Geometric Transformations and NCCS Theory in the Lowest Landau Level

M. Eliashvili and G. Tsitsishvili

Department of Theoretical Physics, A. Razmadze Mathematical Institute, Tbilisi 380093 Georgia
simi@rmi.acnet.ge

Abstract

Chern-Simons type gauge field is generated by the means of the singular area preserving transformations in the lowest Landau level of electrons forming fractional quantum Hall state. Dynamics is governed by the system of constraints which correspond to the Gauss law in the non-commutative Chern-Simons gauge theory and to the lowest Landau level condition in the picture of composite fermions. Physically reasonable solution to this constraints corresponds to the Laughlin state. It is argued that the model leads to the non-commutative Chern-Simons theory of the QHE and composite fermions.

1. Introduction

One of the intriguing features of the quantum Hall effect (QHE) (for a comprehensive introduction see Ref. 1) is that it is a simplest physical realization of the non-commutative spatial geometry (see e.g. Ref. 2). Due to the intense orthogonal magnetic field $B = (0, B_\perp)$, electrons are confined to the lowest Landau level (LLL) and their position coordinates do not commute:

$$[\hat{x}, \hat{y}] = \frac{i}{B_\perp} \equiv -i\theta. \quad (1)$$

This fact had stimulated a considerable number of papers, in which quantum Hall effect is examined from the point of view of the non-commutative quantum field theory [3]−[10].

In the Ref. 3 it was shown, that the Laughlin’s theory [11] of fractional quantum Hall effect (FQHE) for the odd inverse filling factors $\nu^{-1} = 2p + 1$ can be presented as a non-commutative Chern-Simons (NCCS) gauge theory. This assertion is formulated in terms of the fluid mechanics and is based on the use of hydrodynamical variables introduced in Ref. 12.

It would be interesting to substantiate the above assertion using mechanical variables – electron coordinates and momenta. This could make more transparent a transition to the traditional quantum mechanical description of the electron system. Another interesting question is how fits spatial non-commutativity with a composite fermion (CF) picture [13], according which the fractional quantum Hall states are formed by quasiparticles experiencing the reduced magnetic
field $B^*_x = \nu B_\perp$. Hence one has to take into account that in the CF picture the non-commutativity parameter will be $\theta^* = \nu^{-1} \theta$.

In the field theory the composite particle scenario can be introduced by the means of the Chern-Simons (CS) gauge theory [14, 15]. Here the central role is played by the Gauss’s law – constraint binding the CS magnetic field to the electron density and providing a flux attachment mechanism. In Ref. 3 it was obtained the Gauss’s law for the CS theory based on the group of area preserving diffeomorphisms [16] (APD) with a subsequent interpretation of it as a first order truncation of the corresponding non-commutative theory.

In the present paper we discuss these questions considering electrons as charged particles with zero kinetic momenta, i.e. in the LLL. In Ref. 17 it was argued that the standard CS approach and CF picture can be developed starting with some area preserving singular geometric transformations and considering them in the context of QHE. Below we will show, that in reality the area invariance condition leads to the Gauss’s law for CS theory with APD group. We also propose a simple modification of the area transformation rule, which will permit to write down the Gauss’s law directly for the full non-linear NCCS theory. Area transformations induce corresponding changes in the $N$-particle Lagrangian and can be interpreted as a transition to the CF picture in the framework of the NCCS theory.

The lay-out of this paper is as follows. In Section 2 we introduce transformations of the electron Lagrangian and relate them to the area preserving transformations. In remaining sections 3 and 4 the corresponding NCCS theory is considered and the simple solution for the Laughlin state is determined.

**Notations:** In the $x - y$ plane together with the Cartesian coordinates $x^i = x_i$ ($i = 1, 2$) we use the complex ones: $z = x + iy$, $\bar{z} = x - iy$. In the natural units $c = \hbar = 1$ electrons have a charge $e = -1$, mass $m$ and move in the area $\Omega$ in the homogeneous magnetic field pointing down the $\hat{z}$ axis: $B_\perp = \partial_x A_y - \partial_y A_x = -B < 0$. For 2+1 space-time we use coordinates $x^\mu$ ($x^0 = t, x^i$) and metric tensor $\eta_{\mu\nu} = \text{diag}(-1, +1, +1)$.

