_1 < \epsilon < \epsilon_2$ since $\delta$ should be pure imaginary. If we choose the path $A$ in Fig.1 which path the cut the sign of $\rho$ and $p$ is changed, that is, the value $\rho$ and $p$ in (6.29) and (6.30) are changed into those in (6.37) and (6.38) when $a$ is large. On the other hand, if we choose the path $B$ in Fig.1 the sign is not changed.

We now consider the behavior near $\epsilon = 1/12$. Eq.(6.61) shows that the signs of $\rho$ and $p$ at $\epsilon = 1/12$ are not changed since $\rho$ and $p$ have finite values there. By comparing the contribution $\rho_0$ which comes from the effective cosmological constant in the effective gravity on the brane with (6.29) where $\rho_0 = 0$, we find that there is a jump in the value of $\rho_0$ at $\epsilon = 1/12$ (there is no jump for (6.36) which corresponds to the lower $-$ sign in (6.29)). The jump might make a potential barrier at $\epsilon = 1/12$ since $\rho$ corresponds the energy density on the brane.

Figure 1: Cut and branch points on the complex $\epsilon$-plane.
In the similar way one can discuss the other values of Gauss-Bonnet coupling constant and to find the brane matter energy and pressure for such values. As it follows from the discussion in this section this is straightforward while technically a little bit complicated. In the next section we apply the found matter energy and pressure in the consideration of the FRW brane cosmology.

7 FRW Brane Cosmology from Einstein-Gauss-Bonnet gravity

In this section we discuss FRW brane equations for various examples of matter induced by Einstein-GB gravity (see previous section) by using effective potential technique. It is assumed that the bulk spacetime is asymptotically anti-de Sitter. First, we review FRW brane cosmology in five-dimensional Einstein gravity, namely $\epsilon = 0$ (no higher derivative term). If one uses effective potential technique for FRW brane equation (see section 3):

$$H^2 = -\frac{k}{2a^2} + \frac{8\pi G}{3}\rho,$$

one has to rewrite this as

$$\left(\frac{da}{dt}\right)^2 = \frac{-k}{2} - V(a).$$

Here $V(a)$ is the effective potential: $V(a) = -\frac{8\pi G a^2}{3}\rho$ which is proportional to $-1/a^2$. $V(a)$ is plotted in Fig.2. Then the universe can only exist in regions where the line $V(a) = -k/2$ exceeds $V(a)$, so that $H^2 > 0$. For the case of $k = 2$ the spherical (inflationary) brane starts at $a = 0$ and reaches its maximal size at $a_{\text{max}}$ and then it re-collapses. For $k = 0$ or $k = -2$, the brane starts at $a = 0$ and expands to infinity.

Similarly, we consider the cosmology for the higher derivative case. From the analysis of the previous section, one can easily obtain the effective potential $V(a)$ by using energy density $\rho$. For simplicity, we take several limits of $a$ and particular values of $\epsilon$ which were discussed in previous section.
Figure 2: The standard effective potential for the evolution of FRW universe. For $k = 2$, the brane starts at $a = 0$ and reaches its maximum at $a = a_{\text{max}}$ and then it re-collapses.

1. First, we consider large $a$ limit. From Eqs.(6.31), (6.36), (6.37), there appear two types of effective potential:

$$V(a) = -\frac{8\pi G a^2}{3} \rho \to_{a \to \infty} -\frac{\tilde{\mu} \kappa_g^4}{l^2 a^2}. \quad (7.3)$$

and

$$V(a) = \frac{\tilde{\mu} \kappa_g^4}{l^2 a^2} - \frac{\Lambda}{3} a^2 \to_{a \to \infty} -\frac{\Lambda}{3} a^2, \quad (7.4)$$

Here

$$\Lambda = 3 \left\{ \frac{1}{l^2} (12\epsilon - 1)^2 (5 - 12\epsilon)^{-2} - \frac{1 - 2\epsilon}{2\epsilon l^2} \right\}. \quad (7.5)$$

The former case is obtained by taking the upper sign in Eq.(6.29) and the latter is obtained by taking the the lower sign. The former case is similar to the original FRW cosmology in Einstein gravity as we mentioned. From the point of view of the brane:

- $k = 2$, the brane which is sphere (de-Sitter) reaches its maximum size at $a_{\text{max}}$ and then it re-collapses. Note that this case is the
reverse version of “bounce” (see work by Medved in ref.[23]). It can be called the bounce universe when the brane starts at $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands.

