Noncommutative $3D$ harmonic oscillator

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Abstract. We find transformation matrices allowing to express non-commutative three dimensional harmonic oscillator in terms of an isotropic commutative oscillator, following “philosophy of simplicity” approach. Non-commutative parameters have physical interpretation in terms of an external magnetic field. Furthermore, we show that for a particular choice of noncommutative parameters there is an equivalent anisotropic representation, whose transformation matrices are far more complicated. We indicate a way to obtain the more complex solutions from the simple ones.

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String theory results [1], [2], have generated a revival of interest for field theory in a non-commutative geometry [3]. A simpler insight on the role of non-commutativity in field theory can be obtained studying solvable models of non-commutative quantum mechanics [4], [5], [6].

Recently, we have presented [7] the description of the noncommutative harmonic oscillator in two dimensions in terms of an isotropic commutative oscillator in an external magnetic field. This interpretation is made possible by the existence of a simple representation of the noncommutative coordinates in terms of the canonical ones. There are many other possible representations of the noncommutative algebra in terms of two Heisenberg algebras [8]. Nevertheless, all of them fall in two groups: those leading to a set of anisotropic oscillators, and others leading to an isotropic oscillator. This correspondence indicates that, in solving an explicit model, one should always look for the simplest form of the solution. As far as two dimensional models are concerned, choice of particular solution may seem of less importance. However, it becomes very important in higher dimensions where the set of equations is far more complicated and finding a simple way of solving it becomes essential.

In this note we are going to adopt the philosophy of simplicity and point out its advantage in describing the three dimensional noncommutative harmonic oscillator.

As an introduction we are going to give a brief review of the way in which noncommutative system can be transformed into an equivalent commutative form. This approach is shown to be equivalent to the introduction of the Moyal $\ast$-product [9], [10], [11], which is the usual way to introduce noncommutativity. One starts with the set of noncommutative coordinates $(x, p)$ of position and momentum satisfying the following commutation relations

$$\left[ x_k, x_j \right] = i \Theta_{kj} \quad (1)$$
$$\left[ p_k, p_j \right] = i B_{kj} \quad (2)$$
$$\left[ x_k, p_j \right] = i \delta^{kj} \quad (3)$$

where, $\Theta$ and $B$ are matrices whose elements measure the noncommutativity of coordinate and momenta respectively. We shall represent noncommutative variables as a linear combination of commutative coordinates $(\alpha, \beta)$ in a six dimensional phase space

$$\begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} a & b \\ d & c \end{pmatrix} \begin{pmatrix} \vec{\alpha} \\ \vec{\beta} \end{pmatrix} \quad (4)$$

The $6 \times 6$ transformation matrix is written in terms of $3 \times 3$ blocks $a, b, c, d$. The four sub-matrices satisfy

$$a b^T - b a^T = \Theta \quad (5)$$
$$c d^T - d c^T = -B \quad (6)$$
$$c a^T - b d^T = I \quad (7)$$
following from the commutation relations (1), (2), (3). $M^T$ denotes the transposed of a $3 \times 3$ matrix $M$.

As a specific model we choose a three dimensional, noncommutative, harmonic oscillator described by the Hamiltonian

$$H \equiv \frac{1}{2} \left[ p_i^2 + x_i^2 \right]$$

(8)

where, we set classical frequency and mass to unity. One can verify that the attempt to solve the system of equations (5), (6), (7) in full generality (meaning the most general form of the transformation matrices), already in two dimensions, led to a complicated, but still tractable, set of equations [8],[7]. In three dimensions things only get considerably worse. Thus, we apply the above mentioned philosophy of simplicity.

