Approximate Next-to-Leading Order and Next-to-Next-to-Leading Order Corrections

A.P. Contogouris\textsuperscript{a,b,+} and Z. Merebashvili\textsuperscript{a,*,++},

\textit{a. Department of Physics, McGill University, Montreal H3A 2T8, Canada}

\textit{b. Nuclear and Particle Physics, University of Athens, Athens 15771, Greece}

ABSTRACT

For processes involving structure functions and/or fragmentation functions, arguments that, over a range of a proper kinematic variable, there is a part that dominates the next-to-leading order (NLO) corrections are briefly reviewed. The arguments are tested against more recent NLO and in particular complete next-to-next-to-leading order (NNLO) calculations. A critical examination of when these arguments may not be useful is also presented.

\textsuperscript{*}On leave from High Energy Physics Institute, Tbilisi State University, University St. 9, 380086 Tbilisi, Republic of Georgia.

\textsuperscript{+}Email: apcont@physics.mcgill.ca, acontog@cc.uoa.gr

\textsuperscript{++}Email: zaza@physics.mcgill.ca, mereb@sun20.hepi.edu.ge
1. INTRODUCTION

In Perturbative QCD there is now a great effort towards calculating NNLO corrections [1-3]. One reason is that in several cases the NLO corrections are found to be large. Other reasons are that NNLO corrections are expected to increase the stability of predicted cross sections against changes of schemes and scales and that they will lead to more precise determinations of backgrounds towards uncovering signals for new physics.

Although there is no substitute for a complete NNLO calculation, since such calculations are in general very complicated, as a first step one may try approximate ones. Such a step has been presented in [4].

Below we briefly review the arguments of [4]. Sect. 2 mentions the results of certain more recent NLO calculations. Sect. 3 examines applications to the presently existing complete NNLO calculations. Finally, Sect. 4, apart from certain other points, discusses when approximate results may not be useful.

For processes involving structure functions and/or fragmentation functions, in [4] it was argued that, over a range of a proper kinematic variable, there is a part that dominates the NLO; and this was used to explain the fact that, in a number of the then existing NLO calculations, plotted against this kinematic variable, in a wide range, the cross section was almost a constant multiple of the Born.

To briefly review the essential ideas of [4], consider the NLO contribution of the sub-processes $a(p_1) + b(p_2) \rightarrow \gamma(q) + d$ to the large-$p_T$ process $A + B \rightarrow \gamma + X$:

$$
\frac{d\sigma}{d^3p} = \frac{\alpha_s(\mu)}{\pi} \sum_{a,b} \int \frac{dx_a}{x_a} \frac{dx_b}{x_b} F_{a/A}(x_a, M) F_{b/B}(x_b, M) \left[ \hat{\sigma}_B \delta \left( 1 + \frac{t + u}{s} \right) + \frac{\alpha_s(\mu)}{\pi} f\theta \left( 1 + \frac{t + u}{s} \right) \right] + (1 - \delta_{ab})(A \leftrightarrow B, \eta \leftrightarrow -\eta),
$$

(1.1)

where $F_{a/A, b/B}$ are parton momentum distributions to the hadrons $A, B$, $\mu$ and $M$ are the renormalization and factorization scales, $\eta$ the c.m. pseudorapidity,

$$
s = (p_1 + p_2)^2, \quad t = (q - p_1)^2, \quad u = (q - p_2)^2
$$

and $\sigma_B$ and $f$ are functions of $s, t, u$ corresponding to the Born and the higher order correction (HOC). Introducing the dimensionless variables

$$
v = 1 + t/s, \quad w = -u/(s + t)
$$

($s + t + u = sv(1 - w)$), the HOC have the following overall structure:

$$
f(v, w) = f_s(v, w) + f_h(v, w),
$$

where

$$
f_s(v, w) = a_1(v) \delta(1 - w) + b_1(v) \frac{1}{(1 - w)_+} + c_1(v) \left( \frac{\ln(1 - w)}{1 - w} \right)_+ + \left( a_2(v) \delta(1 - w) + b_2(v) \frac{1}{(1 - w)_+} \right) \ln \frac{s}{M^2} + c_2(v) \delta(1 - w) \ln \frac{s}{\mu^2},
$$

(1.3)
where \(1/(1-w)_{+}\) and \((\ln(1-w)/(1-w))_{+}\) are well known distributions. The function \(f_h(v, w)\) contains no distributions and, in general, has a complicated analytic form.

