Abstract

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Scale Dependence of Hadronic Wave Functions and Parton Distributions
In light-cone quantization and light-cone gauge, the hadronic states in QCD are expressed as an expansion of various quark and gluon Fock components [1]. This expansion depends, among others, on the momentum cut-off used to truncate the theory, which is often interpreted as the physical resolution scale. Although the physical observables, such as masses, angular momenta, form factors and cross sections, ought to be independent of the cut-off, many interesting hadronic matrix elements do. A well-known example is the matrix elements of twist-two operators which define the moments of Feynman’s parton distributions [2]. For these quantities, one should be able to trace the scale dependence back to that of the hadronic wave functions.

In this paper, we are interested in how the light-cone Fock components depend on the momentum cut-off. Finding the solution directly from diagonalizing the light-cone hamiltonian is less obvious. Instead we approach the problem by systematically relating the Fock expansion to the matrix elements of a certain class of quark-gluon operators between the QCD vacuum $\left|0\right>$ and the hadron states, taking advantage of the simplicity of $\left|0\right>$ in light-cone quantization. The scale-dependence of the wave function amplitudes can then be traced to the wave function renormalization constants of quark and gluon fields. Following this, we derive the scale dependence of the parton densities from individual Fock components. The scale evolution of these contributions obeys an infinite set of coupled, linear differential-integral equations. When summed over all Fock contributions, we recover the well-known Dokshitzer-Gribov- Lipatov-Altarelli-Parisi (DGLAP) equation for parton densities.

Before starting, let us remind the reader some salient features of light-cone quantization relevant for the following discussion [3]. The light-cone time $x^+$ and coordinate $x^-$ are defined as $x^\pm = 1/\sqrt{2}(x^0 \pm x^3)$. Likewise we define Dirac matrices $\gamma^\pm = 1/\sqrt{2}(\gamma^0 \pm \gamma^3)$. The projection operators for Dirac fields are defined as $P_\pm = (1/2)\gamma^\pm\gamma^\pm$. Any Dirac field $\psi$ can be decomposed into $\psi = \psi_+ + \psi_-$ with $\psi_\pm = P_\pm\psi$. $\psi_\pm$ is a dynamical degrees of freedom and has the canonical expansion,

$$\psi_+(\xi^+ = 0, \xi^-, \xi_\perp) = \int \frac{d^2k_\perp}{(2\pi)^2} \frac{dk^+}{2k^+} \sum_\lambda \left[ b_\lambda(k)e^{-ik\xi^- - i\xi_\perp k_\perp} + d_\lambda(k)e^{ik\xi^- - i\xi_\perp k_\perp} \right].$$

Likewise, for the gluon fields in the light-cone gauge $A^\perp = 0$, $A_\perp$ is dynamical and has the expansion,

$$A_\perp(\xi^+ = 0, \xi^-, \xi_\perp) = \int \frac{d^2k_\perp}{(2\pi)^2} \frac{dk^+}{2k^+} \sum_\lambda \left[ a_\lambda(k)e^{-ik\xi^- - i\xi_\perp k_\perp} + a^\dagger_\lambda(k)e^{ik\xi^- - i\xi_\perp k_\perp} \right].$$

$\psi_-$ and $A^\perp$ are dependent variables.

The key observation in this paper is that the light-cone Fock expansion of a hadron state is completely defined by the matrix elements of a special class equal light-cone time quark-gluon operators between the QCD vacuum and the hadron. These operators are specified as follows: Take the $+$ component of the Dirac field $\psi_+$ and the $+\perp$-component of the gauge field $F^{++\perp}$. We sometimes label the $\perp$-component with index $i = 1,2$. Assume all these fields are at light-cone time $x^+ = 0$, but otherwise with arbitrary dependence on other spacetime coordinates. Products of these fields with the right quantum numbers (spin, flavor, and color) define a set of operator basis. This has been done in the past for light-cone
correlations in which all fields are separated along the light-cone [4].] Clearly, these operators are not gauge-invariant because one cannot gauge-invariantize them by simply inserting string operators along the light-cone. Since all fields are at different points in the transverse directions, the operators do not have singularities requiring special renormalization. Moreover, at equal light-cone time, there is no need to introduce a time-ordering among different fields because the difference is proportional to equal-light-cone-time commutators which are straightforward to evaluate. In fact, to simplify the discussion, we assume all fields in the operators are normal ordered, i.e., the annihilation operators appearing at the right of the creations. We believe that the matrix elements of all these operators between the hadron state and the QCD vacuum yield complete information about the hadron wave function.