2. Electrons in LLL and Geometric Transformations

Consider electrons moving in the $x - y$ plane in the presence of the intense orthogonal magnetic field. In the case of quantum Hall states with the filling factor

$$\nu = \frac{2\pi N}{B \Omega} = \frac{1}{2p + 1}$$

one deals with the incompressible quantum fluid formed by LLL electrons occupying the area $\Omega$. In what follows we mainly consider a finite system of electrons of a limited spatial extent.

In the symmetric gauge where $A_i = \frac{1}{2} B_{ik} x^k$, the $N$-particle LLL wave function must satisfy equations

$$\mathbf{\hat{p}}_z^2 \Psi(x_1, ..., x_N) \equiv \{\mathbf{\hat{p}}_z^2 + A_z(r, \alpha)\} \Psi(x_1, ..., x_N) = 0.$$
Here the locus of particles is described by their position vectors \( \mathbf{x}_\alpha (\alpha = 1, 2, ..., N) \). Solution to the Eq.(3) is of the form

\[
\Psi(\mathbf{x}_1, ..., \mathbf{x}_N) = F(z_1, ..., z_N) e^{-\frac{1}{2} \sum_\alpha |z_\alpha|^2}
\]

(4)

and the dynamical information on the system is encapsulated in the holomorphic function \( F(z_1, ..., z_N) \).

The equation (3) can be viewed as a condition imposed on the physical states by the constraint dynamics (in the Dirac’s [18] sense), and at the classical level these constraints lead to the vanishing kinetic momenta:

\[
\pi^\alpha_i \approx 0.
\]

(5)

Constraints are of the second class with the immediate consequence that the electron coordinates do not commute:

\[
[\hat{x}_\alpha, \hat{y}_\beta] = i \{x_\alpha, y_\beta\} \text{Dirac} = -i \theta \delta_{\alpha\beta}.
\]

(6)

Interesting to note that the Dirac bracket for the particle densities is also non-zero [10]

\[
\{\rho(\mathbf{x}), \rho(\mathbf{x}')\} \text{Dirac} = -\theta \epsilon_{ik} \frac{\partial}{\partial x^i_\alpha} \rho(\mathbf{x}) \frac{\partial}{\partial x^k_\alpha} \rho(\mathbf{x}').
\]

(7)

Constraints \( \pi^\alpha_i \approx 0 \) are provided by the singular Lagrangian [19]

\[
L_N = -\sum_{\alpha=1}^{N} \dot{x}^i_\alpha A_i(x_\alpha) = -\frac{B}{2} \sum_{\alpha=1}^{N} \dot{x}^i_\alpha \epsilon_{ik} x^k_\alpha.
\]

(8)

In the limit of the strong magnetic field one can neglect the kinetic term, i.e. formally put \( m = 0 \). In (8) we have not included different interaction terms (electron-electron, electron-background etc.) or confinement forces and concentrate on the terms which are of first order in time derivatives.

The corresponding canonical Hamiltonian vanishes and the quantum dynamics is completely governed by constraints (5). At the same time one cannot impose operator constraints

\[
\hat{\pi}^\alpha_i |\psi_L\rangle = 0 \quad \langle\psi_L|\hat{\pi}^\alpha_i = 0,
\]

because they do not commute among themselves

\[
0 = \langle\psi_L| [\hat{\pi}^\alpha_1, \hat{\pi}^\alpha_2]|\psi_L\rangle = iB \neq 0.
\]

Instead, constraints can vanish only ”weakly”, i.e.

\[
\hat{\pi}^\alpha_1 |\psi_L\rangle = 0, \quad \langle\psi_L|\hat{\pi}^\alpha_2 = 0.
\]