First of all, we notice that in three dimensions antisymmetric matrices $\Theta$ and $b$ can always be written as

$$\Theta_{ab} \equiv \epsilon_{abc} \theta_c, \quad B_{ab} \equiv \epsilon_{abc} B_c$$

(9)

On physical grounds the isotropic solution, having spherical symmetry, requires the equivalence of all three directions. Therefore, let us impose $\theta_c \equiv \theta$ and $B_c \equiv B, \forall c$. Furthermore, in analogy with two dimensions, let us choose matrices $a, b, c, d$ to be

$$a = a \, I, \quad c = c \, I$$

$$b = b \, K, \quad d = d \, K$$

(10)

where, $I$ is the identity and $K$ is an unknown matrix. Explicit form of $K$ is found to be

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(11)

Inserting the ansatz (10) in (5) and (6) leads to the solutions for the parameters as:

$$b = -\frac{\theta}{a}, \quad d = \frac{B}{c}$$

(12)

The remaining equation (7) gives

$$ac + \frac{B\theta}{ac} = 1$$

(13)

which determines the parameters $c$ and $d$ as:

$$c = \frac{1}{2a} \left( 1 + \sqrt{1 - 4B\theta} \right) \equiv \frac{1}{2a} \left( 1 + \sqrt{\kappa} \right)$$

$$d = \frac{a}{2\theta} \left( 1 - \sqrt{1 - 4B\theta} \right) \equiv \frac{a}{2\theta} \left( 1 - \sqrt{\kappa} \right)$$

(14)

(15)
The three dimensional solutions follows the same pattern as in two dimensions [7]. The two dimensional isotropic representation, is characterized by the presence of a mixed term in the Hamiltonian, which is the reminiscence of the noncommutativity of the system. We find that such term is also present in this case and is of the form

$$H_{\text{mixed}} = \frac{1}{2} (-\theta \alpha_1 \beta_2 + B \alpha_2 \beta_1) + \ldots$$  \hspace{1cm} (16)

In [7] mixed term led to the coupling of the noncommutative parameters to the components of the angular momentum operator. Thus, we were able to interpret the noncommutativity as a “magnetic effect”. In order to reproduce, if possible, the same interpretation in (16) one has to impose the condition

$$B = \theta$$  \hspace{1cm} (17)

which allows to rewrite the mixed term as

$$H_{\text{mixed}} = -\frac{1}{2} \theta_i L_i$$  \hspace{1cm} (18)

where, $\vec{L}$ is the angular momentum operator. We arrive at the isotropic representation of the noncommutative three-$D$ harmonic oscillator

$$H = h_\alpha (\alpha_i)^2 + h_\beta (\beta_i)^2 - \frac{1}{2} \vec{\theta} \cdot \vec{L}$$  \hspace{1cm} (19)

where, $h_\alpha, h_\beta$ are given by

$$h_\alpha \equiv \frac{a^2}{2} \left[ 1 + \frac{1}{4\theta^2} (1 - \sqrt{\kappa})^2 \right]$$  \hspace{1cm} (20)

$$h_\beta \equiv \frac{\theta^2}{2a^2} \left[ 1 + \frac{1}{4\theta^2} (1 + \sqrt{\kappa})^2 \right]$$  \hspace{1cm} (21)

where, $\kappa \equiv 1 - 4\theta^2$. Hamiltonian (20) is invariant under spatial rotations. This fact permits to choose a new set of coordinates with one axis aligned with $\vec{\theta}$. In the rotated frame $\alpha_i \rightarrow R_{ij} \alpha_j, \beta_i \rightarrow R_{ij} \beta_j$. The Hamiltonian (19) takes a simpler looking form

$$H = h_\alpha (\alpha_i)^2 + h_\beta (\beta_i)^2 - \frac{\sqrt{3}}{2} \theta L_\theta$$  \hspace{1cm} (22)

The explicit form of the rotation matrix is given by

$$R = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{pmatrix}$$  \hspace{1cm} (23)