Now denote by \(\sigma_s\) and \(\sigma_h\) the contributions of \(f_s\) and \(f_h\) to \(Ed\sigma/d^3p\) and consider the ratio

\[
L = \sigma_h/(\sigma_s + \sigma_h);
\]

then, at sufficiently large \(x_T\), for fixed total c.m. energy \(\sqrt{S}\), as \(p_T\) (or \(x_T \equiv 2p_T/\sqrt{S}\)) increases, \(|L|\) decreases.

To see the reason, consider a plot of \(x_b\) vs \(x_a\) for \(\eta = 0\) (Fig. 1). The integration region in (1.1) is bounded by \(w = 1, x_a = 1\) and \(x_b = 1\) (hatched region). Now, for \(x\) not too small, \(F_{a/A}(x, M)\) behaves like \((1-x)^n\); with \(A\) = proton, \(n\) is fairly large (\(\geq 3\)); also due to scale violations, \(n\) increases as \(p_T\) increases. Then contributions arising from the region away from \(w = 1\) are supressed by powers of \(1-x_a\) and/or \(1-x_b\). Now, in \(f_s\), the terms proportional to \(\delta(1-w)\) contribute at \(w = 1\) (and so does \(\delta_B\)) whereas the rest give a contribution increasing as \(w \rightarrow 1\). On the other hand, the multitude of terms of \(f_h\) contribute more or less uniformly in the integration region \(\theta(1-w)\) and their contribution \(\sigma_h\) is suppressed. As \(x_T\) increases at fixed \(S\), the integration region shrinks towards \(x_a = x_b = 1\) and the suppression of \(\sigma_h\) increases.

The mechanism is tested by writing the distributions in the form [4(a)]

\[
F_{a/A}(x, M) = F_{b/B}(x, M) = (1-x)^N
\]

and choosing a fictitious \(N >> n\) or choosing \(0 < N << n\). Then the ratio \(L\) with the first choice decreases faster and with the second choice decreases slower then for \(N = n\).

Next we neglect \(f_h(v, w)\) and make the rough approximations \(1/(1-w)_{+} \sim \delta(1-w)\), \((\ln(1-w)/(1-w))_{+} \sim \delta(1-w)\). Furthermore, we note that \(b_1(v), c_1(v), a_2(v), b_2(v), c_2(v)\) and part of \(a_1(v)\) are either proportional to the Born term or contain the Born term times a smooth function of \(v\); the rest of \(a_1(v)\) is also a smooth function of \(v\) (see e.g. Eq. (C.8) of [4(a)] or Eq. (4.11) of [4(b)]). The Born term itself is a smooth function of \(v\). Thus as a first approximation we write

\[
f(v, w) \approx A\delta_B(v)\delta(1-w)
\]

where \(A \approx const\). This results in \(Ed\sigma/d^3p\) of roughly the same shape as \(Ed\sigma_{Born}/d^3p\)

The same argument can be made in terms of the moments of the functions \(\delta(1-w), 1/(1-w)_{+}, (\ln(1-w)/(1-w))_{+}\) and of the functions in \(f_h(v, w)\) [4(a)]. Clearly, \(\sigma_s\) contains all the soft, collinear and virtual contributions to \(Ed\sigma/d^3p\).

At NLO the Bremsstrahlung (Brems) contributions to \(f_s\) are determined via simple formulae [4]: E.g. for \(gg \rightarrow \gamma q\) the Brems contributions arise from products of two graphs \(gq \rightarrow \gamma gg\). If in both graphs the emitted \(g\) arises from initial partons (g or q), the contribution in \(d = 4 - 2\varepsilon\) dimensions is

\[
\frac{d\sigma_{init}}{dvdw} \sim T_0^{(gg)}(v, \varepsilon)N_c \left(\frac{2}{\varepsilon}\right) \left(\frac{v}{1-v}\right)^{-\varepsilon} (1-w)^{-1-2\varepsilon} \left(1 + \varepsilon^2 \frac{\pi^2}{6}\right),
\]

(1.7)
where $T_0^{(gq)}(v,\varepsilon)$ is essentially the Born cross section in $d$ dimensions. If in at least one of the graphs the emitted $g$ arises from the final parton ($g$), then

$$\frac{d\sigma_{\text{fin}}}{dv dw} \sim T_0^{(gq)}(v,\varepsilon) C_F v^{-\varepsilon} (1-w)^{-1-\varepsilon} \tilde{P}_{qq}(\varepsilon)$$

where

$$\tilde{P}_{qq}(\varepsilon) = \frac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)} \int_0^1 y^{-\varepsilon}(1-y)^{-\varepsilon} P_{qq}(y,\varepsilon)$$

and $P_{qq}(y,\varepsilon) = 2/(1-y) - 1 - y - \varepsilon(1-y)$, the split function in $n$ dimensions ($y < 1$).