This is consistent with classical Eqs. (5), as well as the corresponding quantum averages vanish

\[
\langle\Psi_L|\hat{\pi}^\alpha_1 |\Psi_L\rangle = 0.
\]
Remark now, that the formal substitution \( B \rightarrow B^* \equiv \nu B, \Omega \rightarrow \Omega \) amounts to the change of the filling factor \( \nu \rightarrow \nu^* = 1 \). Referred to above substitution can be expressed in terms of the transformations of particle coordinates and velocities

\[
x_i^\alpha \rightarrow \sqrt{\nu}\{x_i^\alpha - \theta^* e^{ik}a_k^\alpha(x_1,...,x_N)\}
\]

(9)

\[
\dot{x}_i^\alpha \rightarrow \sqrt{\nu}\{\dot{x}_i^\alpha - \theta^* e^{ik}\dot{a}_k^\alpha(x_1,...,x_N)\}
\]

(10)

where \( a_k^\alpha \) and \( \dot{a}_k^\alpha \) are yet unspecified functions of particle coordinates.

Under (9) and (10) Lagrangian (8) is transformed to

\[
L_N^* = -\sum_{\alpha=1}^{N} \dot{x}_i^\alpha \{A_i^\alpha(x_\alpha) + a_\alpha^\alpha(x_\alpha)\} + \frac{\theta^*}{2} \sum_{\alpha=1}^{N} \epsilon^{ik} a_\alpha^\alpha \dot{a}_k^\alpha.
\]

(11)

In order to specify functions \( a_\alpha^\alpha \) and \( \dot{a}_k^\alpha \) we appeal to the physical properties of the Hall fluid. First of all, the system of electrons moving in the strong magnetic field forms a special kind of incompressible quantum fluid and incompressibility can be formulated in terms of the area preserving diffeomorphisms, which is the symmetry of non-interacting electrons in the magnetic field [20, 21]. Secondly, according to the CF picture [13] electrons in the quantum Hall state are replaced by the so called composite particles – fermions (bosons) carrying even (odd) number of elementary magnetic flux quantum. Composite particles experience the effective magnetic field \( B^* = \nu B \) and they fill up their own lowest Landau level.

In the present paper we argue that the emergence of the Chern-Simons gauge field can be understood considering the area preserving transformations, i.e. CS field has a geometric origin. In order to clarify this point let us turn to the incompressibility and its geometric manifestation. Particles are restricted to move in the area:

\[
\Omega = \int_D d^2x = (2p + 1)\frac{2\pi}{B}N.
\]

(12)

Consider the map

\[
x^i \rightarrow x'^i = F^i(x),
\]

(13)

which induces the change of the area (12):

\[
\Omega \rightarrow \Omega' = \int_D d^2x J\left(\frac{\partial F^i}{\partial x^k}\right),
\]

(14)

where

\[
J\left(\frac{\partial F^i}{\partial x^k}\right) = \frac{1}{2} \epsilon_{ik} \epsilon^{mn} \partial_m F^i \partial_n F^k
\]

(15)
Introduce a deformed multiplication of two functions

$$(f \circ g)_\tau = fg - i\frac{\tau}{2} \epsilon^{mn} \partial_m f \partial_n g$$

(16)

where $\tau$ is some parameter. Then the area transformation can be presented in the following form

$$\Omega = \frac{i}{\theta} \int_D d^2x \epsilon_{ik} (x^i \circ x^k)_\theta \rightarrow \Omega' = \frac{i}{\theta^*} \int_D d^2x \epsilon_{ik} (F^i \circ F^k)_{\theta^*}.$$  

(17)

Here we admit, that the map (13) is accompanied by the parameter change $\theta \rightarrow \theta^*$.

The area preservation condition looks as follows

$$\Omega_{\theta} \equiv \Omega \rightarrow \Omega' \equiv \Omega_{\theta^*}.$$  

(18)

Let

$$F^i(x) = s(i) \{x^i - \theta^* \epsilon^{ik} a_k(x)\}$$

(19)

where $a_k(x)$ is a deformation field and $s(i)$’s are some constants.