- $k = 0$ or $k = -2$, the brane which is flat or hyperbolic expands to infinity.

In the latter case of Eq.(7.4) there are three kinds of potential which are shown in Figure 3 for $\Lambda \neq 0$ cases. For the case of $\Lambda > 0$:

- $k = 2$, the de-Sitter brane starts at $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands. It can be called the bounce universe as we mentioned above.
- $k = 0$, the flat brane becomes static.
- $k = -2$, the hyperbolic brane expands to infinity.

When $\Lambda < 0$ one can take $k = -2, 0$ because the universe can only exist in the regions where the line $-k/2$ exceeds $V(a)$. Since FRW equation becomes inconsistent when $k = 2$ in the limit $a \to \infty$.

- $k = -2$, the hyperbolic brane reaches its maximum size at $a = a_{\text{max}}$ and then it re-collapses.
- $k = 0$, the flat brane becomes static.

When $\Lambda = 0$, the effective potential is given in Figure 4, and one can take only $k = -2$.

- For $k = -2$ case the brane starts at $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands.

2. Next case corresponds to $a \to 0$ limit for $-2\epsilon\tilde{\mu}(4\epsilon - 1) > 0$. From Eq.(6.40) the effective potential is

$$V(a) \xrightarrow{a \to 0} - \left\{ \frac{2k}{25} \mp \frac{21}{25} \frac{\kappa^2}{2\epsilon l^2} (-2\epsilon\tilde{\mu}(4\epsilon - 1))^{1/2} \right\}$$ (7.6)

In this case the brane exists at $a = 0$. Namely, the brane universe may have a cosmological singularity under the condition $-k/2 \geq \lim_{a \to 0} V(a)$,
Figure 3: The effective potential for the evolution of FRW universe corresponding to Eq.(7.4) with $\Lambda \neq 0$.

Figure 4: The effective potential for the evolution of FRW universe corresponding to Eq.(7.4) with $\Lambda = 0$. For $k = -2$ case, the brane starts from $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands.
that is
\[ k \leq \mp 2 Z_1(\epsilon) , \quad Z_1(\epsilon) \equiv \frac{\kappa_g^2}{2\epsilon l^2} (-2\epsilon \tilde{\mu}(4\epsilon - 1))^{1/2} . \] (7.7)

Otherwise the brane universe has no cosmological singularity. The conditions for \( Z \) that the brane exists at \( a = 0 \) are

- For \( k = 2 \) (the de-Sitter brane) it is \( Z_1(\epsilon) \geq 1 \).
- For \( k = 0 \) (the flat brane) it is \( Z_1(\epsilon) \geq 0 \).
- For \( k = -2 \) (the hyperbolic brane) it is \( Z_1(\epsilon) \geq -1 \).

For \( Z_1(\epsilon) \geq 1 \), all three kinds of brane reach the point \( a = 0 \).

3. If \(-2\epsilon \tilde{\mu}(4\epsilon - 1) < 0\), \( X, \mathcal{H}, \mathcal{G} \) become imaginary when \( a = 0 \). Let us consider \( Y = 0 \) case instead of \( a \to 0 \) limit. When \( Y = 0 \) Eq.(6.46) leads to the following effective potential

\[ V(a) \big|_{Y=0} \to -\frac{2k}{81} + \frac{77}{81} \frac{1}{4\epsilon l^2} \left( \frac{8\epsilon \tilde{\mu}}{4\epsilon - 1} \right)^{1/2} \kappa_g^2 \] (7.8)