As an example, let us consider \( \pi^+ \) meson with momentum \( P^\mu \) along the z-direction. The leading light-cone Fock states consist of a pair of up and anti-down quarks. The light-cone helicity of the \( \pi^+ \) meson is zero, but the light-cone helicity of the quark-antiquark pair can either be zero or \( \pm 1 \). Use \( u_+(\xi^-, \xi^+) \) to represent the up-quark field in the coordinate space and \( \bar{d}_+(0) \) the anti-down-quark field. In the massless limit, the operator \( \bar{d}_+ \gamma^+ \gamma_5 u_+ \) yields a helicity-0 pair, and \( \bar{d}_+ \sigma^{\pm} \gamma_5 u_+ \) a helicity-1 pair. The helicity counting here is based on the chirality of the operators and the relation between chirality and helicity of massless fermions. [One can in principle construct similar operators without the \( \gamma_5 \) matrix; however parity forbids any finite matrix elements of them between the QCD vacuum and pseudo-scalar mesons.] The first operator defines a coordinate amplitude,

\[
\langle 0 | \bar{d}_+(0) \gamma^+ \gamma_5 u_+(\xi^-, \xi^+)| \pi^+(P) \rangle = \phi_0(\xi^-, \xi^+) 2P^+ ,
\]

where we normalize the state covariantly \( \langle P | P' \rangle = 2P^+(2\pi)^3 \delta(P^+ - P'^+) \delta^i(P_\perp - P'_\perp) \). Introducing the Fourier transformation of the amplitude

\[
\phi_0(\vec{k}_\perp, x) = \int d^2 \xi \xi^- e^{i(\vec{k}_\perp \cdot \vec{\xi}_\perp)} \phi_0(\xi^-, \xi^+) ,
\]

we can invert Eq. (3) to find

\[
\langle \pi^+(P) \rangle = \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{dx}{2 \sqrt{x(1-x)}} \phi_0(x, k_\perp) \left[ b_{\bar{d}d}^i(x, \vec{k}_\perp) d_{u\bar{u}}^i(1-x, -\vec{k}_\perp)
- b_{\bar{d}d}^i(x, \vec{k}_\perp) d_{\bar{d}d}^i(1-x, -\vec{k}_\perp) \right] |0\rangle + ...
\]

where \( i \) is the color index. The creation and annihilation operators are normalized according to the commutation relation \( \left[ b(k) , b^\dagger(k') \right]_+ = 2k^+ (2\pi)^3 \delta( k^+ - k'^+) \delta^i(k_\perp - k'_\perp) \). We have assumed here that the full QCD vacuum is a perturbative vacuum in light-cone quantization. In particular, we neglect the sublety of zero-modes which might cause problems at some stage [5].

The operator with helicity-1 quark-anti-quark pair defines the amplitude

\[
\langle 0 | \bar{d}_+(0) \sigma^{\pm} \gamma_5 u_+(\xi^-, \xi^+)| \pi^+(P) \rangle = \partial^i \phi_1(\xi^-, \xi^+) 2P^+ ,
\]

where \( i = 1, 2 \) is an index for transverse directions. Performing a Fourier transformation on the both sides and inverting the equation, we find a light-cone Fock component

\[
\langle \pi^+(P) \rangle = \int \frac{d^2 k_\perp}{(2\pi)^3} \frac{dx}{2 \sqrt{x(1-x)}} \phi_1(x, k_\perp) \left[ (k_1 - ik_2) b_{\bar{d}d}^i(x, \vec{k}_\perp) d_{\bar{d}d}^i(1-x, -\vec{k}_\perp)
+ (k_1 + ik_2) b_{\bar{d}d}^i(x, \vec{k}_\perp) d_{\bar{d}d}^i(1-x, -\vec{k}_\perp) \right] |0\rangle + ...
\]

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The angular momentum content of the wave function is clear: For a quark-antiquark pair carrying helicity $±1$, it couples to an orbital wave function with $J^{\pm}_{\text{3}} = ±1$. Parity determines the relative sign of the two contributions. The phenomenological implications of $\phi_i(x, k_\perp)$ for pion form factors have been discussed in the literature before [6].