Taking into account the definition of the filling factor (2) one gets

$$\frac{1 - s}{s} 2\pi N = -\theta^* \int_D d^2x \epsilon_{ik} D_i a_k.$$  

(20)

Here $s = s(1)s(2)$ and we use the notation

$$D_i a_k = \partial_i a_k - i(a_i \circ a_k)_{\theta^*}.$$  

(21)

Setting $s = \nu = (1 + 2p)^{-1}$ we end up with the equation

$$\int_D d^2x \epsilon_{ik} D_i a_k = -4\pi p N.$$  

(22)

Introduce the local density $\rho(x, t)$, satisfying the condition

$$N = \int_D d^2x \rho(x, t).$$

(23)

In (23) we admit that the density can be time-dependent.

Now the integral relation (22) can be written in the form of local constraint imposed on the deformation field $a_i(x)$

$$\rho(x, t) + \kappa \epsilon^{ik} D_i a_k = 0, \quad 1/\kappa = 4\pi p.$$  

(24)

The last relation resembles the Gauss law known in the Chern-Simons gauge field theory. The main difference is that in (24) we use the "covariant curl" (21), differing from the usual one by the non-linear term $\epsilon^{ik} (a_i \circ a_k)_{\theta^*}.$
Denote the solution of Eq. (24) by \( a_i(x|x_1,\ldots,x_N) \) and formally define
\[
a_\alpha^k = a_k^\alpha(x|x_1,\ldots,x_N)|_{x=x_\alpha} \equiv a_k(x_\alpha) \quad (25)
\]
and
\[
\dot{a}_\alpha^k = \frac{1}{2p} \frac{\partial}{\partial t} a_k(x|x_1,\ldots,x_N)|_{x=x_\alpha} \equiv \frac{1}{2p} \dot{a}_k(x_\alpha). \quad (26)
\]
Substitution of (25) and (26) into Lagrangian (11) yields
\[
L^*_N = -\sum_{\alpha=1}^N \dot{x}_\alpha^i \left\{ A^*_i(x_\alpha) + a_i(x_\alpha) \right\} + \frac{\theta^*}{2} \frac{1}{2p} \sum_{\alpha=1}^N \Delta \epsilon^{ik} a_i(x_\alpha) \dot{a}_k(x_\alpha) \quad (27)
\]
where \( \Delta = \frac{2\pi}{B^*} \).

If we suppose that the sum in (27) may be replaced by the integral, i.e.
\[
\sum_{\alpha=1}^N \epsilon^{ik} \Delta \epsilon^{ik} a_i(x_\alpha) \dot{a}_k(x_\alpha) \rightarrow \int_D d^2x \epsilon^{ik} a_i(x,t) \partial_t a_k(x,t)
\]
we arrive at the Lagrangian
\[
L^* = \int_D d^2x \left\{ -j^i(x,t)[A^*_i(x) + a_i(x,t)] + \frac{\kappa}{2} \epsilon^{ik} a_i(x,t) \partial_t a_k(x,t) \right\}. \quad (29)
\]
Validity of the substitution (28) can be corroborated by the fact that \( \Delta \) is an elementary area occupied by the CF in the magnetic field \( B^* \).