The spectrum of the system is

$$E_{n_+ n_-} = \omega \left( n_+ + n_- + n_0 + \frac{3}{2} \right) + (n_+ - n_-) \frac{\sqrt{3}}{2} \theta$$  \hspace{1cm} (24)

where, $m \equiv n_+ - n_-$ is the “magnetic” eigenvalue of the $L_\theta$ component of the angular momentum operator, and

$$\omega \equiv 2 \sqrt{h_\alpha h_\beta}$$  \hspace{1cm} (25)
\( \omega \) being expressed in terms of units of classical frequency. The explicit solution give the frequency of the harmonic oscillator equal to the classical frequency. The noncommutative effects are pure magnetic field effects in (19). The results are identical to two dimensional case for the special choice \( \theta = B \). In three dimensions, however, this choice is imposed by the form of the mixed Hamiltonian and is the only possible solution which gives isotropy of the Hamiltonian. One can re-write the spectrum in the following way

\[
E_{n_+ n_-} = \omega_+ \left( n_+ + \frac{1}{2} \right) + \omega_- \left( n_- + \frac{1}{2} \right) + \omega \left( n_0 + \frac{1}{2} \right)
\]

(26)

provided the following identifications are in order

\[
\sqrt{3} \theta = (\omega_+ - \omega_-) \\
\omega = \frac{1}{2} (\omega_+ + \omega_-)
\]

(27)

(28)

The above spectrum is the one of three, one-dimensional, anisotropic oscillators. Thus, 3D noncommutative harmonic oscillator has both isotropic and anisotropic commutative representations. In order to prove the existence of solutions for the transformation matrices of the anisotropic representation, without explicitly solving complex set of equations (5), (6), (7), one can proceed in the following way. Let us first find the relation among the commutative coordinates of the two different representations by defining

\[
Q_1 = A_1 \alpha_1 - A_2 \beta_2, \quad Q_2 = -A_1 \alpha_2 + A_2 \beta_1, \quad Q_3 = C \alpha_3 \\
P_1 = A_1 \alpha_2 + A_2 \beta_1, \quad P_2 = -A_1 \alpha_1 - A_2 \beta_2, \quad P_3 = D \beta_3
\]

(29)

(30)

The parameters in (30) are determined by the requirement that the above redefinitions turn the Hamiltonian (19) into its anisotropic form

\[
H = \frac{1}{2} \omega_+ (Q_1^2 + P_1^2) + \frac{1}{2} \omega_- (Q_2^2 + P_2^2) + \frac{1}{2} \omega (Q_3^2 + P_3^2)
\]

(31)

which gives the solutions

\[
A_1 = \sqrt{\frac{\hbar_\alpha}{\omega}}, \quad A_2 = \sqrt{\frac{\hbar_\beta}{\omega}} \\
C = \sqrt{2} A_1, \quad D = \sqrt{2} A_2
\]

(32)

The relation between the anisotropic coordinates \( (\vec{Q}, \vec{P}) \), written as a “column matrix” \( \vec{Q} \), and isotropic ones \( (\vec{\alpha}, \vec{\beta}) \) can be written in matrix form as

\[
\begin{pmatrix}
\vec{\alpha} \\
\vec{\beta}
\end{pmatrix} = 
\begin{pmatrix}
A_2 L_1 & A_2 L_2 \\
-A_1 L_2 & A_1 L_1
\end{pmatrix}
\begin{pmatrix}
\vec{Q} \\
\vec{P}
\end{pmatrix}
\]

(33)
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The above equation is written in the block form with $3 \times 3$ matrices $L_1, L_2$ given by

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (34)$$

The relation between the noncommutative coordinates (after rotation) and the isotropic set of solutions can be written in a block form

$$\begin{pmatrix} \tilde{x} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} a R^T & b K R^T \\ d K R^T & c R^T \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \quad (35)$$

On the other hand, the anisotropic transformation matrices relate the noncommutative coordinates to $\left( \vec{Q}, \vec{P} \right)$ as

$$\begin{pmatrix} \tilde{x} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{d} & \tilde{c} \end{pmatrix} \begin{pmatrix} \vec{Q} \\ \vec{P} \end{pmatrix} \quad (36)$$