It is now straightforward to study the cut-off dependence of $\phi_{Q,1}(x, k_\perp)$. For the moment, we focus on the transverse momentum cut-off $\Lambda$, although a cut-off in $x$ is also needed at $x \to 0$ in general. Besides the explicit cut-off dependence in the wave function amplitudes, the $k_\perp$ integration in Eq. (5) is implicitly bounded by $\Lambda$. In any cut-off scheme, the quark and gluon fields in QCD as well as the strong coupling constant $\alpha_s$ depend on the cut-off. For large $\Lambda$, the dependence of quantum fields on $\Lambda$ can be calculated in perturbation theory because of the asymptotic freedom. In fact, according to the standard renormalization theory, on has

$$\psi_\Lambda(\xi) = Z_F^{1/2}(\Lambda) \tilde{\psi}(\xi), \quad A_\mu^\Lambda(\xi) = Z_A^{1/2}(\Lambda) \tilde{A}_\mu(\xi)$$

where $\tilde{\psi}(\xi)$ and $\tilde{A}_\mu(\xi)$ are independent of $\Lambda$, and $Z_F(\Lambda)$ and $Z_A(\Lambda)$ are the wave function renormalization constants. Although the factorization in the above equation is scheme-dependent, but the $\Lambda$ dependence itself is not.

Going back to Eqs. (3,6), it is now clear that the cut-off dependence of the wave function amplitudes $\tilde{\phi}_i$ comes entirely from the field renormalization,

$$\tilde{\phi}_i^\Lambda(x, k_\perp) = Z_F(\Lambda) \tilde{\phi}_i(x, k_\perp),$$

where $\tilde{\phi}_i(x, k_\perp)$ is independent of $\Lambda$.

We claim that the above feature holds for all components of the pion wave function in the Fock expansion. For instance, the most general two-quark, one-gluon wave function amplitudes can be defined through the matrix elements of the operators,

$$\bar{d}_+(0) \gamma^+ \gamma_5 F^{\pm i}(\eta_-, \eta_\perp) u_+(\xi^- , \xi_\perp),$$

$$\bar{d}_+(0) \sigma^{+i} \gamma_5 F^{\pm j}(\eta_-, \eta_\perp) u_+(\xi^- , \eta_\perp).$$

Their scale dependence comes entirely from the wave function renormalization constants $Z_F(\Lambda) Z_A^{1/2}(\Lambda)$. In general, an $n$-particle Fock wave function amplitude with $n_q$ quark and antiquark and $n_g$ gluon creation operators has an explicit cut-off dependence through $Z_F^{n_q/2}(\Lambda) Z_A^{n_g/2}(\Lambda)$. Once again, all the momentum integrations are cut-off by $\Lambda$.

Knowing the scale dependence of individual components of the hadron wave function, we can calculate the scale dependence of their contributions to hadronic matrix elements.

As an example, we consider in the remainder of the paper Feynman parton distributions, although the discussion applies to generalized parton distributions as well [7]. Deriving the parton evolution equation from light-cone wave functions has been considered in Ref. [8]. Our approach here allows to uncover a set of new equations.