In the Lagrangian (29)
\[
j^i(x,t) = \sum_{\alpha=1}^N \dot{x}_\alpha^i(t) \delta(x - x_\alpha) \quad (30)
\]
is a 2-current and the field \( a_i(x,t) \) is subject to the Gauss law (24)
\[
\Phi \equiv \rho(x,t) + \kappa \epsilon^{ik} D_i a_k = 0 \quad (31)
\]
Remark that the constraint (31) is invariant under the infinitesimal gauge transformations
\[
\delta a_i(x,t) = \partial_i \lambda + i \left\{ (\lambda \circ a_i)_{\theta^*} - (a_i \circ \lambda)_{\theta^*} \right\} \quad (32)
\]
\[
\delta \rho(x,t) = i \left\{ (\lambda \circ \rho)_{\theta^*} - (\rho \circ \lambda)_{\theta^*} \right\} \quad (33)
\]
The Gauss law \( \Phi = 0 \) can be taken into account considering Lagrangian
\[
L = \int_D d^2x \left\{ -j^i(x,t)[A^*_i(x) + a_i(x,t)] + \frac{\kappa}{2} \epsilon^{ik} a_i(x,t) \partial_t a_k(x,t)
\]
\[
- a_0(x,t) \Phi(x,t) \right\}. \quad (34)
\]
Variation with respect to the Lagrange multiplier field $a_0$ imposes the constraint 
(31).

Assuming that on the boundary $a_0(x)|_{\partial D} = 0$ the corresponding Lagrangian density 
can be presented in the following form
\[ L = -J^\mu (A_\mu^* + a_\mu) - \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{i}{3} \varepsilon^{\mu\nu\lambda} (a_\mu \circ a_\nu \circ a_\lambda)_{\theta^*}. \] (35)

Here $J^\mu(x)$ is a 3-current consisting of the density $\rho$ and 2-current $j^i$.

Lagrangian (35) is equivalent to the one given in Ref. 3, but is based on the 
use of mechanical variables and as it is claimed in Ref. 3, this Lagrangian can 
be considered as an approximation to the NCCS Lagrangian.

Transition to NCCS theory can be performed simply substituting the deformed product (16) by the Moyal-Weyl star product
\[ (f \circ g)_{\theta^*} = \left( f \cdot g \right)_{\theta^*} = e^{-i \frac{\tau^2}{\nu^2} \sum_{i<k} \partial_i a_k} f(x) \cdot g(x) |_{\xi = \eta = 0} + O(\tau^2) \] (36)

This step can be accomplished, taking into account that the LLL condition 
forces the area $\Omega$ to be a part of a noncommutative $x-y$ plane. The expression
\[ \frac{i}{\theta} \int_D d^2x \epsilon_{ik}(x^i \star x^k)_{\theta^*} = \Omega_{\theta^*} \] (37)

can be used for an heuristic definition of the area and its transformation under 
the map (13) accompanied by the change of the non-commutativity parameter $\theta \rightarrow \theta^* = \nu^{-1} \theta$:
\[ \Omega_{\theta} \rightarrow \Omega_{\theta^*} = \frac{i}{\theta^*} \int_D d^2x \epsilon_{ik}(F^i(x) \star F^k(x))_{\theta^*}. \] (38)

All the consideration given above can be repeated with the minor modification of the product: $(f \circ g)_{\theta^*} \rightarrow (f \star g)_{\theta^*}$. For example for the ”covariant curl” one must take
\[ D^i_\ast a_k = \partial_i a_k - i(a_i \ast a_k)_{\theta^*}. \] (39)

instead of (21).

Proceeding in this way we arrive at the NCCS Lagrangian
\[ \mathcal{L}_{NCCS} = -J^\mu (A_\mu^* + a_\mu) - \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} a_\mu \star \left\{ \partial_\nu a_\lambda - \frac{2}{3} a_\nu \ast a_\lambda \right\}. \] (40)

In this expression and hereafter we use the star product with parameter $\theta^*$ 
$(f \star g \equiv (f \ast g)_{\theta^*})$.