Expanding

$$(1-w)^{-1-\varepsilon} = -\frac{1}{\varepsilon} \delta(1-w) + \frac{1}{(1-w)_{+}} - \varepsilon \left( \frac{\ln(1-w)}{1-w} \right)_{+} + \mathcal{O}(\varepsilon^2)$$

as well as $(v/(1-v))^{-\varepsilon}$ and $v^{-\varepsilon}$ in powers of $\varepsilon$ we determine the contributions. The singular terms $\sim 1/\varepsilon^2$ and $1/\varepsilon$ cancel by adding the loop contributions and proper counterterms.

### 2. FURTHER NLO CALCULATIONS

In addition to the examples presented in Refs. [4], the following are some NLO studies supporting the ideas of Sect. 1:

(a) Large $p_T$ $W$ and $Z$ production in $p\bar{p}$ collisions [5]. At $\sqrt{S} = 0.63$ and 1.8 TeV, for $p_T \geq 80$ GeV the cross sections $d\sigma/dp_T^2$ are also almost a constant multiple of the LO (Figs. 7 and 8 of [5]).

(b) The production of two isolated direct photons in $p\bar{p}$ collisions [6]. At $\sqrt{S} = 1.8$ TeV, when the $p_T$ of each photon exceeds 10 GeV the shape of the NLO QCD cross section $d\sigma/dp_T$ differs little from that of the Born (Fig. 2 of [6]).

Regarding NLO results for polarized reactions we mention the following:

(a) Polarized deep inelastic Compton scattering [7], in particular the contribution of the subprocess $\vec{\gamma}\vec{q} \rightarrow \gamma q$ to large $p_T$ direct photon production in polarized $\gamma - p$ collisions ($\vec{\gamma}\vec{p} \rightarrow \gamma + X$). At $\sqrt{S} = 27$ and 170 GeV, for $x_T \geq 0.15$, it is $\vert L \vert < 0.28$ and for sufficiently large $x_T$, $L$ decreases as $x_T \rightarrow 1$ (Fig. 4 of Ref. [7]). Also, denoting by $\sigma^{(k)}$ the $\mathcal{O}(\alpha_s^k)$, $k = 0, 1$, contributions of $\vec{\gamma}\vec{q} \rightarrow \gamma q$ to $Ed\sigma/d^3p$, for $0.2 \leq x_T \leq 0.8$ the factor $K_{\gamma q} = (\sigma^{(0)} + \sigma^{(1)})/\sigma^{(0)}$ is found to differ little from a constant.

(b) Large $p_T$ direct $\gamma$ production in longitudinally polarized hadron collisions [8,9]. Here of interest are the $\mathcal{O}(\alpha_s^k)$, $k = 1, 2$, contributions of the subprocess $\vec{g}\vec{q} \rightarrow \gamma q$. As $x_T$ increases, the ratio $-\sigma_h/\sigma_s$ steadily decreases (Fig. 10 of [8]). The factor $K_{gq} = (\sigma^{(1)} + \sigma^{(2)})/\sigma^{(1)}$ is not constant, but increases moderately (Fig. 2 of [8]).
are contributions from the extra subprocesses $\bar{g}g \rightarrow q\gamma$, $q\bar{q} \rightarrow q\gamma$, and $q\bar{q}' \rightarrow q\gamma'$, where $q, \bar{q}$ are either of different quark flavor or of the same flavor but interacting via exchange of a gluon. In general, these are found to be relatively small (Figs. 3, 4 and 5 of [8]). The reason is that the extra subprocesses possess no terms involving distributions (no loops and vanishing contributions of the type (1.7) and (1.8)).

The considerations of Sect. 1 explain also the following fact: Taking as example large $pT \sim \sqrt{S}$, at NLO, apart from the HOC of the dominant subprocess $g\bar{g} \rightarrow \gamma q$, there are contributions from the extra subprocesses $\bar{g}g \rightarrow q\bar{q}\gamma$, $q\bar{q} \rightarrow q\gamma$, and $q\bar{q}' \rightarrow q\gamma'$, where $q, \bar{q}'$ are either of different quark flavor or of the same flavor but interacting via exchange of a gluon. In general, these are found to be relatively small (Figs. 3, 4 and 5 of [8]). The reason is that the extra subprocesses possess no terms involving distributions (no loops and vanishing contributions of the type (1.7) and (1.8)).