Consider, for example, the two-particle wave-function contribution to the $u$ quark distribution in the pion. We have

$$u_2(x, \Lambda) = \int_0^\Lambda k_\perp d^2k_\perp (2\pi)^3 \left[ |\phi_0^\Lambda(x, k_\perp)|^2 + k_\perp^2 |\phi_1^\Lambda(x, k_\perp)|^2 \right].$$

Since at large $k_\perp$, $\phi_0(x, k_\perp)$ goes like $1/k_\perp^2$ and $\phi_1(x, k_\perp)$ like $1/k_\perp^4$ modulo logarithms [1], the $k_\perp$ integration is convergent and hence $\Lambda$ can be taken to infinity. Thus the only $\Lambda$
dependence in $u_2(x, \Lambda)$ comes from the wave function renormalization factor $Z_F$. This yields

$$\frac{d}{d \ln \Lambda^2} u_2(x, \Lambda) = -2\gamma_F \frac{\alpha_s(\Lambda)}{4\pi} u_2(x, \Lambda),$$

(12)

where $\gamma_F$ is the anomalous dimension of $Z_F$ and is gauge-dependent. [In physical gauges, it is positive-definite.] In light-cone gauge,

$$\gamma_F = 2C_F \int_0^1 dy \frac{1 + (1 - y)^2}{y}$$

(13)

where $C_F = 4/3$. The above integral diverges at $y = 0$ [9], and we regulate the integral by cutting it off at $y = \epsilon$. Physics of course must be independent of any cut-off.

Equation (12) indicates that the two-particle Fock state contribution to the up-quark distribution gradually diminishes as $\Lambda \rightarrow \infty$. The physics is simple: As $\Lambda$ gets larger, the state is probed at shorter distance, and it becomes increasingly difficult for the meson to remain in the two-particle Fock component because of the radiation. As a consequence, the three-particle Fock amplitude increases at the leading-logarithmic level. In the light-cone gauge, $\gamma_F$ diverges at small $x$, and the radiation rate will depend on the cut-off $\epsilon$.

Consider now the three-particle Fock component contribution to the $u$-quark distribution. We write schematically,

$$u_3(x, \Lambda) = \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{d^2 k'_{\perp}}{(2\pi)^2} \frac{dx'}{(2\pi)^3} \frac{\phi^A(x, k_\perp, x', k'_{\perp}, \Lambda)^2}{\phi^A(x, k_\perp, x', k'_{\perp}, \Lambda)^2},$$

(14)

where we have not considered individual quark-antiquark-gluon helicities and orbital angular momentum projections although this can be done straightforwardly. As discussed before, the wave function $\phi^A(x, k_\perp, x', k'_{\perp})$ has an explicit dependence on $\Lambda$ through the wave function renormalization constant $Z_F Z_A^{1/2}$. Additional dependence comes from integrations over the transverse momenta $k_{\perp}$ and $k'_{\perp}$.

We take derivative with respect to $\Lambda$ using the chain rule. The derivative with respect to the wave function renormalization yields,

$$\frac{d}{d \ln \Lambda^2} u_3(x, \Lambda) = -(2\gamma_F + \gamma_A) \frac{\alpha_s(\Lambda)}{4\pi} u_3(x, \Lambda) + ... .$$

(15)

where $\gamma_A$ is the anomalous dimension of the gluon wave function renormalization. The physics of this part of the scale evolution is the same as the two-particle Fock component case: The splitting of the partons leads to the decrease of the probability for the pion to remain in the three-particle Fock state.

The integrations over $\vec{k}_\perp$ and $\vec{k}'_{\perp}$ do not yield divergences in general. In fact, there is no overall divergence (divergence arising from when all transverse momenta going to infinity at the same rate) because the power counting indicates that when two-momenta going to infinity at the same time, the integrals have negative superficial degree of divergence. However, there are subdivergences. These subdivergences arise from one-loop diagrams shown in Fig 1. The physics of these diagrams is that there are three-particle Fock amplitudes which are generated from the radiation of the two-particle Fock amplitude. Therefore, the result is proportional to $u_2(x, \Lambda)$. 

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First consider the loop integral from the $d$-quark line shown in Fig. 1a. In this case, the $k'_L$ integration is divergent whereas the $k_L$ integration is finite. It is easy to see that when $\Lambda$ changes, the result 3-particle distributions also changes. From this diagram, one finds,

$$\frac{d}{d\ln \Lambda^2} u_3(x, \Lambda) = \ldots + \gamma_F \frac{\alpha_s(\Lambda)}{4\pi} u_2(x, \Lambda) .$$

(16)

where the plus sign indicates that the three-particle Fock component receives a contribution from the radiation of the two-parton states.