3. Chern-Simons Theory in LLL

Up to now we have considered area preserving geometric transformations, 
which are supposed to satisfy the constraint equation
\[ \rho(x) + \kappa e^{ik} (\partial_ia_k - ia_i \ast a_k) = 0. \] (41)
This constraint have been related to the field theory Lagrangian (40). Up to the surface terms this Lagrangian density is equivalent to
\[ L = -J^i (A_i^* + a_i) + a_0 \{ J_0 - \kappa (\varepsilon^{ik} \partial_i a_k - i \varepsilon^{ik} a_i \ast a_k) \} + \frac{K}{2} \varepsilon^{ik} a_i \dot{a}_k. \] (42)

For the interior points of the area \( \Omega \) the Euler-Lagrange equation for the gauge field \( a_\mu \) reads
\[ J_\mu = -\kappa \{ \varepsilon_{\mu\nu\lambda} \partial_\nu a_\lambda - i \varepsilon_{\mu\nu\lambda} a_\nu \ast a_\lambda \} \] (43)
and in particular leads to the constraint
\[ \Pi_0 = -J_0 + \kappa \{ \varepsilon^{ik} \partial_i a_k - i \varepsilon^{ik} a_i \ast a_k \} \approx 0. \] (44)

The 3-current is not conserved in the usual sense as well as
\[ \partial_\mu J_\mu = i \kappa \varepsilon_{\mu\nu\lambda} \partial_\nu (a_\nu \ast a_\lambda). \] (45)

Remind, that the 3-current is given by
\[ J_0(x) = \rho(x,t) \quad J^i(x) = \sum_{\alpha=1}^{N} \dot{x}_\alpha \delta(x - x_\alpha(t)). \] (46)

The field \( a_0 \) is Lagrange multiplier providing constraint (44) and one can set \( a_0 = 0 \). In the complex coordinates Lagrangian reads
\[ L = \int_D d\mathbf{x} L = -\sum_{a=1}^{N} \dot{z}_a(t) \{ A_\ast^z(x_a) + a_\ast^z(x_a) \} - \sum_{a=1}^{N} \dot{\bar{z}}_a(t) \{ A^\ast_{\bar{z}}(x_a) + a_{\bar{z}}(x_a) \} + 2i \kappa \int_D d\mathbf{x} a_{\bar{z}}(x) \dot{a}_z(x) \] (47)
and the Gauss law (44) is given by
\[ \rho(x,t) + 2i \kappa (\partial_z a_\ast - \partial_{\bar{z}} a_{\bar{z}}) - 4 \kappa a_\ast \ast a_{\bar{z}} = 0. \] (48)

Lagrangian (47) is a first order in particle velocities and generates the system of second-class constraints
\[ \Pi_{\bar{z}}^\alpha = p_{\bar{z}}^\alpha + A_{\bar{z}}^\alpha(x_\alpha) + a_{\bar{z}}(x_\alpha) \approx 0, \] (49)
\[ \Pi_{z}^\alpha = p_z^\alpha + A_z^\alpha(x_\alpha) + a_z(x_\alpha) \approx 0. \] (50)

Chern-Simons field \( a_\mu(x) \) must be quantized. The equal time canonical commutation relation reads
\[ [\hat{a}_z(x), \hat{a}_{\bar{z}}(x')] = \frac{1}{2\kappa} \delta(x - x'). \] (51)
Choosing the holomorphic polarization [22] we set
\[
\hat{a}_z(x) = \frac{1}{2\kappa} \frac{\delta}{\delta a_z(x)}.
\] (52)

The quantum state vector must satisfy the Gauss law (48)
\[
\hat{\Pi}_0(x) \Phi[a_z; x_1, \ldots, x_N] = 0
\] (53)
where
\[
\hat{\Pi}_0(x) = \hat{J}_0(x) + 2i\kappa \left\{ \partial_z a_z(x) - \frac{1}{2\kappa} \partial_z \frac{\delta}{\delta a_z(x)} \right\} - 2a_z(x) * \frac{\delta}{\delta a_z(x)}. \] (54)

Together with (53) the state vector is subjected to the constraint (50)
\[
\hat{\Pi}_\alpha \Phi = \left\{ \hat{p}_\alpha^2 + A_\alpha^\gamma(x_\alpha) + \frac{1}{2\kappa} \frac{\delta}{\delta a_z(x_\alpha)} \right\} \Phi = 0. \] (55)