Similarly to the case 2 the brane reaches the singularity at \( Y = 0 \), only if the condition \(-k/2 \geq V(a)|_{Y=0} \) is fulfilled, i.e.

\[ k \leq -Z_2(\epsilon) , \quad Z_2(\epsilon) \equiv \frac{1}{2\epsilon l^2} \left( \frac{8\epsilon \tilde{\mu}}{4\epsilon - 1} \right)^{1/2} \kappa_g^2 = -Z_2(\epsilon) . \] (7.9)

is satisfied. The conditions for \( Z_2(\epsilon) \) that the brane reaches the singularity at \( Y = 0 \) are

- For \( k = 2 \) (the de-Sitter brane) it is \( Z_2(\epsilon) \leq -2 \).
- For \( k = 0 \) (the flat brane) it is \( Z_2(\epsilon) \leq 0 \).
- For \( k = -2 \) (the hyperbolic brane) it is \( Z_2(\epsilon) \leq 2 \).

Then, if \( Z_2(\epsilon) \leq -2 \), all three kinds of brane reach the singularity at \( Y = 0 \).

4. The most simple case is \( \epsilon = 1/4 \) whose effective potential is zero. Then one can take \( k = -2, k = 0 \) only. The hyperbolic brane starts at \( a = 0 \) and expands to infinity and the flat brane cannot move.
5. Next, we consider $\epsilon = 1/4 - \delta^2$ case where $\delta^2 > 0$ and $|\delta| \ll 1$. From Eq.(6.53), the effective potential looks as

$$V(a) = \mp \frac{\kappa^2}{l^2} \left\{ 2l \left( \frac{k}{2a^2} + \frac{1}{l^2} \right)^{1/2} - 7 \right\} (2\tilde{\mu})^{1/2} \delta. \tag{7.10}$$

Under the condition $\frac{k}{2a^2} + \frac{1}{l^2} \geq 0$, if we take the upper sign of $\mp$, the situation is same as in Fig.2,

- $k = 2$, the de-Sitter brane starts at $a = 0$ and reaches its maximum size at $a_{\text{max}}$ and then it re-collapses. That is the reverse version of “bounce” universe.
- $k = 0$ or $k = -2$, the flat or hyperbolic brane starts at $a = 0$ and expands to infinity.

Taking the lower sign of $\mp$, the situation is shown in Fig.4. In this case only hyperbolic brane exists. The hyperbolic brane starts at $a = \infty$ and reaches its minimum size at $a = a_{\text{min}}$ and then it re-expands.

6. Finally, we consider the case of $\epsilon = 1/12$. One can get the effective potential from Eq.(6.59) but it has very complicated form. It is easier to consider only the limits of $a \to \infty$ or $a \to 0$. In the limit of $a \to \infty$ the effective potential follows from Eq.(6.59) as

$$V(a) \xrightarrow{a \to \infty} \left( \frac{3}{l^2} \pm \frac{2}{l^2} \right) a^2. \tag{7.11}$$

Since $\frac{3}{l^2} \pm \frac{2}{l^2}$ is always positive, one should take only $k = -2$ or $k = 0$.

- $k = -2$, the hyperbolic brane reaches maximum at $a_{\text{max}}$ and then it re-collapses.
- $k = 0$, the flat brane cannot move.

In the limit of $a \to 0$, the effective potential can be derived from Eq.(6.62) as

$$V(a) \xrightarrow{a \to 0} \pm \frac{42}{25} \frac{\sqrt{\mu}}{l^2} \kappa_g^2. \tag{7.12}$$
Under the condition of $-k/2 \geq V(a)$, the brane reaches the point $a = 0$, namely, the brane universe has the cosmological singularity. The conditions for $\epsilon$ that the brane reaches the point $a = 0$ are

- For $k = 2$ (the de-Sitter brane) it is $\frac{25}{42} \leq \sqrt{\frac{\tilde{\mu}}{\tilde{\nu}} \kappa_g^2}$.
- For flat brane it is $0 \leq \sqrt{\frac{\tilde{\mu}}{\tilde{\nu}} \kappa_g^2}$.
- For $k = -2$ (the hyperbolic brane) it is $\frac{25}{42} \geq \sqrt{\frac{\tilde{\mu}}{\tilde{\nu}} \kappa_g^2}$.