Finally, let us consider the gluon radiation from for the $u$-quark line as shown in Fig. 1b. The integration over $\vec{k}_L$ is now divergent whereas the one over $\vec{k}'_L$ is finite. The $\vec{k}_L$ integration can be done using the standard light-cone perturbation theory,

$$\frac{d}{d\ln \Lambda^2} u_3(x, \Lambda) = \ldots + \frac{\alpha_s(\Lambda)}{2\pi} C_F \int_x^1 \frac{dy}{y} \frac{1 + y^2}{1 - y} u_2 \left( \frac{x}{y}, \Lambda \right) ,$$

(17)

where the divergence at $y = 1$ must be regulated. Adding everything together, we obtain the complete evolution equation for $u_3$ is

$$\frac{d}{d\ln \Lambda^2} u_3(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -(2\gamma_F + \gamma_A) u_3(x, \Lambda) + \gamma_F u_2(x, \Lambda) \right. \right.$$

$$+ 2C_F \int_x^1 \frac{dy}{y} \frac{1 + y^2}{1 - y} u_2 \left( \frac{x}{y}, \Lambda \right) \right] .$$

(18)

This is an inhomogeneous equation with a driving term $u_2$.

Going to wave function amplitudes with four and more partons posts no special difficulty, except one has to take into account the mixing with the singlet contribution. Take the
example of four-parton amplitudes for which three flavor structures $u\bar{d}gg$, $u\bar{d}\pi u$, and $u\bar{d}\bar{d}d$ must be considered separately. For $u\bar{d}gg$, the gluon radiation from $u$, $\bar{d}$, an $g$ of the three-parton component $u\bar{d}g$ yields,

\[
\frac{d}{d\ln \Lambda^2} u_4^u \bar{d}gg(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -2(\gamma_F + \gamma_A) u_4^u \bar{d}gg(x, \Lambda) \\
+ (\gamma_F + \gamma_{A1}) u_3(x, \Lambda) + 2C_F \int_x^1 \frac{dy}{y} \frac{1 + y^2}{1 - y} u_3 \left( \frac{x}{y}, \Lambda \right) \right], \tag{19}
\]

where $\gamma_{A1}$ is the part of the gluon anomalous dimension from the gluon loop. On the other hand, for $u\bar{d}\pi u$, the gluon splitting into $u\bar{u}$ pair yields,

\[
\frac{d}{d\ln \Lambda^2} u_4^u \bar{d}\pi u(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -4\gamma_F u_4^u \bar{d}\pi u(x, \Lambda) + \gamma_{A2} u_3(x, \Lambda) \\
+ 2 \int_x^1 \frac{dy}{y} T_F(y^2 + (1 - y)^2) g_3 \left( \frac{x}{y}, \Lambda \right) \right], \tag{20}
\]

where $T_F = 1/2$ and $\gamma_{A2}$ is the part of the gluon anomalous dimension from the $\bar{u}u$ loop. $g_3(x, \Lambda)$ is the gluon distribution from the $\bar{u}u$ Fock amplitude. Finally, for $u\bar{d}\bar{d}d$, the gluon splitting into $d\bar{d}$ pair yields,

\[
\frac{d}{d\ln \Lambda^2} u_4^u \bar{d}\bar{d}d(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -4\gamma_F u_4^u \bar{d}\bar{d}d(x, \Lambda) + \gamma_{A3} u_3(x, \Lambda) \right]. \tag{21}
\]

where $\gamma_{A3}$ is the part of the gluon anomalous dimension from the $\bar{d}d$ loop. When adding all the contributions ($\gamma_A = \gamma_{A1} + \gamma_{A2} + \gamma_{A3}$), we have,

\[
\frac{d}{d\ln \Lambda^2} u_4(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -4\gamma_F u_4^u \bar{d}\pi u(x, \Lambda) - 2(\gamma_F + \gamma_A) u_4^u \bar{d}gg(x, \Lambda) + (\gamma_F + \gamma_A) u_3(x, \Lambda) \\
+ 2 \int_x^1 \frac{dy}{y} \left\{ C_A \frac{1 + y^2}{1 - y} u_3 \left( \frac{x}{y}, \Lambda \right) + T_F(y^2 + (1 - y)^2) g_3 \left( \frac{x}{y}, \Lambda \right) \right\} \right]. \tag{22}
\]