Equation (45) gives
\[
\left\{ \partial_\mu \hat{j}^\mu(x) - \partial_t \left[ a_z(x) * \frac{\delta}{\delta a_z(x)} \right] \right\} \Phi = 0. \] (56)

Consider a simplest case of constant (\(a_z\)-independent) functionals
\[
\frac{\delta}{\delta a_z(x)} \Phi_0 = 0
\] (57)
Remark, that in the subspace of constant wave functionals current is conserved
\[
\partial_\mu \hat{j}^\mu(x) \Phi_0 = 0
\] (58)
and one can set \(\hat{\rho}(x) = \hat{J}_0(x) = \sum_{\alpha=1}^N \delta(x - x_\alpha)\). Now the Gauss law is reduced to the equation
\[
\left\{ \sum_{\alpha=1}^N \delta(x - x_\alpha) + 2i\kappa \partial_z a_z \right\} \Phi_0 = 0
\] (59)

Solving the last equation one gets
\[
a_z(x) = \frac{i}{2\pi \kappa} \sum_{\alpha=1}^N \frac{1}{z - z_\alpha}. \] (60)

This is a complex connection [23] used in the holomorphic gauge quantization of the non-Abelian CS fields (see e.g. Ref. 24). Remark, that the use of this non-Hermitian connection requires introduction of compensating measure in scalar products [25, 26].
A comment is in order here. It regards the status of the variables \( z \) and \( \bar{z} \) in the transformation (9). The proper approach is to handle coordinates \( z \) and \( \bar{z} \) as independent and impose the reality condition \( z^* = \bar{z} \) at the end of calculations.

The wave function \( \Phi_0 \) depends on the particle coordinates, and this dependence can be read out from the LLL condition (55):

\[
\{ \hat{p}^0_z + A^*_z(x_\alpha) \} \Phi_0 = 0.
\]

The corresponding solution looks as follows

\[
\Phi_0(x_1, ..., x_N) \sim F(z_1, ..., z_N)e^{-\frac{B^\star}{4} \sum_{\alpha=1}^N |z_\alpha|^2}
\]

with a holomorphic function \( F \).

4. Laughlin Wave Function

Wave function of the system of electrons satisfies LLL condition (3). The wave function (62) also belongs to the LLL, but with respect to the reduced magnetic field \( B^\star = \nu B \). One of the principal assertions of CF approach is that function \( F \) in (62) gives the holomorphic part of the total LLL wave function (4)

\[
\Psi(x_1, ..., x_N) = e^{-2pB^\star \sum_{\alpha=1}^N |z_\alpha|^2} \Phi_{CF}.
\]

For the electrons in the LLL the filling factor can be defined by the ratio

\[
\nu = \frac{N(N-1)}{2J} = \frac{J^*}{J}.
\]

where \( J \) is a total angular momentum of the system of electrons. Strictly speaking definition (64) is valid in the thermodynamical limit, but we suppose its validity in our case.

Write down the operator equation

\[
\hat{J} = (2p + 1) \hat{J}^*,
\]

which corresponds to the fact, that the CF in the magnetic field \( B^\star \) occupies the site, which is \( (2p + 1) \) times larger than Landau site for the original electron in the magnetic field \( B \).

Angular momentum operator for electrons in LLL is given by

\[
\hat{J} = \frac{2}{B} \sum |\hat{p}^\alpha_z - A^z(x_\alpha)||\hat{p}^\alpha_{\bar{z}} - A^z(x_\alpha)|,
\]

and classically on the constrained manifold

\[
J(\pi^a_\alpha = 0) = \frac{B}{2} \sum_{\alpha=1}^N |z_\alpha|^2.
\]
One easily verifies that analogous expression for the composite particles is given by

\[ J^\star (\Pi^2 = 0) = \frac{2}{B^*} \sum_{\alpha=1}^{N} [p_{\bar{z}} - A^*_z(x_\alpha) + a_{\bar{z}}(x_\alpha)] [p_{z} - A^*_z(x_\alpha) + a_{z}(x_\alpha)] \]

\[ = \frac{B^*}{2} \sum_{\alpha=1}^{N} |z_\alpha|^2. \]  