3. NNLO CALCULATIONS

NNLO calculations have been carried for Drell-Yan (DY) production of lepton pairs, $W^\pm$ and $Z$, and for the deep inelastic (DIS) structure functions $F_i(x, Q^2)$, $j = 1, 2$ and the longitudinal part. Now the parts involving distributions contain also terms of the type $(\ln(1-w)/(1-w))_+$, with $i = 2$ and 3 and $w$ a proper dimensionless variable. The subsequent calculations are carried using the updated $\overline{\text{MS}}$ CTEQ5M1 set of [12], one of the most recent sets of NLO parton distributions [13]. We present results for $\mu = M = \sqrt{Q^2}$.

Beginning with DY, we are interested in the process $pp \rightarrow \gamma^* + X \rightarrow l^+l^- + X$ and to the cross section

$$d\sigma(\tau, S)/dQ^2 \equiv \sigma(\tau, S)$$

where $\tau = Q^2/S$ with $\sqrt{S}$ the total c.m. energy of the initial hadrons and $\sqrt{Q^2}$ the $\gamma^*$ mass [14,15]. Here we deal with the subprocess $q + \bar{q} \rightarrow \gamma^*$ and its NLO and NNLO corrections [14]. For DY, $w \sim \tau$. We use number of flavors $n_f = 4$.

Denote by $\sigma^{(k)}(\tau, S)$, $k = 0, 1, 2$, the $\mathcal{O}(\alpha_s^k)$ part of $\sigma(\tau, S)$, by $\sigma^{(k)}_s$ the part of $\sigma^{(k)}$ arising from distributions and by $\sigma^{(k)}_h$ the rest. Defining

$$L^{(k)}(\tau, S) = \sigma^{(k)}_h(\tau, S)/\sigma^{(k)}(\tau, S)$$

Fig. 2 shows $L^{(k)}$, $k = 1, 2$, as functions of $\tau$ for $\sqrt{S} = 20$ GeV. Clearly, for $\tau > 0.3$: $L^{(1)} \leq 0.17$ and $L^{(2)} \leq 0.33$.

It is of interest also to see the percentage of $\sigma^{(k)}_h$ of the total cross section determined up to $\mathcal{O}(\alpha_s^k)$. Fig. 2 also shows the ratios $\sigma^{(1)}_h/(\sigma^{(0)} + \sigma^{(1)})$ and $\sigma^{(2)}_h/(\sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)})$ for the same $\sqrt{S}$; clearly, for $\tau \geq 0.2$ both ratios are less than 0.1.

Now we turn to DIS [16,17]. Here we deal with the sum

$$\Sigma(x, Q^2) = u_v(x, Q^2) + d_v(x, Q^2),$$

where $u_v$ and $d_v$ are the $u$-valence and $d$-valence quark distributions in the proton. We will deal with the subprocess $q + \gamma^* \rightarrow q$ and the nonsinglet part of its NLO and NNLO corrections [16]. For DIS, $w \sim x$. 4
Denote by $\Sigma^{(k)}(x,Q^2)$, $k = 0, 1, 2$, the $O(\alpha_s^k)$ contribution, by $\Sigma_s^{(k)}$ the part of $\Sigma^{(k)}$ arising from distributions and by $\Sigma_h^{(k)}$ the rest. Defining

$$L^{(k)}(x,Q^2) = \frac{\Sigma_h^{(k)}(x,Q^2)}{\Sigma^{(k)}(x,Q^2)}$$

(3.4)

Fig. 3 presents $L^{(k)}(x,Q^2)$, $k = 1, 2$, as functions of $x$ for $\sqrt{Q^2} = 5$ GeV. Now, for $x \leq 0.5$ $L^{(1)}$ is not small, but this is due to the fact that $\Sigma_s^{(1)}$ changes sign and $\Sigma_h^{(1)}$ stays $> 0$, so at $x \approx 0.3$, $\Sigma^{(1)}$ vanishes. On the other hand, at $x \geq 0.6$, $L^{(2)}$ is less than 0.28.