Comparing (36) to (35) with the help of (33) one obtains solutions

$$\tilde{a} = a A_2 \tilde{L}_1 - b A_1 \tilde{L}_4, \quad \tilde{b} = a A_2 \tilde{L}_2 + b A_1 \tilde{L}_3 \quad (37)$$

$$\tilde{c} = d A_2 \tilde{L}_4 + c A_1 \tilde{L}_1, \quad \tilde{d} = d A_2 \tilde{L}_3 - c A_1 \tilde{L}_2 \quad (38)$$

where the $3 \times 3$ matrices $\tilde{L}_i, i = 1, 2, 3, 4$, are found to be

$$\tilde{L}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & \sqrt{3} & -\sqrt{2} \\ \sqrt{2} & -\sqrt{3} & -\sqrt{2} \\ \sqrt{2} & 0 & 2\sqrt{2} \end{pmatrix}, \quad \tilde{L}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{3} & -\sqrt{2} & 0 \\ \sqrt{3} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 \end{pmatrix} \quad (39)$$

$$\tilde{L}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -\sqrt{3} & -\sqrt{2} \\ \sqrt{2} & 0 & 2\sqrt{2} \\ \sqrt{2} & \sqrt{3} & -\sqrt{2} \end{pmatrix}, \quad \tilde{L}_4 = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 \\ -\sqrt{3} & -\sqrt{2} & 0 \end{pmatrix} \quad (40)$$

Exploiting the explicit solutions (20), (21) and (32), one can re-write the anisotropic set of solutions in terms of isotropic ones

$$\tilde{a} = \sqrt{ac} \left( \tilde{L}_1 + \frac{\theta}{ac} \tilde{L}_4 \right), \quad \tilde{b} = \sqrt{ac} \left( \tilde{L}_2 - \frac{\theta}{ac} \tilde{L}_3 \right)$$

$$\tilde{a} = \tilde{c}, \quad \tilde{d} = -\tilde{b} \quad (41)$$

One can verify that the above set of solutions satisfies basic requirements (5), (6), (7). As already advocated, a comparison of the isotropic solutions to the anisotropic ones confirms the validity of the “philosophy of simplicity” approach.

One may wonder if there are other solutions leading to the same isotropic representation. Let us assume that the transformation matrices are of the same form as before, but with
different matrix elements. For example, the matrices are $\mathbf{a}$ and $\mathbf{c}$ are $a_{ij} = a^{(i)} \delta_{ij}$ (no summation over the $i$ index), while matrices $\mathbf{b}$ and $\mathbf{d}$ are given by

$$
\begin{align*}
\mathbf{b} &= \begin{pmatrix} 0 & b_{12} & 0 \\
0 & 0 & b_{23} \\
b_{31} & 0 & 0 
\end{pmatrix}, & \mathbf{d} &= \begin{pmatrix} 0 & d_{12} & 0 \\
0 & 0 & d_{23} \\
d_{31} & 0 & 0 
\end{pmatrix}
\end{align*}
$$

(42)

Without going into details, the Hamiltonian following from the above solution is the generalization of (19) with different coefficients $h_i, i = 1, \ldots, 6$ multiplying canonical coordinates. The isotropy of the system requires the equivalence of those coefficients for the coordinates $\alpha$ and $\beta$ respectively. This requirement inevitably leads to the condition (17). Thus, we conclude that there are no other isotropic solutions different from those described in this paper. We have thus shown that the three-D noncommutative harmonic oscillator can be represented as an isotropic oscillator coupled to an external magnetic field, generated by space non-commutativity. This representation is based on a very simple set of transformation matrices relating noncommutative to canonical coordinates. Alternative representation is also possible in terms of three 1D anisotropic harmonic oscillators. The second set of solutions is far more complicated and difficult to obtain solving (5), (6), (7). Nevertheless, we have described an indirect way of finding these solutions. Their explicit form was needed to support the philosophy of simplicity approach described in this paper.

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