(68)

Using this observation and Eq.(60) we write down the angular momentum for the system of composite particles

\[ \hat{J}^\star = \frac{2}{B^*} \sum_{\alpha=1}^{N} \left\{ \hat{p}_{\bar{z}} - A^*_z(x_\alpha) \right\} \left\{ \hat{p}_{z} - A^*_z(x_\alpha) + 2ip \sum_{\beta=1, \beta \neq \alpha}^{N} \frac{1}{z_\alpha - z_\beta} \right\}. \]  

(69)

The sought for LLL composite fermion wave function is the angular momentum eigenstate

\[ \hat{J}^\star \Phi_{CF} = \frac{N(N-1)}{2} \Phi_{CF}, \]  

(70)

and satisfies the Knizhnik-Zamolodchikov [27] equation

\[ \left\{ \hat{p}_{\bar{z}}^\alpha - A_z^*(x_\alpha) + 2ip \sum_{\beta=1, \beta \neq \alpha}^{N} \frac{1}{z_\alpha - z_\beta} \right\} \Phi_{CF} = -i \sum_{\beta=1, \beta \neq \alpha}^{N} \frac{1}{z_\alpha - z_\beta} \Phi_{CF}. \]  

(71)

The corresponding solution is given by

\[ \Phi_{CF} = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^{2p+1} e^{-\frac{B^*}{4} \sum_{\alpha=1}^{N} |z_\alpha|^2} \]  

(72)

yielding the final result – Laughlin wave function

\[ \Psi = \prod_{\alpha < \beta} (z_\alpha - z_\beta)^{2p+1} e^{-\frac{B^*}{4} \sum_{\alpha=1}^{N} |z_\alpha|^2} \]  

(73)

5. Conclusions

In the present paper we have considered a system of electrons in the lowest Landau level with the aim to obtain the non-commutative version of the CS description of quantum Hall effect in terms of particle variables. We have introduced the area preserving singular geometric transformations and conclude that the area preservation condition when interpreted in terms of the QHE filling factor yields the Gauss law in CS theory with APD as a gauge group. Geometric transformations are generated by the gauge fields and the Gauss law is invariant.
with respect to generalized gauge transformations. In that part we reproduce corresponding conclusions given in Ref. 3.

As a further step we have proposed the modification of the area transformation rule. This modification is presented as an heuristic tool without any special justification. Despite of that, this Ansatz leads to the Gauss law in the form adopted in the NCCS theory and permits to write down corresponding Lagrangian. This Lagrangian describes particles in the effective (reduced) magnetic field and interacting with the non-commutative CS field. Developed scheme corresponds to the composite fermion picture in the non-commutative CS theory.

The quantum state vector depends on the gauge field configurations and particle coordinates. Dynamics is governed by the system of constraints imposed on the state vector. These conditions are Gauss law and constraints expressing vanishing of kinetic momenta of particles in the LLL. We have examined the self-consistent solution corresponding to the constant (gauge field independent) wave functional. In that case one may identify the particle density as a time component of the conserved local 3-current and consideration is reduced to the linear CS theory in the holomorphic gauge.

The detailed form of the wave function was determined considering the total angular momentum of the system of composite fermions. This permitted to express the corresponding solution in the form of the Laughlin wave function.

The proposed scheme seems to be equivalent to the hydrodynamical formulation, but the use of mechanical variables permits to reconstruct the structure of the many-electron wave function in the composite fermion approach.

Acknowledgements

Authors thank P. Sorba and A. Tavkhelidze for the interest and encouraging remarks. M.E. is grateful to P. Sorba for his hospitality at LAPTH (Annecy), where the part of the present work was done. Work was supported in part by the grant INTAS-GEORGIA 97-1340 and by SCOPES under grant 7GEPJ62379.

References


[18] P. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964),