The effect of neglecting $\sigma_h^{(k)}$ in DY or $\Sigma_h^{(k)}$ in DIS is shown in Fig. 4. In DY, denoting

$$K_s = \frac{\sigma^{(0)} + \sigma_s^{(1)} + \sigma_s^{(2)}}{\sigma^{(0)}}$$

$$K = \frac{\sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)}}{\sigma^{(0)}}$$

(3.5)

we show $K_s(K)$ by solid (dashed) line at $\sqrt{S} = 20$ GeV (upper part). Clearly, as $\tau \to 1$, $K_s \to K$, and for $\tau > 0.3$ the error is less than 14%. In DIS, denoting

$$K_s = \frac{\Sigma^{(0)} + \Sigma_s^{(1)} + \Sigma_s^{(2)}}{\Sigma^{(0)}}$$

$$K = \frac{\Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)}}{\Sigma^{(0)}}$$

(3.6)

we show $K_s$ and $K$ at $\sqrt{Q^2} = 5$ GeV (lower part). Again, as $x \to 1$, $K_s \to K$. Now, in spite of the fact that $L^{(k)}$ is, in general, not small, $K_s$ differs from $K$ even less. The reason is that the NLO and NNLO corrections are smaller than in DY, and so are $\Sigma_s^{(k)}/\Sigma^{(0)}$.

4. CONCLUDING REMARKS

The above discussion and examples show that for processes involving structure and/or fragmentation functions, for not too small values of a proper kinematic variable ($x_T$ for large-$p_T$ reactions, $\tau$ for DY, $x$ for DIS), one may retain only that part of the differential cross section arising from distributions [18]. At NNLO the range of this variable is larger for DY than for DIS (Figs. 2 and 3). Yet, regarding the $K$-factor, which determines the physically important quantity, DIS is somewhat advantageous (Fig. 4).

As we go to NNLO, in view of the presence of terms of the type $(\ln'(1-w)/(1-w))_+$ with $i = 2$ and 3, the rough approximation of replacing $(\ln(1-w)/(1-w))_+$ by $\delta(1-w)$ is becoming worse. In general, this implies that with NNLO corrections, the shape of a physical quantity should deviate more than that of the Born term.

It is desirable (and nontrivial) to extend the formulas (1.7) and (1.8) to the NNLO and perhaps even higher orders.

The question now is when the arguments of Sec. 1 may not be useful. Such a case is when, over a wide range of $w$, $\sigma_h^{(k)}$ is comparable and of opposite sign to $\sigma_s^{(k)}$. Then
$\sigma_s^{(k)} + \sigma_h^{(k)}$ is small and $L^{(k)}$ is large in absolute value. Even then, for $w$ very near 1, $|L^{(k)}|$ should decrease, but in that case threshold resummation [19,20] is important, and the approximation is not useful. Of course, in such a case, the correction $|\sigma^{(k)}| = |\sigma_s^{(k)} + \sigma_h^{(k)}|$ will be small. The point, however, is that we do not see how one can determine such a case without calculating $\sigma_h^{(k)}$.

ACKNOWLEDGEMENTS

We would like to thank E. Basea and G. Grispos for checking certain of our results. The work was also supported by the Natural Sciences and Engineering Research Council of Canada and by the Secretariat for Research and Technology of Greece.

REFERENCES


[18] At NLO, certain of the terms of \( f_h(v,w) \) are also given by simple analytic formulas similar to (1.7) and (1.8). See S. Papadopoulos, Ph.D. Thesis (McGill Univ. 1989).


FIGURE CAPTIONS

Fig. 1 The integration region in the expression (1.1) for c.m. pseudorapidity \( \eta = 0 \).

Fig. 2 The ratios \( L^{(2)} = \frac{\sigma_h^{(2)}}{\sigma^{(2)}} \) and \( \frac{\sigma_h^{(2)}}{(\sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)})} \) (solid lines) as well as \( L^{(1)} = \frac{\sigma_h^{(1)}}{\sigma^{(1)}} \) and \( \frac{\sigma_h^{(1)}}{(\sigma^{(0)} + \sigma^{(1)})} \) (dash-dotted lines) for Drell-Yan lepton-pair production versus \( \tau = Q^2/S \) at \( \sqrt{S} = 20 \) GeV.

Fig. 3 The ratios \( L^{(2)} = \frac{\Sigma_h^{(2)}}{(\Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)})} \) (solid lines), where \( \Sigma \equiv u_v + d_v \), as well as \( L^{(1)} = \frac{\Sigma_h^{(1)}}{(\Sigma^{(0)} + \Sigma^{(1)})} \) (dash-dotted lines) for \( q + \gamma^* \rightarrow q \) versus \( x \) at \( Q^2 = 25 \) GeV^2.

Fig. 4 \( K \)-factors: Approximate \( K_s \) (solid lines) and exact \( K \) (dotted lines) for the Drell-Yan case of Fig. 2 (upper part) and for the case of Fig. 3 (lower part).
Fig. 1
Fig. 2
Fig. 4