Otherwise the brane universe has no cosmological singularity.

In the same way, one can study FRW brane cosmology from Einstein-GB gravity for other values of GB coupling constant.

There is an important lesson which follows from the results of this section. It is indicated by recent astrophysical data that currently our observable universe has small and positive cosmological constant. In other words, the universe is in de Sitter phase. It would be nice to have some mechanism which would predicted the de Sitter universe as some preferrable state of FRW brane cosmology. Unfortunately, despite the number of attempts such mechanism was not found in brane-world approach to five-dimensional Einstein gravity. As it follows from our analysis it is unlikely that such mechanism exists in brane-world approach to higher derivative gravity too.

8 Discussion

In summary, we discussed FRW brane equations in the situation when brane is embedded in the five-dimensional (A)dS black hole. One of the main points of such discussion is the possibility in all cases under consideration (Einstein, Einstein-Maxwell, quantum-corrected or Einstein-GB gravity) to rewrite the FRW equations in the form similar to two-dimensional CFT entropy. In the first part of the paper (sections 2,3,4,5) we review the analogy between FRW cosmological equations and two-dimensional CFT entropy (so-called Cardy-Verlinde formula), two ways where Cardy-Verlinde formula appears in gravity theory, the presentation of brane equations of motion in five-dimensional (A)dS BH in the FRW form and then in Cardy-Verlinde form. The modification of such FRW presentation in case when five-dimensional Maxwell field or four-dimensional (brane) quantum fields are present is also reviewed.
Mainly, the first way (formal re-writing of FRW equation) in Cardy-Verlinde
formula appearance is reviewed. However, using holographic duality between
bulk BH and dual CFT entropies, the second way of Cardy-Verlinde formula
appearance is given in brane-world too.

In the second part of this work (sections 6,7) we investigate the brane
matter induced by five-dimensional AdS BH in Einstein-GB gravity. The
corresponding FRW brane equations are written. The novelty of this study
consists in the fact that brane matter energy and pressure significantly de-
pend on the choice of GB coupling constant. In particular, in some cases
dual brane matter is not CFT (the possibility of (A)dS/non-CFT corre-
spondence?). Moreover, DEC and WEC are not always satisfied because for
some values of GB coupling constant the brane matter energy and pressure
are negative. The presentation of FRW brane equations in the form similar
to two-dimensional CFT entropy formula is again possible. Finally, FRW
brane cosmology is studied using the effective potential technique. The num-
ber of brane universes: de Sitter, hyperbolic or flat with various dynamics
(expanding, contracting, first expanding then contracting, etc) is explicitly
constructed. The brief analysis of singularity (when it appears) is also pre-
sented.

It is clear that CV representation of FRW (brane) cosmology has some
holographic origin. However, the details of such holography remain to be
found. Moreover, as quite much is known about two-dimensional CFT, it is
possible that FRW equation representation as two-dimensional CFT entropy
may have various applications in the modern cosmology. In particular, it
may give an idea about why dS brane cosmology seems to be so fundamental
in our universe? In this respect it is quite important to search for various
modifications of CV formulation.

The questions discussed in this work maybe also important for the estab-
lishment of bulk/non-CFT correspondence. For example, in section three it
has been shown how thermodynamic entropy of bulk AdS BH maybe used
to get the holographic CV description of dual four-dimensional CFT cosmol-
ogy. It turns out that similar procedure does not work when it is applied to
bulk (A)dS BH in higher derivative gravity (if Riemann tensor squared term
presents) [18, 9]. Taking into account that still dual (non-CFT) description of
induced brane matter is possible (see section six) this looks quite promising.
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