To keep the evolution simple, we also consider the anti-up quark distribution at this order. The only contribution is from $u\bar{d}u\bar{u}$ component for which we have

\[
\frac{d}{d\ln \Lambda^2} \overline{u}_4(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -4\gamma_F \overline{u}_4(x, \Lambda) + 2 \int_x^1 \frac{dy}{y} T_F(y^2 + (1 - y)^2) g_3 \left( \frac{x}{y}, \Lambda \right) \right]. \tag{23}
\]

Therefore, if we define the valence up-quark distribution $u_\nu = u - \overline{u}$, then

\[
\frac{d}{d\ln \Lambda^2} u_\nu(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -4\gamma_F u_\nu \bar{d}\pi u(x, \Lambda) - 2(\gamma_F + \gamma_A) u_\nu \bar{d}gg(x, \Lambda) \\
+ (\gamma_F + \gamma_A) u_3(x, \Lambda) + 2 \int_x^1 \frac{dy}{y} \frac{1 + y^2}{1 - y} u_\nu \left( \frac{x}{y}, \Lambda \right) \right]. \tag{24}
\]

without the complication from the $g \to g\pi$ kernel.

It is not difficult to see that the evolution equation for $u_{\nu n}$ from Fock states with $n$ partons is

\[
\frac{d}{d\ln \Lambda^2} u_{\nu n}(x, \Lambda) = \frac{\alpha_s(\Lambda)}{4\pi} \left[ -\sum_{i=1}^n \gamma_i u_{\nu n}(x, \Lambda) + \sum_{i=1}^{n-1} \gamma_i u_{\nu n-1}(x, \Lambda) \\
+ 2C_F \int_x^1 \frac{dy}{y} \frac{1 + y^2}{1 - y} u_{\nu n-1} \left( \frac{x}{y}, \Lambda \right) \right], \tag{25}
\]

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where the sum over \( n - 2 \gamma_i \) excludes one \( \gamma_F \) and one \( \gamma_A \). The first term in Eq. (25) describes the depletion of the \( n \)-particle Fock component due to the gluon radiation into \( n + 1 \)-particle component; the second and third terms describe the increase of the \( n \)-particle component due to the gluon emission of the \( n - 1 \)-particle component. The difference between the later two comes from whether the gluon is radiated from the active particle or the spectators. The total \( u_v(x) \) distribution is a sum over all possible Fock components,

\[
    u_v(x) = \sum_{i=2}^{\infty} u_{iv}(x).
\]

Summing over all the equations for the individual Fock components, we recover the standard DGLAP equation,

\[
    \frac{d}{d \ln \Lambda^2} u_v(x, \Lambda) = \frac{\alpha_s(\Lambda)}{2\pi} \int_x^1 \frac{dy}{y} \frac{1 + y^2}{1 - y} u_v \left( \frac{x}{y}, \Lambda \right)
\]

which is an important check. Using the same procedure, one can derive evolution equations for other types of parton distributions, such as the singlet quark and gluon distributions, quark helicity and transversity distributions, as well as higher-twist distributions.

In summary, we find that the light-cone wave functions of hadrons in QCD can be entirely determined by the matrix elements of a class of quark-gluon operators. From this, we derive an infinite set of evolution equations for the parton distributions contributed by individual \( n \)-particle Fock components. These equations are consistent with the well-known DGLAP equation and are useful for phenomenological studies of hadronic structures and model buildings. They should also provide important constraints for wave functions derived from light-cone quantization.

Note added: After this paper was finished, we were informed that Ref. [10] has studied the evolution of the 2 and 3 particle Fock components, using an explicit light-front calculation.

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